Semigroups of Polyhedral Lattice Points: Convexity, Combinatorics, and Algebra

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CHENGYANG WANG
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Approved:

Jesús A. De Loera, Chair

Fu Liu

Anne Schilling

Committee in Charge

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Abstract

This dissertation explores problems of convexity, combinatorics, and algebra associated with semigroups of polyhedral lattice points.

In Chapter 2, we first attempt to generalize and extend three well-known convexity theorems, including Helly theorem, Tverberg theorem, and Colorful Carathéodory theorem, to affine semigroups. We define a novel notion, chromatic representations of semigroup elements, this is in contrast to the colorful theory developed by Bárány et al. Later, we focus on one-dimensional affine semigroups, numerical semigroups, and study the number of chromatic solutions in numerical semigroups.

In Chapter 3, we generalize the classical Hilbert functions and Hilbert series of a semigroup algebra to have weightings. We list three ways to add weightings, q-weighting, r-weighting, and s-weighting, and study their relationships. We find that the q-weighting can derive other weightings. Later, we specialize to the special family of semigroup algebras, the Ehrhart rings. We study and extend the properties of h^*-nonnegativity and Ehrhart–Macdonald reciprocity for the Ehrhart series under these three weightings.

In Chapter 4, we focus on the Ehrhart functions under the s-weighting and give a practical method to evaluate the s-weighted Ehrhart function. Specifically, we construct a new polytope, the weight-lifting polytope, and build a connection between the s-weighted Ehrhart function and the classical Ehrhart function. Later, we present several applications and experiments of our method in combinatorial representation theory and number theory.

In Chapter 5, we discuss a long-standing conjecture, Kakeya's conjecture, which brings a surprising connection between numerical semigroups and symmetric polynomials. We give partial results, prove the conjecture for two variables, and outline a general computer proof for an arbitrary number of variables.

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CHAPTER 1

Introduction

Semigroups are one of the simplest algebraic structures in mathematics but they have some unexplored results. They appear in many areas of mathematics. In probability theory, semigroups are associated with the Markov process [**62**]. In functional analysis, operator semigroups provide solutions of linear constant coefficient ordinary differential equations in Banach spaces [**51**]. In knot theory, semigroups are used to understand the knot structures [**34**].

This dissertation focuses on the semigroups of lattice points in polyhedral cones and their connections to convexity, combinatorics, and algebra. We present four contributions in this introduction, proofs and technical details will appear in the following chapters. This dissertation contains new versions of some results I published in [**41**], [**37**], and two more papers that are in preparation.

1.1. Key concepts and preliminaries

A **semigroup** is a set with an associative binary operation. For example, a subset of integers with the minimum or maximum operation forms a semigroup. In our context, we only consider semigroups with the standard addition operation.

Definition 1.1.1. *For an integer matrix $\mathbf{A} \in \mathbb{Z}^{d\times n}$, the* ***affine semigroup generated by A****, denoted as $\mathrm{Sg}(\mathbf{A})$, is the additive semigroup of all non-negative integer combinations of the column vectors of $\mathbf{A}$, i.e.,*

$$\mathrm{Sg}(\mathbf{A}) = \{\mathbf{b} \in \mathbb{R}^d : \mathbf{A}\mathbf{x} = \mathbf{b},\ \textit{for some } \mathbf{x} \in \mathbb{Z}^n, \mathbf{x} \geq 0\}.$$

For a given set of integral vectors $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, we also use $S = \mathrm{Sg}(V)$ to denote the affine semigroup generated by the matrix with column vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

EXAMPLE 1.1.1. Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \end{pmatrix}$, then $\mathrm{Sg}(\mathbf{A}) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \dots \right\}$. See Figure 1.1.

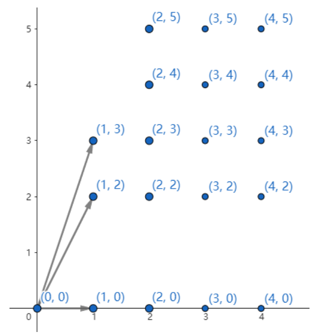

FIGURE 1.1. The visualization of a part of the affine semigroup in Example 1.1.1.

Affine semigroups lie in the intersection of algebraic geometry, combinatorics, commutative algebra, convex discrete geometry, and number theory. They are the combinatorial building blocks of toric varieties [**33**], and they find countless applications in optimization and number theory [**14**, **15**, **19**, **21**, **25**, **29**, **47**, **78**, **80**].

If we relax the condition from the non-negative integer combinations to non-negative real combinations, then we can associate an affine semigroup with a polyhedral cone.

DEFINITION 1.1.2. *For an affine semigroup generated by the integer matrix* $\mathbf{A} \in \mathbb{Z}^{d\times n}$*, its* ***associated (convex polyhedral) cone****, denoted as* $\mathrm{Cone}(\mathbf{A})$*, is the set consisting of all non-negative real combinations of the columns of* $\mathbf{A}$*, i.e.,*

$$\mathrm{Cone}(\mathbf{A}) = \{\mathbf{b} \in \mathbb{R}^d : \mathbf{Ax} = \mathbf{b} \textit{ for some } \mathbf{x} \in \mathbb{R}^n, \mathbf{x} \geq 0\}.$$

EXAMPLE 1.1.2. Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \end{pmatrix}$ (same as Example 1.1.1), then the associated cone of $\mathrm{Sg}(\mathbf{A})$, $\mathrm{Cone}(\mathbf{A})$, is the shaded unbounded region in Figure 1.2.

There are some properties of affine semigroups which are important to us. An affine semigroup is **pointed** if it has no nontrivial subgroup. This is equivalent to $\mathrm{Cone}(\mathbf{A})$ containing no positive dimensional linear subspace of $\mathbb{R}^d$. For example, the affine semigroup in Example 1.1.1 is pointed

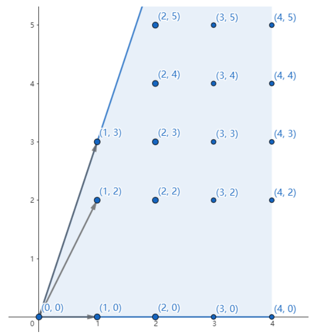

FIGURE 1.2. The visualization of a part of the cone in Example 1.1.2.

since it only contains a subgroup $\{(0,0)\}$. However, if we consider $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, then the affine semigroup $\mathrm{Sg}(\mathbf{B})$ is not pointed since it contains a nontrivial subgroup $\mathbb{Z} \times \{0\}$. An affine semigroup is **trivial** if it only contains the zero element.

DEFINITION 1.1.3. *An affine semigroup $S = \mathrm{Sg}(\mathbf{A})$ is **normal** with respect to an integral lattice $\mathcal{L}$ if $\mathrm{Sg}(\mathbf{A}) = \mathrm{Cone}(\mathbf{A}) \cap \mathcal{L}$.*

EXAMPLE 1.1.3. The affine semigroup in Example 1.1.1 is not normal to the standard integer lattice $\mathbb{Z}^2$. However, if we include more generators and consider $\mathbf{B} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$, then the affine semigroup $\mathrm{Sg}(\mathbf{B})$ is normal with respect to the lattice $\mathbb{Z}^2$.

In this dissertation, we would like to introduce some colors to distinguish the generators of an affine semigroup.

DEFINITION 1.1.4. *An ℓ-**coloring** on the column vectors of a matrix $\mathbf{A}$ is a partition $\{\mathcal{I}_i\}_{i=1}^{\ell}$ of the n column indices set of $\mathbf{A}$. A column vector $\mathbf{v}_i$ is associated with the color c if the index $i \in \mathcal{I}_c$. $\mathbf{A}_c$ denotes the submatrix of $\mathbf{A}$ with color c.*

DEFINITION 1.1.5. *For an ℓ-coloring $\{\mathcal{I}_i\}_{i=1}^{\ell}$ on the matrix $\mathbf{A}$, a **colored affine semigroup**, denoted as $\mathrm{Sg}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_\ell)$, is the affine semigroup generated by $\mathbf{A}$ with the given coloring on the generating column vectors.*

The colored affine semigroup has the same algebraic structure as the affine semigroup, however, the coloring changes the way of representing an element using the generators. For a solution vector

$\mathbf{x}$ of $\mathbf{Ax} = \mathbf{b}$, the **support** of $\mathbf{x}$, denoted $\mathrm{supp}(\mathbf{x})$, is the set of coordinate indices i such that $x_i \neq 0$. For a solution vector $\mathbf{x}$ of $\mathbf{Ax} = \mathbf{b}$ and an ℓ-coloring $\{\mathcal{I}_i\}_{i=1}^{\ell}$, the vector $\mathbf{x}$ **uses** a color c if $i \in \mathrm{supp}(\mathbf{x})$ for some $i \in \mathcal{I}_c$.

DEFINITION 1.1.6. *For a solution vector* $\mathbf{x}$ *of* $\mathbf{Ax} = \mathbf{b}$ *and an* ℓ*-coloring* $\{\mathcal{I}_i\}_{i=1}^{\ell}$*, the solution* $\mathbf{x}$ *is:*

(a) k***-chromatic*** *if* $\mathbf{x}$ *uses at least* k *different colors;*

(b) ***monochromatic*** *if* $\mathbf{x}$ *is not* 2*-chromatic (i.e.,* $\mathrm{supp}(\mathbf{x}) \subseteq \mathcal{I}_c$ *for some* c*);*

(c) ***chromatic*** *if* $\mathbf{x}$ *uses all available colors (i.e.,* $|\mathrm{supp}(\mathbf{x}) \bigcap \mathcal{I}_c| \geq 1$ *for all* c*); and*

(d) ***colorful*** *if no 2 columns of identical color are used (i.e.,* $|\mathrm{supp}(\mathbf{x}) \bigcap \mathcal{I}_c| \leq 1$ *for all* c*).*

Some terms in Definition 1.1.6 have subtle distinctions. Example 1.1.4 shows some differences, and in particular neither chromatic nor colorful implies the other.

EXAMPLE 1.1.4. Let $\mathbf{A} = [9\ 16\ 11\ 14\ 12\ 13]$ and $b = 70$. Let $\mathcal{I}_1 = \{1, 2\}$, $\mathcal{I}_2 = \{3, 4\}$, $\mathcal{I}_3 = \{5, 6\}$ being a 3-coloring of $\mathbf{A}$. Consider

$$\mathbf{Ax} = b, \quad \mathbf{x} \geq 0, \quad \mathbf{x} \in \mathbb{Z}^6.$$

The solution $\mathbf{x} = (6, 1, 0, 0, 0, 0)$ is monochromatic. The solution $\mathbf{x} = (3, 1, 0, 1, 0, 1)$ is chromatic since each color is used, but not colorful since two distinct columns from $\mathcal{I}_1$ are used. The solution $\mathbf{x} = (0, 1, 0, 2, 0, 2)$ is both chromatic and colorful since exactly one column is used from each color. Lastly, the solution $\mathbf{x} = (0, 0, 2, 0, 4, 0)$ is colorful, 2-chromatic, but not chromatic.

Among all semigroups, one special class of affine semigroups consisting of only natural numbers receives a lot of interest. For an integral n-dimensional vector $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$, the vector $\mathbf{a}$ is **primitive** if $\gcd(a_1, a_2, \ldots, a_n) = 1$. A **numerical semigroup** [**73**] is an affine semigroup generated by $\mathbf{a}$, where the matrix $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{>0}$ is a positive primitive integral n-dimensional vector. For example, Let $\mathbf{a} = (4, 6, 7)$, then $\mathrm{Sg}(\mathbf{a}) = \{0, 4, 6, 7, 8, 10, 11, 12, 13, \ldots\}$ is a numerical semigroup.

Numerical semigroups are associated with many interesting and important invariants. For a numerical semigroup $\mathrm{Sg}(\mathbf{a})$, A **gap** element is a natural number which cannot be generated by $\mathbf{a}$ and the **gaps**, denoted as $\mathsf{G}(\mathrm{Sg}(\mathbf{a}))$, is the set of all gap elements. For example, let $\mathbf{a} = (4, 6, 7)$, then $\mathsf{G}(\mathrm{Sg}(\mathbf{a})) = \{1, 2, 3, 4, 5, 9\}$ is the gaps of numerical semigroup $\mathrm{Sg}(\mathbf{a})$. For a numerical semigroup $\mathrm{Sg}(\mathbf{a})$, the **Frobenius number**, denoted as $\mathsf{F}(\mathbf{a})$ or $\mathsf{F}(\mathrm{Sg}(\mathbf{a}))$, of this numerical semigroup is the largest integer in the gaps [**72**]. For example, let $\mathbf{a} = (4, 6, 7)$ and $\mathsf{G}(\mathrm{Sg}(\mathbf{a})) = \{1, 2, 3, 4, 5, 9\}$, then $\mathsf{F}(\mathbf{a}) = \mathsf{F}(\mathrm{Sg}(\mathbf{a})) = 9$ is the Frobenius number of numerical semigroup $\mathrm{Sg}(\mathbf{a})$.

Similarly, we will consider **colored numerical semigroup** which is both a numerical semigroup and a colored affine semigroup. We define an important invariant related to colored numerical semigroup. For a colored numerical semigroup $\mathrm{Sg}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_\ell)$, the k**-chromatic Frobenius number**, denoted as $\mathsf{CF}_k(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_\ell)$, is the largest integer b such that $\mathbf{Ax} = b$ has no k-chromatic integral solution.

Besides numerical semigroups, another special class of affine semigroups arising from polytopes is also a focused area of research. For a set $S \subset \mathbb{R}^d$, the **conic hull** of S, denoted as $\mathrm{Cone}(S)$, is the set of all conical combinations of S, i.e.,

$$\mathrm{Cone}(S) = \left\{ \sum_{i=1}^{k} \alpha_i \mathbf{x}_i \mid \mathbf{x}_i \in S, \alpha_i \in \mathbb{R}_{\geq 0}, k \in \mathbb{N} \right\}.$$

Given a polytope $\mathcal{P} \subset \mathbb{R}^d$, we will identify $\mathbb{R}^d$ as the level 1 hyperplane of $\mathbb{R}^{d+1}$, therefore, the polytope is naturally lifted to $\mathbb{R}^{d+1}$. The **polytopal cone generated by** $\mathcal{P}$, denoted as $C_\mathcal{P}$, is the conic hull of the lifted $\mathcal{P}$. The **polytopal semigroup**, denoted as $\mathrm{Sg}_\mathcal{P}$, is $C_\mathcal{P} \cap \mathbb{Z}^d$. The minimal generating set of $\mathrm{Sg}_\mathcal{P}$ is called the **Hilbert basis** of $\mathrm{Sg}_\mathcal{P}$.

EXAMPLE 1.1.5. We give an example of a polytopal cone in dimension three. See Figure 1.3.

Then we shift our attention to algebra. Fix a coefficient field $\mathbb{K}$, the Laurent polynomial ring with d variables over the field $\mathbb{K}$, $R_d^{\pm} = \mathbb{K}[\mathbf{t}, \mathbf{t}^{-1}]$, is the ambient ring that we consider in this dissertation. Its monomials are throughout abbreviated by $\mathbf{t}^{\mathbf{a}} := t_1^{a_1} \cdots t_d^{a_d}$, $\mathbf{a} = (a_1, \ldots, a_d)$ in $\mathbb{Z}^d$. We often think of monomials as lattice points and rely on their lattice geometry for our analysis. Affine semigroups are also very common and important ingredient in $\mathbb{K}$-algebra. For an affine semigroup $M \subseteq \mathbb{Z}^d$, the **semigroup algebra**, denoted as $\mathbb{K}[M]$, is $\mathbb{K}[\mathbf{t}^{\mathbf{a}} \mid \mathbf{a} \in M] \subseteq R_d^{\pm}$. Let G be an Abelian group, A G**-grading** on a ring R is a decomposition $R = \bigoplus_{g \in G} R_g$ of Abelian groups such that $R_g R_h \subset R_{g+h}$ for all $g, h \in G$, and R is called a G**-graded ring**. In this dissertation, we

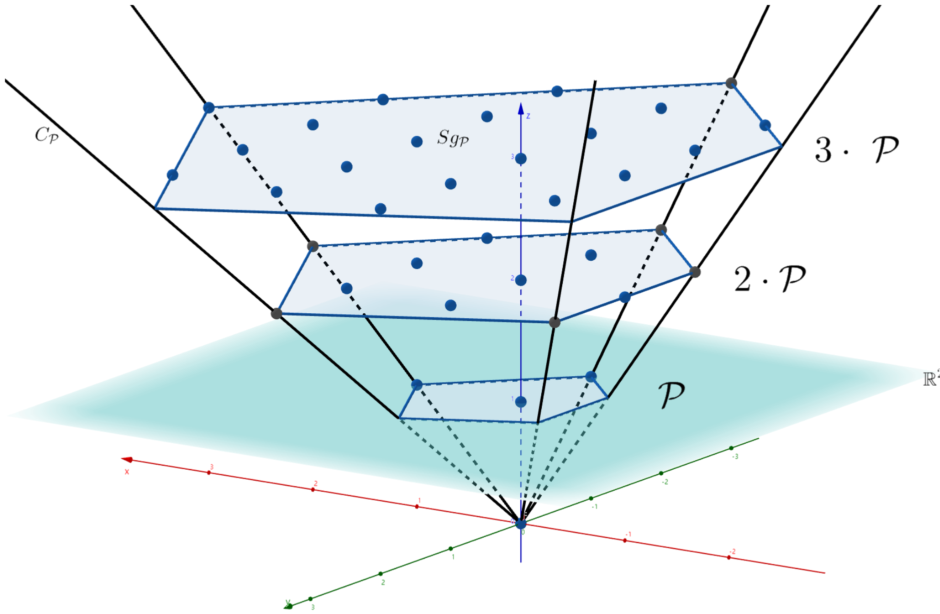

FIGURE 1.3. The visualization of a part of a polytopal cone and a polytopals semigroup.

are mainly interested in a $\mathbb{Z}$-graded semigroup algebra, $\mathbb{K}[M]$ with a $\mathbb{Z}$-grading ϕ. The theory of graded semigroup algebras, in particular their Hilbert functions, has been fundamental in algebraic combinatorics, commutative algebra, and algebraic geometry (see the books [**79**, **83**]).

Graded semigroup algebras arising from the polytopes are of great interest.

DEFINITION 1.1.7. *For a polytope $\mathcal{P}\subset\mathbb{R}^d$, the* ***Ehrhart ring*** *of $\mathcal{P}$, denoted as $A(\mathcal{P})$, is the graded semigroup algebra of the semigroup $\mathrm{Sg}_{\mathcal{P}}$ with a $\mathbb{Z}$-grading $\phi((\mathbf{x},n))=n$. Specifically,*

$$A(\mathcal{P})=\mathbb{K}[\mathbf{t}^{(\mathbf{a},n)}\mid\mathbf{a}\in n\mathcal{P}\cap\mathbb{Z}^d].$$

EXAMPLE 1.1.6. Let $\mathcal{P}=[0,2]$, then

$$A(\mathcal{P})=\begin{array}{llllll} \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \oplus\mathbb{K}\cdot t_2^2 & \oplus\mathbb{K}\cdot t_1t_2^2 & \oplus\mathbb{K}\cdot t_1^2t_2^2 & \oplus\mathbb{K}\cdot t_1^3t_2^2 & \oplus\mathbb{K}\cdot t_1^4t_2^2 & \\ \oplus\mathbb{K}\cdot t_2 & \oplus\mathbb{K}\cdot t_1t_2 & \oplus\mathbb{K}\cdot t_1^2t_2 & & & \\ \oplus\mathbb{K}\cdot 1 & & & & & \end{array}.$$

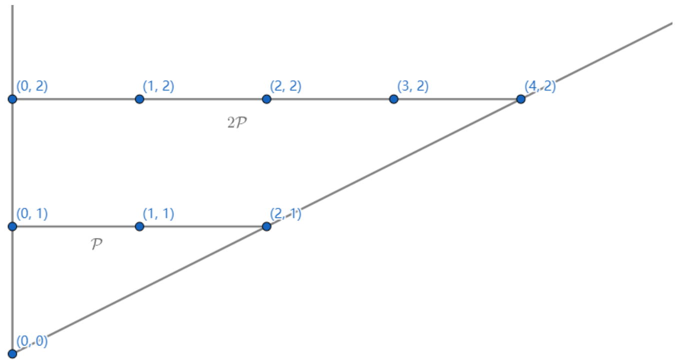

FIGURE 1.4. The visualization of the polytopal semigroup.

The Hilbert function and the Hilbert series are critical tools to understand these graded algebras. For a graded semigroup algebra $\mathbb{K}[M]$ with a $\mathbb{Z}$-grading ϕ, the **multivariate Hilbert function** is

$$E_M(\mathbf{t}, n) = \sum_{\mathbf{a}\in M, \phi(\mathbf{a})=n} \mathbf{t}^{\mathbf{a}}$$

and the **Hilbert function** is

$$E_M(\mathbf{1}, n) = \sum_{\mathbf{a}\in M, \phi(\mathbf{a})=n} 1.$$

The **multivariate Hilbert series** is

$$F_M(\mathbf{t}, x) = \sum_{\mathbf{a}\in M} \mathbf{t}^{\mathbf{a}} x^{\phi(\mathbf{a})} = \sum_{n\geq 0} E_M(\mathbf{t}, n) x^n$$

and the **Hilbert series** is

$$F_M(\mathbf{1}, x) = \sum_{\mathbf{a}\in M} x^{\phi(\mathbf{a})} = \sum_{n\geq 0} E_M(\mathbf{1}, n) x^n.$$

If the graded semigroup algebra is an Ehrhart ring, then the Hilbert function and the Hilbert series are also called **Ehrhart function and Ehrhart series**, which are investigated in *Ehrhart theory* [**20**, **48**]. More generally, in the work by Okounkov [**69**], Kaveh and Khovanskii [**60**] and Lazarsfeld and Mustaţă [**63**] graded semigroup algebras come from lattice points of a convex body, the Newton-Okounkov body.

In this dissertation, we will introduce three weighted versions of the Hilbert function and Hilbert series. These weighted versions carry much more information, and from them, one can recover the basic form we showed above.

Definition 1.1.8. *Let $\mathbb{K}[M]$ be a graded semigroup algebra with a $\mathbb{Z}$-grading ϕ. For $i = 1, \ldots, r$, let $w_i \colon \mathbb{R}^d \to \mathbb{R}$, $\mathbf{a} \mapsto w_i(\mathbf{a})$ be a weight. We now define weightings using these w_i's:*

(1) The q-weighted multivariate Hilbert function and q-weighted multivariate Hilbert series of M relative to $w_1, \ldots, w_r$, denoted as $E_M^{q,w_1,\ldots,w_r}$ and $F_M^{q,w_1,\ldots,w_r}$ respectively, are given by

$$E_M^{q,w_1,\ldots,w_r}(\mathbf{q}, \mathbf{t}, n) := \sum_{\mathbf{a} \in M, \phi(\mathbf{a}) = n} q_1^{w_1(\mathbf{a})} \cdots q_r^{w_r(\mathbf{a})} \mathbf{t}^{\mathbf{a}},$$

$$F_M^{q,w_1,\ldots,w_r}(\mathbf{q}, \mathbf{t}, x) := \sum_{n=0}^{\infty} E_M^{q,w_1,\ldots,w_r}(\mathbf{q}, \mathbf{t}, n) x^n.$$

The q-weighted Hilbert function and q-weighted Hilbert series of M relative to $w_1, \ldots, w_r$ are given by,

$$E_M^{q,w_1,\ldots,w_r}(\mathbf{q}, \mathbf{1}, n) = \sum_{\mathbf{a} \in M, \phi(\mathbf{a}) = n} q_1^{w_1(\mathbf{a})} \cdots q_r^{w_r(\mathbf{a})},$$

$$F_M^{q,w_1,\ldots,w_r}(\mathbf{q}, \mathbf{1}, x) = \sum_{n=0}^{\infty} E_M^{q,w_1,\ldots,w_r}(\mathbf{q}, \mathbf{1}, n) x^n.$$

(2) The r-weighted multivariate Hilbert function and r-weighted multivariate Hilbert series of M relative to $w_1, \ldots, w_r$, denoted $E_M^{r,w_1,\ldots,w_r}$ and $F_M^{r,w_1,\ldots,w_r}$ respectively, are given by

$$E_M^{r,w_1,\ldots,w_r}(\mathbf{q}, \mathbf{t}, n) := \sum_{\mathbf{a} \in M, \phi(\mathbf{a}) = n} \prod_{i=1}^{r} \left(\sum_{j=0}^{w_i(\mathbf{a})} q_i^j \right) \mathbf{t}^{\mathbf{a}},$$

$$F_M^{r,w_1,\ldots,w_r}(\mathbf{q}, \mathbf{t}, x) := \sum_{n=0}^{\infty} E_M^{r,w_1,\ldots,w_r}(\mathbf{q}, \mathbf{t}, n) x^n.$$

The r-weighted Hilbert function and r-weighted Hilbert series of M relative to $w_1, \ldots, w_r$ are given by

$$E_M^{r,w_1,\ldots,w_r}(\mathbf{q}, \mathbf{1}, n) = \sum_{\mathbf{a}\in M, \phi(\mathbf{a})=n} \prod_{i=1}^{r} \left(\sum_{j=0}^{w_i(\mathbf{a})} q_i^j \right),$$

$$F_M^{r,w_1,\ldots,w_r}(\mathbf{q}, \mathbf{1}, x) = \sum_{n=0}^{\infty} E_M^{r,w_1,\ldots,w_r}(\mathbf{q}, \mathbf{1}, n) x^n.$$

When there is only one weight function $w\colon \mathbb{R}^d \to \mathbb{R}$, $a \mapsto w(a)$. We define the third weighting using w:

(3) The s-weighted multivariate Hilbert function and s-weighted multivariate Hilbert series of M relative to w, denoted $E_M^{s,w}$ and $F_M^{s,w}$ respectively, are given by

$$E_M^{s,w}(\mathbf{t}, n) := \sum_{\mathbf{a}\in M, \phi(\mathbf{a})=n} w(\mathbf{a})\mathbf{t}^{\mathbf{a}}, \quad \textit{and} \quad F_M^{s,w}(\mathbf{t}, x) := \sum_{n=0}^{\infty} E_M^{s,w}(\mathbf{t}, n) x^n.$$

The s-weighted Hilbert function and s-weighted Hilbert series of M relative to w are given by

$$E_M^{s,w}(n) := \sum_{\mathbf{a}\in M, \phi(\mathbf{a})=n} w(\mathbf{a}), \quad \textit{and} \quad F_M^{s,w}(x) := \sum_{n=0}^{\infty} E_M^{s,w}(n) x^n.$$

Example 1.1.7. Let $M = \mathbb{N}$ and $w(a) = a^2$,

(1) $F_M^{q,w}(q, t, x) = 1 + q^1 t^1 x^1 + q^4 t^2 x^2 + q^9 t^3 x^3 + \ldots.$
(2) $F_M^{r,w}(q, t, x) = 1 + (q^1 + 1) t^1 x^1 + (q^4 + q^3 + \ldots + 1) t^2 x^2 + \ldots.$
(3) $F_M^{s,w}(q, t, x) = 1 + 1 t^1 x^1 + 4 t^2 x^2 + 9 t^2 x^2 + \ldots.$

Hilbert function and series obtain many fascinating results in Ehrhart rings, see [**20**, **23**, **48**, **83**] and references therein. For example, the Ehrhart series always has a rational form, the numerator of the rational form always has non-negative coefficients [**77**] and Ehrhart–Macdonald reciprocity [**76**] holds. In Section 1.3.2, we will discuss in detail what properties are preserved after weighting.

Lastly, we temporally switch our attention from semigroups to symmetric functions. Later in Section 1.5, we will explore the mysterious relationship between semigroups and symmetric functions. Let $R_n = \mathbb{K}[x_1, x_2, \ldots, x_n]$ denote the polynomial ring in n indeterminate variables over a field $\mathbb{K}$ with characteristics 0, the **ring of symmetric polynomials**, denoted as Λ_n, is the invariant space $R_n^{S_n}$ of the polynomial ring R_n under the standard permutation of symmetric group S_n.

There are many well-known basic building blocks of symmetric polynomials in Λ_n, here we introduce two of them. The **elementary symmetric polynomials** in n variables $x_1, \ldots, x_n$, written $e_k(x_1, \ldots, x_n)$ for $k = 1, \ldots, n$ are defined by

$$e_k(x_1, \ldots, x_n) = \sum_{1 \le j_1 < \ldots < j_k \le n} x_{j_1} \ldots x_{j_k}.$$

The **power sum symmetric polynomial** of degree k in n variables $x_1, \ldots, x_n$, written $p_k(x_1, \ldots, x_n)$ for $k = 0, 1, \ldots$, is the sum of all k-th powers of the variables. Formally,

$$p_k(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^k.$$

Definition 1.1.9. *For a ring of symmetric polynomials in n variables Λ_n, a set of n symmetric polynomials, $\{f_1, f_2, \ldots, f_n\}$, forms a* ***fundamental system*** *(of the fraction field of Λ_n), if any rational symmetric function of x_i's can be expressed as a rational function of f_i's.*

Example 1.1.8. $\{p_1, p_2 \ldots, p_n\}$ is a fundamental system of Λ_n.

Now we present our primary contributions in the following sections.

1.2. Convexity theorems in affine semigroups

The Affine semigroup is the algebraic-combinatorial analog of its associated (convex polyhedral) cone, so we explore the following question in this dissertation.

Question 1.2.1. *How far can one generalize the convex geometry theorems of Helly, Tverberg, and Carathéodory to affine semigroups?*

Helly's theorem, a basic result in convex geometry, states that given a finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ if every collection of $d+1$ sets in $\mathcal{F}$ intersects, then the entire family intersects [**35**]. Helly-type theorems appear in many variations [**12, 35, 38**]; for example, Doignon's theorem, an integer version of Helly's theorem, states that if every collection of 2^d sets in $\mathcal{F}$ intersects at an integer point, then the whole family intersects at an integer point [**46**]. In Section 2.1, we prove a new variation for affine semigroups.

Theorem 1.2.1 (A Helly theorem for affine semigroups). *For each $m \in \mathbb{Z}_{\geq 0}$, there exists a constant $N(m) \in \mathbb{Z}_{\geq 1}$ such that the following holds: given any finite family $\mathcal{F} = \{S_1, \ldots, S_n\}$ of affine semigroups in $\mathbb{Z}^m$, and letting C_i be the associated cone of S_i for each i, if the intersection of any $N(m)$ affine semigroups in $\mathcal{F}$ is nontrivial, then $S_1 \cap \cdots \cap S_n$ is nontrivial. More specifically,*

(1) if each S_i is pointed and $C_1, \ldots, C_n$ do not cover $\mathbb{R}^m$, then $N(m) = m$;
(2) if each S_i is pointed and $C_1, \ldots, C_n$ cover $\mathbb{R}^m$, then $N(m) = m + 1$; and
(3) if some S_i is not pointed, then $N(m) = 2m$.

Carathéodory's theorem states that given a pointed cone $C \subseteq \mathbb{R}^d$, every element $x \in C$ is generated by at most d extreme rays of C. In wide contrast to Theorem 1.2.1, obtaining a variant of Carathéodory's theorem for affine semigroups (i.e., a bound on the number of generators needed to generate any given element) is much more complicated. In particular, for general affine semigroups, it is impossible to obtain such a bound in terms of ambient dimension d alone; one must also take into account, for instance, the coordinates of the semigroup generators (see [**3**] and all the references there). A special case of particular interest is when $S = \mathrm{Sg}(\mathbf{A})$ is normal, every element of S can be generated by at most $2d - 2$ generators [**75**], though this bound is not tight [**24**].

Colorful variations of Helly's, Carathéodory's, and Tverberg's theorems have been a key topic in combinatorial convexity [**11**, **38**]. In this vein, we consider colored affine semigroup and some notions of a colored affine semigroup which we introduced in Section 1.1. Item (a) and item (c) of Definition 1.1.6 also appear in [**13**, **71**]. It is worth mentioning that the same type of ideas are studied for real solutions in [**10**, **13**, **66**]; here, we require integer solutions, and the theory becomes more subtle.

We briefly argue that these notions in Definition 1.1.6 arise naturally when modeling manufacturing diversity requirements. The notion of colorful has already been connected to linear programming and game theory in [**38**, **66**]. When dealing with indivisible goods, this kind of integer programming requires affine semigroups. Imagine your company produces batteries with three ingredient providers (call them red, green, and blue). They each sell the same resources or ingredients to you, which are represented by vectors (say different types of metals or chemicals). However, due to trade agreements, one cannot produce a battery with parts coming from one provider alone (no monochromatic solutions are allowed). Since batteries must be built with parts from at least two providers, solutions then have to be 2-chromatic. Or regulations can be even more strict, requiring batteries to be built with ingredients from all three providers (chromatic solutions). Another possible type of restriction is that a company may only contribute at most one ingredient to the creation of your product (colorful solutions). In some scenarios, it should be possible to purchase the same ingredient from different providers to cover demand. As such, we allow the same column to appear more than once, but with a different color.

Colorful versions of Helly's and Tverberg's theorems for affine semigroups follow from Theorem 1.2.1 (Corollary 2.1.1 and Corollary 2.1.2), but obtaining a colorful version of Carathéodory's theorem for affine semigroups turns out to be a bit more subtle.

With the above definitions in hand, we recall a colorful variation of Carathéodory's theorem due to Bárány. Given $d+1$ nonempty subsets $T_1, \ldots, T_{d+1} \subseteq \mathbb{R}^d$, Bárány's theorem states that any point $x \in \operatorname{conv}(T_1) \cap \cdots \cap \operatorname{conv}(T_{d+1})$ can be expressed as the convex combination of $d+1$ points, with one point from each T_j [**10**]. Considering each set T_j as a color class, Bárány's theorem has the following interpretation: given a colored generating matrix $\mathbf{A}$ and an element $\mathbf{b} \in \operatorname{Cone}(\mathbf{A})$, if a monochromatic solution exists for each color, then a colorful solution exists. In discrete convexity, the colorful Carathéodory theorem has been intensely studied [**42**, **43**, **65**].

Returning once again to affine semigroups, suppose an element $\mathbf{b}$ of a colored affine semigroup has a monochromatic solution for each color. Can one guarantee $\mathbf{b}$ also has a colorful solution? What about a chromatic solution? Note, an answer of "yes" to either question would constitute a variant of Bárány's theorem for affine semigroups. Turns out, the answer to the latter question is indeed "yes" for all but finitely many $\mathbf{b}$ (Theorem 1.2.3), but the former question has an overwhelmingly negative answer, as the following result demonstrates in two different ways. Note that the families described therein can be easily lifted to higher dimensions.

THEOREM 1.2.2. *Bárány's **colorful** Carathéodory theorem fails to extend to affine semigroups.*

(a) There exist colored affine semigroups with arbitrarily many colors in $\mathbb{R}^3$, formed by a family $\mathcal{F}$ of normal affine semigroups and an element $\mathbf{b}$ such that $\mathbf{b}$ has a monochromatic solution for every color but yet has no colorful solutions and no chromatic solutions (in fact, every solution for $\mathbf{b}$ is monochromatic).

(b) There exist colored affine semigroups with arbitrarily many colors in $\mathbb{R}^4$, formed by a family $\mathcal{F}$ of normal affine semigroups and infinitely many elements $\mathbf{b}$ such that $\mathbf{b}$ has a monochromatic solution for every color and has no colorful solution.

We now turn our attention to chromatic solutions. Theorem 1.2.2(a) demonstrates the "all but finitely many" hypothesis in Theorem 1.2.3 cannot be dropped. We note that this hypothesis may seem unnatural to those in convexity theory. Still, such theorems arise frequently in semigroup theory, where the finitely many exceptions can be attributed to the important notion of *gaps* or *holes* describing exceptions in semigroup membership [**50**].

THEOREM 1.2.3 (A **chromatic** Carathéodory theorem for affine semigroups). *In any colored affine semigroup S, all but finitely many elements $\mathbf{b} \in S$ with a monochromatic solution for each color also have a chromatic solution.*

In Section 2.3, we focus on numerical semigroups [**73**], a special case of affine semigroups. They are often referred to as a knapsack problem [**61**], numerical semigroups are fundamental and look simple, but are often a source of very challenging problems [**1**]. One old and classical problem is the Frobenius coin-exchange problem, which asks for the largest monetary amount that cannot be obtained using only coins of specified denominations. Here, we consider the chromatic Frobenius problem, i.e., the chromatic Frobenius number. Our main results in this direction are as follows.

THEOREM 1.2.4. *Fix $k \geq 1$ and a colored numerical semigroup $S = \mathrm{Sg}(\mathbf{A}_1, \ldots, \mathbf{A}_\ell)$.*

(1) There are only finitely many elements of S that are not k-chromatic, and as such, the k-chromatic Frobenius number $\mathsf{CF}_k(S)$ is well-defined.

(2) The problem of computing the colored Frobenius number $\mathsf{CF}_k(S)$ is NP-hard.

(3) The number of distinct k-chromatic solutions of a positive integer b coincides with a quasi-polynomial function in b for sufficiently large b.

1.3. Weighted Hilbert function and series

We have introduced weighted Hilbert functions and series in Definition 1.1.8. Let us look at some examples and review prior work on these three manners of assigning weights. First of all, the s-weighted Hilbert series is the most studied so far. If $w = 1$, we recover the classical Hilbert function and Hilbert series of $\mathbb{K}[M]$, respectively. But even in general, the s-weighted Hilbert series is the power series expansion of a rational function; see [**22**]. In the case of the semigroup M coming from dilations of a polytope we have s-weighted Ehrhart functions and s-weighted Ehrhart series, which have been developed in several papers, see [**8**, **9**, **22**, **27**] and references therein. To the best of our knowledge research on q-weightings is less expansive, the first paper on the subject comes from Chapoton [**30**] where he only looks at the q-weights of degree 1, thus when w is linear.

1.3.1. What do we know in general? In Section 3.1.1, we show that q-weighted multivariate Hilbert series is the most general weighted Hilbert series of what we considered.

PROPOSITION 1.3.1. *Both r-weighted multivariate Hilbert series and s-weighted Hilbert series can be recovered from q-weighted multivariate Hilbert series.*

In general, if weight is not a linear function, then its weighted Hilbert series can have a complicated compact form.

EXAMPLE 1.3.1. Let $M = \mathbb{N}$ and $w(a) = a^2$, then

$$F_M^{q,w}(q,t,x) = 1 + q^1t^1x^1 + q^4t^2x^2 + q^9t^3x^3 + \dots.$$

Interestingly, if we specialize $t = 1$ and $x = 1$, then it agrees with a special case of the Lambert series for Liouville's function, and the sum is related to the Jacobi theta function.

As the next result shows, the hypothesis that w_i is linear for $i = 1, \dots, r$ is essential to prove that the q-weighted graded semigroup algebra is finitely generated.

PROPOSITION 1.3.2. *Let M be $\mathbb{N}$ and $w(a) = a^2$, then the ring*

$$A_q^w[M] = \mathbb{K}[q^{a^2}x^a \mid a \in \mathbb{N}]$$

is not Noetherian. In particular, $A_q^w[M]$ is not finitely generated as a $\mathbb{K}$-algebra.

When weights are linear and the given semigroup M has a nice structure, then we can conclude that all the Hilbert series we considered are rational functions just as in the traditional case.

THEOREM 1.3.1. *If $w_i(\mathbf{a}) = \mathbf{v}_i^{\intercal}\mathbf{a} + b_i$'s are linear weights and the multivariate Hilbert series of the original semigroup algebra $\mathbb{K}[M]$ is a rational function. The q-weighted multivariate Hilbert series, the q-weighted Hilbert series, the r-weighted multivariate Hilbert series, the r-weighted Hilbert series, and the s-weighted Hilbert series are all rational functions.*

1.3.2. Weighted Ehrhart functions and series. In Section 3.2, we specialize in Ehrhart rings and the Ehrhart series.

Ehrhart theory has been the subject of much attention for its applications in Combinatorics and Commutative Algebra. Several beautiful weighted generalizations have been presented. For example, Brion and Vergne [**22**] presented in 1997 a generalization of Ehrhart's theorem in the context of Euler-Maclaurin formulas where the points are counted with "weight" given by a function f, i.e., $E_{\mathcal{P}}^{s,f}(n) = \sum_{\alpha \in n\mathcal{P} \cap \mathbb{Z}^d} f(\alpha)$. Later Chapoton introduced the q-analogue of Ehrhart functions. In what follows we will investigate when some classical results in Ehrhart theory extend to this weighted version.

1.3.2.1. *Nonnegativity of h^* coefficients.* It is a famous result of R.P. Stanley in Ehrhart theory that the numerator of the rational function representing the Hilbert series is a polynomial of nonnegative coefficients. From Theorem 1.3.1, we know that especially when $\mathbb{K}[M]$ is an Ehrhart ring of a polytope $\mathcal{P}$, then all Hilbert series we considered are rational functions. Could it be that Stanley's result extends to the weighted Ehrhart series? Inspired by the fact that the numerator of the Ehrhart series always has nonnegative coefficients, Chapoton investigated this in [**30**] the numerator of univariate q-weighted Hilbert series and noticed that such nonnegativity result does not hold anymore.

Here, we give a sufficient, but not necessary, geometric condition to show when the nonnegativity result holds.

DEFINITION 1.3.1. *For a lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ and r linear functions $w_i \colon \mathbb{R}^d \to \mathbb{R}$ ($i = 1, \ldots, r$), we say a triangulation $\mathcal{T}$ of the polytope $\mathcal{P}$ is $(w_1, \ldots, w_r)$-compatible if every simplex $S \in \mathcal{T}$ satisfies that the multiset of weight vectors on the vertices of S is identical.*

EXAMPLE 1.3.2. Let $w(a_1, a_2) = a_1 + a_2$, the square $ABCD$ has a w-triangulation $\{ABC, ADC\}$, since both simplices have the weight vector $(0, 1, 2)$. However, the diamond $ABEC$ does not have a w-triangulation, since the triangle ABC has the weight vector $(0, 1, 2)$, the triangle BCE has the weight vector $(1, 2, 3)$ or the triangle ABE has the weight vector $(0, 1, 3)$, the triangle ACE has the weight vector $(0, 2, 3)$.

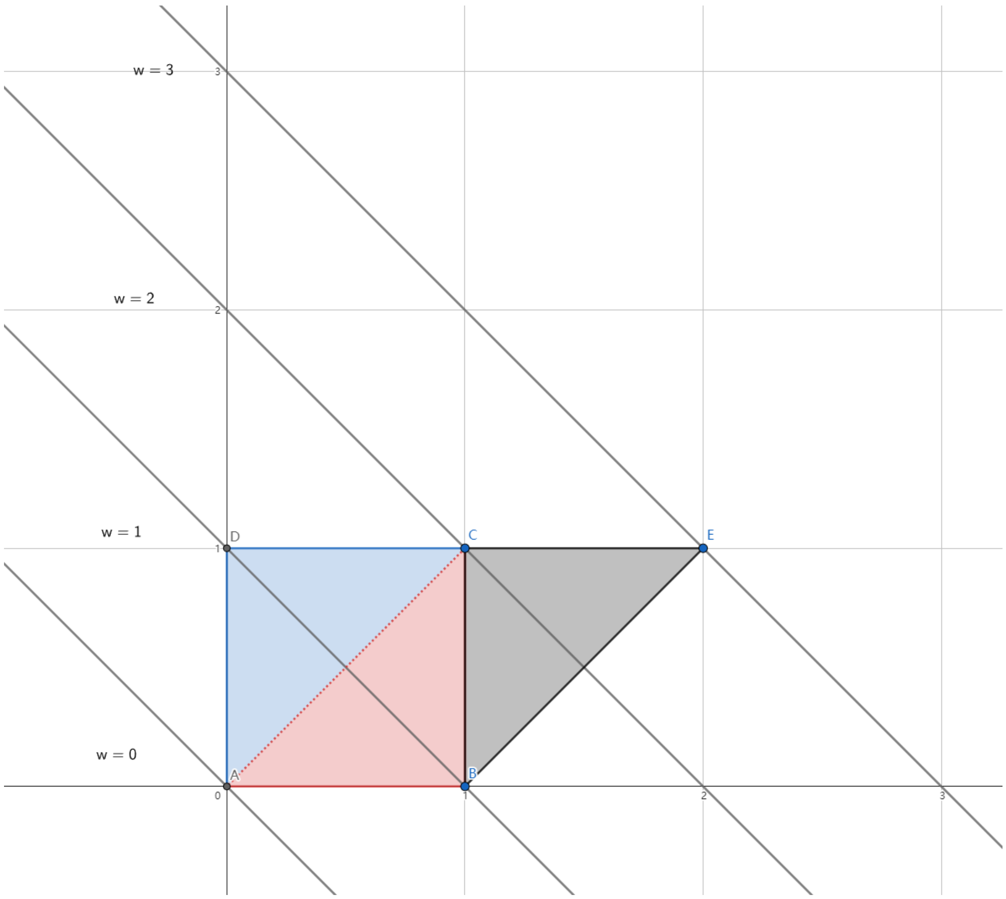

FIGURE 1.5. Example of compatible triangulation and non-compatible triangulation.

THEOREM 1.3.2. *If w_i's are linear weights and a lattice polytope $\mathcal{P}$ has a $(w_1, \ldots, w_r)$-compatible triangulation $\mathcal{T}$, then the numerator of the rational form of its q-weighted Hilbert series has positive coefficients. Nevertheless, this is only a sufficient condition for nonnegativity.*

Example 1.3.3 shows that the condition of Theorem 1.3.2 is not necessary.

EXAMPLE 1.3.3. Let the polytope be $\mathcal{P} = \text{conv}((0,0,0),(1,0,0),(1,1,0),(1,1,1),(2,1,1))$ and a linear function be $w(t_1,t_2,t_3) = t_1+t_2+t_3$, we calculate that its q-weighted Hilbert series is:

$$\frac{1+q^2x}{(1-x)(1-qx)(1-q^3x)(1-q^4x)}.$$

The q-weighted Hilbert series satisfies the nonnegative coefficients property but the polytope has no w-compatible triangulation.

1.3.2.2. *Reciprocity for weighted Ehrhart series.* When the graded semigroup algebra $\mathbb{K}[M]$ has the Ehrhart ring structure, we are inspired by the Chapoton's reciprocity result for the q-weighted Hilbert series for only one weight. We prove the reciprocity properties for both the q-weighted Hilbert series and the s-weighted Hilbert series.

DEFINITION 1.3.2. *For an Ehrhart ring $\mathbb{K}[M] = A(\mathcal{P})$ for a polytope $\mathcal{P} \subseteq \mathbb{R}^d$.*

(1) We define interior q-weighted multivariate Hilbert series as

$$F_{M^\circ}^{q,w_1,\ldots,w_r}(\mathbf{q},\mathbf{t},x) := \sum_{\mathbf{a}\in\text{Cone}(\mathcal{P}^\circ)\cap\mathbb{Z}^{d+1}} q_1^{-w_1(-\mathbf{a})}\cdots q_r^{-w_r(-\mathbf{a})}\mathbf{t}^{\mathbf{a}}x^{a_{d+1}}.$$

We define interior q-weighted Hilbert series as

$$F_{M^\circ}^{q,w_1,\ldots,w_r}(\mathbf{q},\mathbf{1},x) = \sum_{\mathbf{a}\in\text{Cone}(\mathcal{P}^\circ)\cap\mathbb{Z}^{d+1}} q_1^{-w_1(-\mathbf{a})}\cdots q_r^{-w_r(-\mathbf{a})}x^{a_{d+1}}.$$

(2) We define interior s-weighted Hilbert series as

$$F_{M^\circ}^{s,w}(x) = \sum_{\mathbf{a}\in\text{Cone}(\mathcal{P}^\circ)\cap\mathbb{Z}^{d+1}} w(-\mathbf{a})x^{a_{d+1}}.$$

EXAMPLE 1.3.4. Let $\mathcal{P} = [0,2]$ and $w(a_1,a_2) = a_1^2a_2$, then

(1) $F_{M^\circ}^{q,w}(\mathbf{q},\mathbf{t},x) = q^1t_1^1t_2^1x^1 + q^2t_1^1t_2^2x^2 + q^8t_1^2t_2^2x^2 + q^{18}t_1^3t_2^2x^2 + \ldots.$

(2) $F_{M^\circ}^{q,w}(\mathbf{q},\mathbf{1},x) = q^1x^1 + q^2x^2 + q^8x^2 + q^{18}x^2 + \ldots.$

(3) $F_{M^\circ}^{s,w}(x) = (-1)x^1 + (-2-8-18)x^2 + \ldots.$

THEOREM 1.3.3. *If w_i's are linear weights and $\mathbb{K}[M]$ is an Ehrhart ring of a polytope $\mathcal{P}$ of dimension d, then the q-weighted multivariate Hilbert(Ehrhart) series satisfies the reciprocity property, i.e.,*

$$F_M^{q,w_1,\ldots,w_r}(\mathbf{q}^{-1},\mathbf{t}^{-1},x^{-1}) = (-1)^{d+1}F_{M^\circ}^{q,w_1,\ldots,w_r}(\mathbf{q},\mathbf{t},x).$$

Lastly, we look at the reciprocity in the case of the s-weighting of Ehrhart rings.

THEOREM 1.3.4. *Let $\mathcal{P} \subset \mathbb{R}^d$ be a rational polytope and $h(a) = \prod_{i=1}^{d+1}\sum_{j=1}^{k_i} P_{ij}(a_i)\gamma_{ij}^{a_i}$ with P_{ij}'s are polynomials and γ_{ij} are nonzero complex numbers, then*

(1) $F_M^{s,h}(\mathbf{t},x)$ and $F_{M^\circ}^{s,h}(\mathbf{t},x)$ are rational s-weighted multivariate Hilbert series,

(2) they satisfy the reciprocity relation,

$$F_M^{s,h}\left(\mathbf{t}^{-1},x^{-1}\right) = (-1)^{d+1}F_{M^\circ}^{s,h}\left(\mathbf{t},x\right).$$

1.4. s-weighted Ehrhart theory

A computational problem arising throughout the mathematical sciences is to compute or at least estimate,

$$E_{\mathcal{P}}^{s,w}(n) = \sum_{\mathbf{x}\in n\mathcal{P}\cap\mathbb{Z}^d} w(\mathbf{x}). \tag{1.1}$$

One can prove $E_{\mathcal{P}}^{s,w}(n)$ is a quasi-polynomial in the sense that it is a function in the variable n which is a sum of monomials up to degree $d+e$, where $e = \deg w$, but whose coefficients α_i are periodic functions of $n \in \mathbb{N}$:

$$E_{\mathcal{P}}^{s,w}(n) = \sum_{i=0}^{d+e}\alpha_i n^i.$$

The leading coefficient of $E_{\mathcal{P}}^{s,w}(n)$ is given by the integral of w over the polytope $\mathcal{P}$. These integrals were studied in [**16**], [**17**] and more recently in [**8**].

We will illustrate many important examples of such s-weighted Ehrhart problems in Section 4.2. For now, note they appear in enumerative combinatorics [**5**], algebraic combinatorics [**7, 30**], statistics [**31, 44**], and in symbolic integration and optimization [**8, 39**], among others.

We now outline the main contributions. The main theorem is a surprisingly simple way to evaluate the function $E_{\mathcal{P}}^{s,w}(n)$ where $\mathcal{P}$ is a rational polytope and $w(\mathbf{x})$ is a very general weight function. The key idea is that we build a new polytope, the *weight-lifting polytope* $\mathcal{P}^*$, for which these functions become simply $E_{\mathcal{P}^*}(n)$, in other words, just a "standard" lattice point counting function. This way (often) the s-weighted Ehrhart polynomial $\mathcal{P}$ is equivalent to the (usual) Ehrhart polynomial of $\mathcal{P}^*$. Clearly, $\mathcal{P}^*$ will depend on both $\mathcal{P}$ and w:

THEOREM 1.4.1 (The existence of weight-lifting polytopes). *Let $\mathcal{P}$ be a rational convex polytope in the form $\{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$, where $\mathbf{A} \in \mathbb{Z}^{s\times d}, \mathbf{b} \in \mathbb{Z}^s$. Let $\mathcal{Q}(x_1, \ldots, x_d)$ be the parametric family of rational convex polytopes parameterized by $x_1, \ldots, x_d$, given by*

$$\mathcal{Q}(x_1, \ldots, x_d) = \left\{ \mathbf{y} \mid \mathbf{C}\mathbf{y} = \sum_{i=1}^{d} x_i \mathbf{d_i} + \mathbf{e}, \mathbf{y} \geq \mathbf{0} \right\},$$

where $\mathbf{C} \in \mathbb{Z}^{r\times e}, \mathbf{d_i}, \mathbf{e} \in \mathbb{Z}^r$. Using $\mathcal{Q}$ define $w(\mathbf{x})$ to be the multivariate Ehrhart quasi-polynomial function in n variables that counts the number of lattice points in the parametric polytope $\mathcal{Q}(x_1, \ldots, x_d)$ when x_i are chosen integers, i.e.,

$$w(x_1, \ldots, x_n) = |\mathcal{Q}(x_1, \ldots, x_d) \cap \mathbb{Z}^e|.$$

(a) There is a weight-lifting polytope $\mathcal{P}^* \subset \mathbb{R}^{d+e}$ defined by

$$\mathcal{P}^* = \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \middle| \mathbf{A}^* \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ -\mathbf{e} \end{pmatrix}, \mathbf{x} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0} \right\}$$

where

$$\mathbf{A}^* = \begin{bmatrix} & \mathbf{A} & & \mathbf{0} \\ \mathbf{d_1} \;\; \mathbf{d_2} & \cdots & \mathbf{d_n} & -\mathbf{C} \end{bmatrix},$$

for which the summation of the lattice points of $\mathcal{P}$ weighted by w equals the number of lattice points of $\mathcal{P}^*$.

(b) Moreover, when $\mathbf{e} = 0$, the construction is parametric in the sense that the weight w is a homogeneous function, then for all $n \in \mathbb{N}$, we have $(n\mathcal{P})^* = n(\mathcal{P}^*)$, and

$$E_{\mathcal{P}}^{s,w}(n) = |(n\mathcal{P})^* \cap \mathbb{Z}^{d+e}| = |n(\mathcal{P}^*) \cap \mathbb{Z}^{d+e}| = E_{\mathcal{P}^*}(n)$$

Remark 1.4.1. To the best of our knowledge the first version of Theorem 1.4.1 appeared in print in work by Ardila and Brugallé (see [**7**, Section 4]), but in [**7**] the weights $w(\mathbf{x})$ were special polynomials and in that case, some of the consequences we show were not possible. In Section 4.1, we present a direct constructive/algorithmic proof of Theorem 1.4.1 and describe several interesting special cases depending on the type of Ehrhart quasi-polynomials (in particular, we recover the results of [**7**]).

Remark 1.4.2. The second half of Theorem 1.4.1 uses special weights that are by construction non-negative. But we note that most of the proof of the theorem works even when $w(\mathbf{x})$ takes negative or zero values over $\mathcal{P}$. The function $E_{\mathcal{P}}^{s,w}(n)$ still makes sense, but what we obtain is not a traditional Ehrhart polynomial, because, for example, the leading coefficient could be negative, and volumes are never negative.

Remark 1.4.3. Theorem 1.4.1 says the weight $w(\mathbf{x})$ can be any Ehrhart quasi-polynomial. In Section 4.1, we carefully discuss many ways to express polynomials in terms of these quasi-polynomial weights. A key point is that Theorem 1.4.1 is more versatile and expressive because it applies to more functions than just polynomial weights. In fact, Section 4.1 shows w can have many different representations (e.g., polynomials), some more efficient than others. To demonstrate the power in Section 4.2 we present applications to Combinatorial Representation Theory and Number Theory.

Corollary 1.4.1 below is a notable new consequence of Theorem 1.4.1 that can be applied to many problems of interest. For example, these ideas can be applied to the integration and maximization of Kostka numbers, Littlewood–Richardson coefficients, and any other combinatorial invariant that is given by an Ehrhart quasi-polynomial.

Corollary 1.4.1. *Let w be weight obtained from an Ehrhart quasi-polynomial function of a parametric polyhedron $\mathcal{Q}$, whose parameters are defined over the lattice points of a polytope $\mathcal{P}$. Here $\mathcal{P}, \mathcal{Q}, w$ are just as in Theorem 1.4.1. Using the weight-lifting polytope construction of Theorem 1.4.1 one can integrate and maximize w over $\mathcal{P}$ as follows:*

- *One can compute the integral $\int_{\mathcal{P}} w(\mathbf{x})d\mathbf{x}$ reformulated as a volume computation of the weight-lifting polytope $\mathcal{P}^*$.*

- *One can solve the maximization problem and determine $\max_{\mathbf{x}\in\mathcal{P}\cap\mathbb{Z}^d} w(\mathbf{x})$. It reduces to counting the lattice points of a finite sequence of weight-lifting polytopes which contain each other and can be read from $\mathcal{P}^*$ efficiently.*

We sketch the proof of Corollary 1.4.1 in Section 4.1.

1.5. Kakeya's conjecture

In the ring of symmetric polynomials Λ_n, we consider an interesting question which is finding a set, $\{f_1, f_2, \ldots, f_n\}$, such that it is sufficient to use only these f_i's to generate all symmetric polynomials in n variables. If only addition and multiplication are allowed, then there are several well-studied candidates such as symmetric elementary polynomials, symmetric homogeneous polynomials, symmetric power sum polynomials, and Schur polynomials. However, if addition, multiplication, and division are all allowed, then the complexity of the question increases rapidly since the division operation enlarges the way of generating symmetric polynomials and there will be more candidates. Here, we only focus on the symmetric power sum polynomials to reduce the difficulty. According to the definition we provided in previous Section 1.1, we restate the question formally in this way.

QUESTION 1.5.1. *Given n positive integers, $c_1 < c_2 < \ldots < c_n$, decide whether the set $\{p_{c_1}, p_{c_2}, \ldots, p_{c_n}\}$ form a fundamental system or not?*

Besides the system, we give earlier in Example 1.1.8. Borchard gave a system $\{p_1, p_3, \ldots, p_{2n-1}\}$ by excluding the first $n-1$ even indices. Vahlen [**82**] extended it by excluding the first few multiples of ν and gave a system $\{p_1, \ldots, p_{\nu-1}, p_{\nu+1}, \ldots, p_{2\nu-1}, p_{2\nu+1}, \ldots, p_{k\nu-1}, p_{k\nu+1}, \ldots, p_\mu\}$. Ludwig [**84**] took a different approach by excluding some large indices and gave a system $\{p_1, \ldots, p_k, p_{l_{k+1}}, \ldots, p_{l_n}\}$ where $k+1 \le l_{k+1} < \ldots < l_n \le 2k+1$ and $k < n < 2k+1$. Kakeya [**54, 55**] gave a relatively complete description of the fundamental system. Specifically, Kakeya proved that if the set of positive integers $\{c_1, c_2, \ldots, c_n\}$ forms a gap of a numerical semigroup, then $\{p_{c_1}, p_{c_2}, \ldots, p_{c_n}\}$ forms a fundamental system. In the same paper, Kakeya conjectured that this gap set condition might be the equivalent characterization of the fundamental system.

Conjecture 1.5.1 (Kakeya's Conjecture). *$\{p_{c_1}, p_{c_2}, \ldots, p_{c_n}\}$ forms a fundamental system if and only if the index set $\{c_1, c_2, \ldots, c_n\}$ forms a gap of a numerical semigroup.*

Kakeya's conjecture reveals an unexpected and mysterious relationship between the theory of symmetric polynomials and the theory of numerical semigroups.

We will present an equivalent condition on when $\{p_{c_1}, p_{c_2}, \ldots, p_{c_n}\}$ forms a fundamental system. Kakeya found this criterion and used it to prove his results. However, his proof contains a small gap, and we will fill in this gap here.

Theorem 1.5.1 (Kakeya's criterion). *For any positive integer n, suppose the coefficient field $\mathbb{E}$ containing Λ_n is algebraically closed, then the following statements are equivalent:*

(1) The set $\{p_{c_1}, p_{c_2}, \ldots, p_{c_n}\}$ forms a fundamental system.

(2) The series $Log\left(\frac{1+\sum_{i=1}^{n}\alpha_i t^i}{1+\sum_{i=1}^{n}e_i t^i}\right)$ can be identified as 0 by checking only coefficients of n terms $t^{c_1}, t^{c_2}, \ldots,$ and t^{c_n}.

(3) The system of n polynomial equations with A_i unknowns,

$$
\begin{cases}
G_{c_1}(A_1, A_2, \ldots, A_n) & = 0 \\
G_{c_2}(A_1, A_2, \ldots, A_n) & = 0 \\
 & \vdots \\
G_{c_n}(A_1, A_2, \ldots, A_n) & = 0
\end{cases}, \tag{1.2}
$$

has the unique solution in $\mathbb{E}^n$, namely, $(A_1, A_2, \ldots, A_n) = (0, 0, \ldots, 0)$.

(4) Polynomials can be distinguished by their coefficients at the terms $t^{c_1}, t^{c_2}, \ldots,$ and t^{c_n} under the formal logarithm map.

(5) The system of n polynomial equations with α_i unknowns,

$$
\begin{cases}
F_{c_1}(\alpha_1, \alpha_2, \ldots, \alpha_n) & = (-1)^{c_1}\frac{p_{c_1}}{c_1} \\
F_{c_2}(\alpha_1, \alpha_2, \ldots, \alpha_n) & = (-1)^{c_2}\frac{p_{c_2}}{c_2} \\
 & \vdots \\
F_{c_n}(\alpha_1, \alpha_2, \ldots, \alpha_n) & = (-1)^{c_n}\frac{p_{c_n}}{c_n}
\end{cases}, \tag{1.3}
$$

has the unique solution in $\mathbb{E}^n$, *namely,* $(\alpha_1, \alpha_2, \ldots, \alpha_n) = (e_1, e_2, \ldots, e_n)$.

Theorem 1.5.1 motivates us to study those polynomial equations.

DEFINITION 1.5.1 (Kakeya variety). *For any positive integer* n, n *positive integers* $c_1 < c_2 < \ldots < c_n$ *and algebraically closed field* $\mathbb{E}$,

- *Kakeya variety* $Ka_1(c_1, c_2, \ldots, c_n)$ *of the form 1 in* $\mathbb{E}$ *is the set of solutions of Equation* (1.2).
- *Kakeya variety* $Ka_2(c_1, c_2, \ldots, c_n)$ *of the form 2 in* $\mathbb{E}$ *is the set of solutions of Equation* (1.3).

CONJECTURE 1.5.2. *The dimension of any Kakeya variety is* 0.

In Section 5.2, we attempt to solve Conjecture 1.5.1 by searching for nontrivial solutions of Equation (1.2) or Equation (1.3) and showing directly $\mathrm{Frac}(\Lambda_n) \neq \mathbb{K}(p_{c_1}, p_{c_2}, \ldots, p_{c_n})$. Combining the above two strategies, we prove the conjecture when there are two variables.

CHAPTER 2

Convexity in (Colored) Affine Semigroups

This chapter first discusses technical details about how convexity theorems can be extended to colored affine semigroups, and then we take a closer look at the fundamental properties of colored numerical semigroups.

2.1. Helly and Tverberg theorems for semigroups

To prove Theorem 1.2.1, we recall three fundamental results about affine semigroups that, together, ensure a nontrivial intersection of affine semigroups occurs precisely when their associated cones intersect nontrivially (Proposition 2.1.1).

LEMMA 2.1.1 ([**23**, Corollary 2.11(a)]). *The intersection of two affine semigroups is again an affine semigroup.*

REMARK 2.1.1. If a semigroup $S = \mathrm{Cone}(\mathbf{A}) \cap \Lambda$ for some lattice Λ, then the minimal generating set of S, called the *Hilbert basis* of $\mathbf{A}$, can be computed [**26**]. If the generators of two affine semigroups S_1 and S_2 are given via matrices $\mathbf{A}$ and $\mathbf{B}$, then one can compute the generators of $S_1 \cap S_2$ by constructing a rational cone

$$C = \{(\mathbf{x}, \mathbf{y}) \geq 0 : \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{y} = 0\},$$

finding its Hilbert basis, and then mapping each Hilbert basis element $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{A}\mathbf{x}$.

LEMMA 2.1.2. *For any affine semigroup $S \subset \mathbb{Z}^d$, if* $\mathrm{Cone}(S)$ *contains an integral point* $\mathbf{p}$*, then* $k\mathbf{p} \in S$ *for some positive integer k.*

LEMMA 2.1.3. *For affine semigroups, taking finite intersections commutes with taking the associated cone: if $S_1, \ldots, S_n \subset \mathbb{Z}^d$ are affine, then* $\mathrm{Cone}(\bigcap_{i=1}^n S_i) = \bigcap_{i=1}^n \mathrm{Cone}(S_i)$*.*

Proposition 2.1.1. *The intersection $\bigcap_i S_i$ of affine semigroups $S_1, \ldots, S_n \subset \mathbb{Z}^d$ contains a non-zero element if and only if $\bigcap_i \mathrm{Cone}(S_i)$ contains a non-zero element.*

Proof. Apply all parts of Lemma 2.1.1, Lemma 2.1.2, and Lemma 2.1.3. □

Proof of Theorem 1.2.1. Let $C_i = \mathrm{Cone}(S_i)$ for each i, and let $\mathcal{G} = \{C_1, \ldots, C_n\}$. By Proposition 2.1.1, it suffices to show in each case that $C_1 \cap \cdots \cap C_n$ is nontrivial.

a). Consider the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$. If a convex cone C contains a non-zero element, then C will intersect the unit sphere. Hence, instead of proving the family $\mathcal{G}$ intersects at a non-zero element, it suffices to prove $\mathcal{G}' = \{C_1 \cap \mathbb{S}^{d-1}, \ldots, C_n \cap \mathbb{S}^{d-1}\}$ has nonempty intersection. Since $\mathcal{G}$ does not cover $\mathbb{R}^d$, $\mathcal{G}'$ does not cover $\mathbb{S}^{d-1}$. Therefore, for any point $q \in \mathbb{S}^{d-1}$ not covered by $\mathcal{G}'$, there exists a homeomorphism $f : \mathbb{S}^{d-1} \setminus \{q\} \to \mathbb{R}^{d-1}$. under which it suffices to prove the family

$$\mathcal{G}'' = \{f(C_1 \cap \mathbb{S}^{d-1}), \ldots, f(C_n \cap \mathbb{S}^{d-1})\}$$

has nonempty intersection. To this end, we employ a topological variant of Helly's theorem [**59**], which states that for a finite family of closed sets in $\mathbb{R}^d$, if the intersection of every $d+1$ members is contractible, then the intersection of the family is contractible.

Now, each C_i is a rational polyhedral cone and therefore topologically closed, and if S_i has only the trivial subgroup, then C_i is pointed. Since each C_i is closed and pointed, so is any intersection of the C_i's. We can conclude that each set $f(C_i \cap \mathbb{S}^{d-1})$ is closed in $\mathbb{R}^{d-1}$, and in particular that the intersection of any $N = d$ of the sets in $\mathcal{G}''$ is nonempty and contractible. As such, applying the aforementioned topological Helly's theorem to $\mathcal{G}''$ completes the proof.

b). If each S_i has only the trivial subgroup, then each C_i is pointed, and thus $C_i \setminus \{0\}$ is convex for each i. As such, if every $N = d+1$ of the C_i's intersects nontrivially, then the claim in this case follows from Helly's theorem for convex sets in $\mathbb{R}^d$.

c). In this case, we employ a j-dimensional variant of Helly's theorem [**35**], which states that for a family of finite convex sets in $\mathbb{R}^d$, if the intersection of every $2d$ members is at least 1-dimensional, then the intersection of the family is at least 1-dimensional. This can be applied directly, as any intersection of rational cones that contains a nonzero point must be at least 1-dimensional.

In each of the above cases, $C_1 \cap \cdots \cap C_n$ contains a non-zero element. $\square$

We now illustrate that each choice of N in Theorem 1.2.1 is the best possible.

EXAMPLE 2.1.1. Let e_i be i-th standard basis in $\mathbb{R}^d$.

a). Let $E = \{e_1, \ldots, e_d\}$, and consider the affine semigroups $S_i = \mathrm{Sg}(E \setminus \{e_i\})$. The intersection of any $d-1$ contains a non-zero element, as $e_i \in \bigcap_{j \neq i} S_j$ for each i, but the intersection of all d affine semigroups is trivial.

b). Let P be any d-simplex with the origin in its interior and vertices set denoted $V = \{v_1, \ldots, v_{d+1}\}$, and consider the affine semigroups $S_i = \mathrm{Sg}(V \setminus \{v_i\})$. We can verify that $v_i \in \bigcap_{j \neq i} S_j$ for each i, but $\bigcap_j S_j$ is trivial.

c). Let $E = \{e_1, -e_1, \ldots, e_d, -e_d\}$. Consider the affine semigroups

$$S_{i,+} = \mathrm{Sg}(E \setminus \{-e_i\}) \qquad \text{and} \qquad S_{i,-} = \mathrm{Sg}(E \setminus \{e_i\})$$

for each i. Any $2d-1$ of the above affine semigroups share a non-zero element, as

$$\pm e_i \in S_{1,+} \cap S_{1,-} \cap \cdots \cap S_{i,\pm} \cap \cdots \cap S_{d,+} \cap S_{d,-}$$

for each i, but the only point common to all $2d$ is the origin.

We close this section with two corollaries of Proposition 2.1.1. The first is an analog of the colorful Helly's theorem [**35**], which asserts that given $d+1$ finite families $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ of convex sets, if for every choice of transversal $S_1 \in \mathcal{F}_1$, $S_2 \in \mathcal{F}_2$, $\ldots$, $S_{d+1} \in \mathcal{F}_{d+1}$, the intersection $S_1 \cap \cdots \cap S_{d+1}$ is nonempty, then for some j, the sets in $\mathcal{F}_j$ have nonempty intersection. The second is an analog of Tverberg's theorem [**81**], which states that for any set D of $(d+1)(r-1)+1$ points in $\mathbb{R}^d$, there exists a point p (not necessarily in D) and a partition of D into r blocks, such that p belongs to the convex hull of each block. Note that both are "partial" analogs, as all affine semigroups therein are required to be pointed.

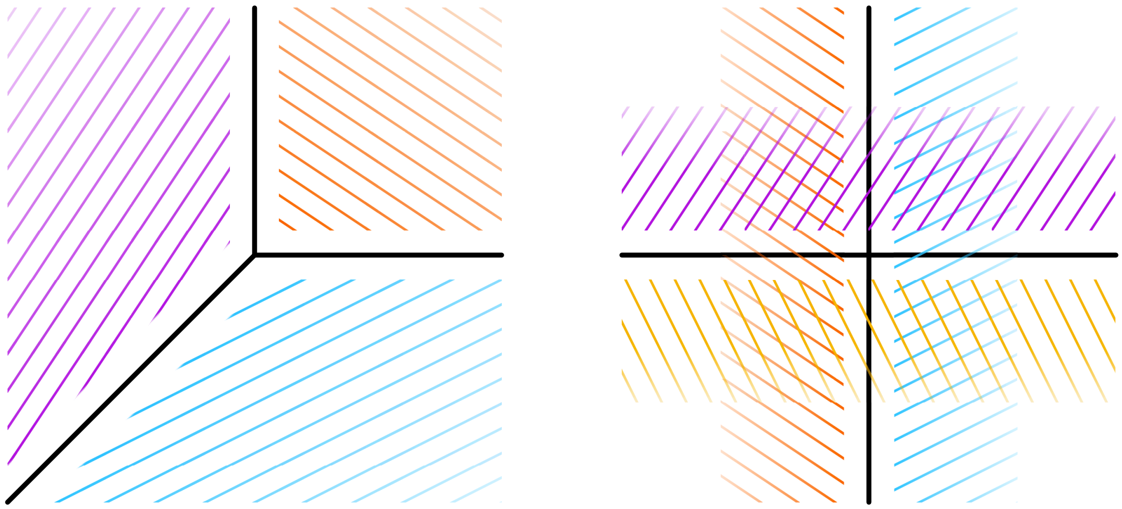

Figure 2.1. Depiction of the families with $d = 2$ in Example 2.1.1(b) (left) and Example 2.1.1(c) (right).

Corollary 2.1.1. *Let $\mathcal{F}_1, \ldots, \mathcal{F}_N$ be finite families of pointed affine semigroups in $\mathbb{Z}^d$. If for every choice of a transversal $S_1 \in \mathcal{F}_1$, $S_2 \in \mathcal{F}_2$, $\ldots$, $S_{d+1} \in \mathcal{F}_{d+1}$, the intersection $S_1 \cap \cdots \cap S_{d+1}$ contains a non-zero element, then there is a family $\mathcal{F}_j$ such that all semigroups in $\mathcal{F}_j$ intersect at a non-zero element.*

Proof. Following the proof of Theorem 1.2.1(b), replacing each affine semigroup with its associated cone with the origin removed yields $d+1$ families of convex sets in $\mathbb{R}^d$, to which one can readily apply colorful Helly's theorem [**38**]. $\square$

Corollary 2.1.2. *Fix a pointed affine semigroup $S = \mathrm{Sg}(A) \subset \mathbb{Z}^d$ given by $|A| = k$ generators. If $k \geq d(r-1)+1$, then there exists a r-coloring of S such that some element $p \in S$ has a monochromatic solution of every color.*

Proof. By Proposition 2.1.1, we must show that there exists a partition $A_1, \ldots, A_r$ of A such that some non-zero element $p \in \mathrm{Cone}(A)$ lies in $\mathrm{Cone}(A_i)$ for each i. Since S has only the trivial subgroup, $\mathrm{Cone}(S)$ is pointed, so by taking a cross-section of $\mathrm{Cone}(A)$, it is equivalent to show that given a set D of k points in $\mathbb{R}^{d-1}$, there exists a r-coloring of D and a point p that lies in the convex hull of each color class. Since $k \geq ((d-1)+1)(r-1)+1$, this is exactly the statement of Tverberg's theorem. $\square$

	g_i	g_i'	g_i''
S_1	(0,1,2)	(1,7,9)	(2,9,9)
S_2	(0,3,4)	(1,9,11)	(2,5,5)
S_3	(0,7,8)	(1,13,15)	(2,-3,-3)
S_4	(0,15,16)	(1,21,23)	(2,-19,-19)
S_5	(0,31,32)	(1,37,39)	(2,-51,-51)
S_6	(0,63,64)	(1,69,71)	(2,-115,-115)

TABLE 2.1. The family of semigroups in Proposition 2.2.1 with $n = 6$.

PROBLEM 2.1.1. *Generalize Corollary 2.1.1 and Corollary 2.1.2 to families of (not necessarily pointed) affine semigroups.*

2.2. Carathéodory type theorems for semigroups

The semigroup version of the colorful Carathéodory theorem fails strongly. We provide two counterexamples; Table 2.1 contains an example of one, and Example 2.2.1 illustrates another.

PROPOSITION 2.2.1. *Fix $n \geq 1$, and consider the family of semigroups $\mathcal{F}_n = \{S_i = \mathrm{Sg}(g_i, g_i', g_i'')\}$, where*

$$g_i = (0, 2^i - 1, 2^i), \quad g_i' = (1, n + 2^i - 1, n + 2^i + 1), \quad g_i'' = (2, 2(n - 2^i) + 1, 2(n - 2^i) + 1)$$

and $1 \leq i \leq n$. Letting $p = (3, 3n - 1, 3n + 2)$, we have $p \in S_i$ for each i, and the only expressions for p as a sum of generators from across the S_i's are those of the form

$$p = g_i + g_i' + g_i''$$

for each i.

PROOF. Consider an arbitrary expression for p as a sum of generators from the S_i's. We claim any expression for p must have the form $p = g_i + g_j' + g_k''$, where i, j, and k are not necessarily distinct. Indeed, some generator g_j' must appear, since the first coordinate of p is odd, and from there, some generator g_i must appear since the last 2 coordinates of p differ by 3. The first coordinate of p then forces the third and final generator in the expression to have the form g_k''. This proves the claim. Examining second coordinates in any such expression, we see $2^i + 2^j = 2^{k+1}$, which is impossible unless $i = j = k$. $\square$

Example 2.2.1. Let $\mathcal{I}_1 = \{1,2,3\}$, $\mathcal{I}_2 = \{4,5,6\}$, $\mathcal{I}_3 = \{7,8,9\}$, and

$$A = \left[\begin{array}{ccc|ccc|ccc} 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 32 & 63 & 1 & 33 & 61 & 3 & 35 & 57 \\ 1 & 34 & 63 & 2 & 35 & 61 & 4 & 37 & 57 \end{array}\right].$$

The element $p = (3, 95, 98)$ has monochromatic solutions

$$(1,1,1,0,0,0,0,0,0), \quad (0,0,0,1,1,1,0,0,0), \quad \text{and} \quad (0,0,0,0,0,0,1,1,1),$$

but no 3-chromatic solutions.

Proof of Theorem 1.2.2. Proposition 2.2.1 implies part (a) upon noting that all affine semigroups therein are normal since their generating matrices have determinant -1 (see [**14**, Chapter 8, Corollary 2.6]). For part (b), for each family $\mathcal{F}_n$ in Proposition 2.2.1, consider the family

$$\mathcal{F}' = \{S \times \mathbb{Z}_{\geq 0} : S \in \mathcal{F}_n\},$$

of semigroups of the form

$$S \times \mathbb{Z}_{\geq 0} = \mathrm{Sg}((g_1, 0), \ldots, (g_r, 0), (0, 1)) \qquad \text{whenever} \qquad S = \mathrm{Sg}(g_1, \ldots, g_r).$$

For each $k \geq 1$, the element (p, k) lies in $S \times \mathbb{Z}_{\geq 0}$ for each $S \in \mathcal{F}$, and the only expressions for (p, k) as a sum of generators of the semigroups in $\mathcal{F}'$ are obtained by concatenating an expression for $(p, 0)$ with k copies of $(0, 1)$. According to the proof of Proposition 2.2.1, the only way to generate $(p, 0)$ is $(g_i, 0) + (g_i', 0) + (g_i'', 0)$ for some index i. This solution violates the condition of being colorful because three different vectors of the same color are part of the expression for (p, k). Note that in the previous construction, $(0, 1)$ appeared many times with different colors. The definition of colorful is violated because the vectors $(g_i, 0), (g_i', 0), (g_i'', 0)$ are of the same color. □

The family in the above proof can be adjusted to use different vectors, of different colors, by replacing the instances of $(0, 1)$ with vectors from $(0, 1), (0, 2), (0, 3), \ldots$, so that the infinitely many vectors $(p, k\,\mathrm{lcm}(1, 2, 3, \ldots, s))$ still do not have a colorful representation since $(p, 0)$ does not.

Proof of Theorem 1.2.3. Let $S = \mathrm{Sg}(A_1, \ldots, A_\ell)$ and $S' = \bigcap_{i=1}^{\ell} \mathrm{Sg}(A_i)$. If an element $b \in S$ has a monochromatic solution of each color, then $b \in S'$, so it suffices to prove there are only finitely many elements of S' with no chromatic solution. By Lemma 2.1.1, S' is finitely generated, say with minimal generating set G. Therefore, each $b \in S'$ can be written as $b = \sum_{g \in G} \lambda_g g$ with each $\lambda_g \in \mathbb{Z}_{\geq 0}$. We will prove that if $\sum_g \lambda_g \geq \ell$, then b has a chromatic solution. In fact, under this assumption, we can collect terms in this sum to form an expression $b = s_1 + \cdots + s_\ell$ as a sum of ℓ nonzero elements of S'. Each s_i thus has a monochromatic solution in S of color i, and concatenating these monochromatic solutions yields a chromatic solution for b. $\square$

2.3. Colored numerical semigroups

In this section, we turn our attention to colored numerical semigroups and the chromatic Frobenius problem. Before restricting to this case, however, we prove the following general result, which forms the backbone of the proof of Theorem 1.2.4 but holds for any colored affine semigroup.

Theorem 2.3.1. *For a colored affine semigroup $S = \mathrm{Sg}(A_1, \ldots, A_\ell)$, the set*

$$S(A, k) = \{Ax : x \text{ is } k\text{-chromatic}\}$$

equals the union of finitely many translated copies of S.

Proof. Consider the map $\varphi : \mathbb{Z}_{\geq 0}^n \to \mathrm{Sg}(A)$ sending each standard basis vector e_i to the i'th column Ae_i of A, and let

$$E = \varphi^{-1}(S(A, k)) = \{x \in \mathbb{Z}_{\geq 0}^n : x \text{ is } k\text{-chromatic}\}.$$

Note that E is closed under the additive action of $\mathbb{Z}_{\geq 0}$, as $x + e_i$ is nonzero in every entry that x is nonzero. By Dickson's lemma [**45**], any subset of $\mathbb{Z}_{\geq 0}^n$ has finitely many minimal elements under the component-wise partial order, so

$$E = (\mathbb{Z}_{\geq 0}^n + x_1) \cup \cdots \cup (\mathbb{Z}_{\geq 0}^n + x_r)$$

for some $x_1, \ldots, x_r \in E$. Applying φ to the above equality completes the proof. $\square$

2.3.1. Chromatic Frobenius numbers. For the remainder of this section, fix a colored numerical semigroup $S = \mathrm{Sg}(A_1, \ldots, A_\ell)$, where $A_1, \ldots, A_\ell$ partition $A = \{a_1, \ldots, a_n\}$ with $\gcd(A) = 1$. The set of *gaps* of S, denoted $G(A) = \mathbb{Z}_{\geq 0} \setminus S$, is then a finite set with $\mathsf{F}(A) = \max(G(A))$ (this follows from Bézout's identity, see [**73**]). Analogously, the *k-chromatic gaps* are the integers in the set $G(A, k) = \mathbb{Z}_{\geq 0} \setminus S(A, k)$, so that $\mathsf{CF}_k(A_1, \ldots, A_\ell) = \max(G(A, k))$.

The following provides upper and lower bounds for $\mathsf{CF}_k(A_1, \ldots, A_\ell)$, and in particular verifies $G(A, k)$ is a finite set, as claimed in Theorem 1.2.4(1).

Corollary 2.3.1. *The colored Frobenius number satisfies*

$$\min(m(A, k)) - 1 \leq \mathsf{CF}_k(A_1, \ldots, A_\ell) \leq \min(m(A, k)) + \mathsf{F}(A),$$

where $m(A, k) = \bigcup_{|I|=k} \sum_{i \in I} A_i$.

Proof. By Theorem 2.3.1, $S(A, k)$ equals the union of finitely many translations of copies of S. More can be said, as

$$S(A, k) = \bigcup_{v \in m(A,k)} S + v.$$

Therefore, if $b \leq \min(m(A, k)) - 1$, then $b \notin S(A, k)$ since $b \notin S + v$ for any $v \in m(A, k)$, and if $b > \min(m(A, k)) + \mathsf{F}(A)$, then $b \in S(A, k)$ since $b - \min(m(A, k)) \in S$. □

Proof of Theorem 1.2.4(1). Apply Corollary 2.3.1. □

We also obtain the following chromatic generalization of the well-known formula $\mathsf{F}(a, b) = ab - (a + b)$, which holds whenever $\gcd(a, b) = 1$.

Corollary 2.3.2. *We have*

$$\mathsf{CF}_\ell(\{a_1\}, \ldots, \{a_\ell\}) = a_1 + \cdots + a_\ell + \mathsf{F}(A),$$

and in particular $\mathsf{CF}_2(\{a_1\}, \{a_2\}) = a_1 a_2$.

Proof. Proceeding as in the proof of Corollary 2.3.1, if each A_i is a singleton,

$$S(A, \ell) = S + a_1 + \cdots + a_\ell,$$

and as such, $\mathsf{CF}_\ell(\{a_i\},\ldots,\{a_\ell\}) = a_1+\cdots+a_\ell+\mathsf{F}(A)$. When $\ell = 2$, this then yields $\mathsf{CF}_2(\{a\},\{b\}) = \mathsf{F}(A) + a + b = ab$. □

Remark 2.3.1. The chromatic Frobenius number is not always represented as the Frobenius number and some generators from each color class. For instance, $\mathsf{CF}_2(\{a,c\},\{b\}) = ab$ whenever $c > ab$.

Before proving Theorem 1.2.4(2), we prove the following lemma.

Lemma 2.3.1. *If $1 \le k < \ell$, then $\mathsf{CF}_k(A) \le \mathsf{CF}_{k+1}(A)$. Moreover, if $\gcd(A \setminus A_i) = 1$, so that $\mathsf{F}(A \setminus A_i)$ and $\mathsf{CF}_{\ell-1}(A_1,\ldots,A_{i-1},A_{i+1},\ldots,A_\ell)$ both exist, then*

$$\begin{aligned}\mathsf{CF}_\ell(A_1,\ldots,A_\ell) &\le \mathsf{CF}_{\ell-1}(A_1,\ldots,A_{i-1},A_{i+1},\ldots,A_\ell) + \min A_i\\ &\le \mathsf{CF}_\ell(A_1,\ldots,A_\ell) + \mathsf{F}(A \setminus A_i) + 1.\end{aligned}$$

Proof. The first claim follows from the fact that $S(A,k) \subseteq S(A,k+1)$.

In what follows, let $A = (A_1,\ldots,A_\ell)$ and $B = (A_1,\ldots,A_{i-1},A_{i+1},\ldots,A_\ell)$. For the first inequality, we must prove that if $b > \mathsf{CF}_{\ell-1}(B) + \min A_i$, then $b \in S(A,\ell)$. Since $b - \min A_i > \mathsf{CF}_{\ell-1}(B)$, we know $b - \min A_i \in S(B,\ell-1)$, so we can write

$$b - \min A_i = a_1' + \cdots + a_{i-1}' + a_{i+1}' + \cdots + a_\ell' + c,$$

where $c \in \mathrm{Sg}(A \setminus A_i)$ and each $a_k' \in A_k$. This implies $b \in S(A,\ell)$.

For the final inequality, we must prove that if $b > \mathsf{CF}_\ell(A) + \mathsf{F}(A \setminus A_i) - \min A_i + 1$, then $b \in S(B,\ell-1)$. Since $b - (\mathsf{F}(A \setminus A_i) - \min A_i + 1) \in S(A,\ell)$, we can write

$$b - (\mathsf{F}(A \setminus A_i) - \min A_i + 1) = a_1' + \cdots + a_i' + \cdots + a_\ell' + c,$$

where $c \in \mathrm{Sg}(A)$ and each $a_k' \in A_k$. Notice $a_i' - \min A_i + 1 > 0$ and $c \ge 0$, which imply

$$c' = a_i' - \min A_i + 1 + \mathsf{F}(A \setminus A_i) + c > \mathsf{F}(A \setminus A_i)$$

and in particular $c' \in \mathrm{Sg}(A \setminus A_i)$. Hence,

$$b = a_1' + \cdots + a_{i-1}' + a_{i+1}' + \cdots + a_\ell' + c' \in S(B,\ell-1),$$

as desired. $\square$

Proof of Theorem 1.2.4(2). For each $k \in \mathbb{Z}_{>0}$, let $P(k)$ be the statement that computing $\mathsf{CF}_k(A_1, A_2, \ldots, A_\ell)$ is NP-hard for all $\ell \geq k$. We will prove the statement by induction on k. First, when $k = 1$, the colored Frobenius number coincides with the classical Frobenius number. Hence, the statement $P(1)$ is true since the computation complexity of the classical Frobenius number is NP-hard [**72**].

For the inductive step, supposing the statement $P(m)$ is true, we will find a polynomial-time reduction to prove $P(m+1)$. We do so by proving that there exists a natural number b, which can be found in polynomial time, such that:

a). if $\ell > k$, then $\mathsf{CF}_{k+1}(2A_1, \ldots, 2A_\ell, \{b\}) = 2\mathsf{CF}_k(A_1, \ldots, A_\ell) + b$ (here, $2A = A + A$); and

b). if $\ell = k$, then $\mathsf{CF}_{\ell+1}(A_1, \ldots, A_\ell, \{b\}) = \mathsf{CF}_\ell(A_1, \ldots, A_\ell) + b$.

Indeed, the above claims immediately yield a polynomial-time reduction, so the statement $P(m+1)$ is true.

For simplicity, let $A = (A_1, \ldots, A_\ell)$. We will prove claim (a) by proving the following statement: if $\ell > k$, then for any odd b with $b > 2\mathsf{CF}_{k+1}(A) - 2\mathsf{CF}_k(A)$ and $b > 2\mathsf{CF}_k(A)$,

$$\mathsf{CF}_{k+1}(2A_1, \ldots, 2A_\ell, \{b\}) = 2\mathsf{CF}_k(A_1, \ldots, A_\ell) + b.$$

If a number $p > 2\mathsf{CF}_k(A) + b$, then by the choice of b, $p > 2\mathsf{CF}_{k+1}(A)$.

When p is even, then $\frac{p}{2} > \mathsf{CF}_{k+1}(A)$. By the definition of colored Frobenius numbers, $\frac{p}{2}$ has a $(k+1)$-chromatic solution in the colored numerical semigroup $\mathrm{Sg}(A_1, A_2, \ldots, A_\ell)$. Hence we can construct a $(k+1)$-chromatic solution of p in the colored numerical semigroup $\mathrm{Sg}(2A_1, 2A_2, \ldots, 2A_\ell, \{b\})$.

When p is odd, then $\frac{p-b}{2} > \mathsf{CF}_k(A)$. By the definition of colored Frobenius numbers, $\frac{p-b}{2}$ has a k-chromatic solution in the colored numerical semigroup $\mathrm{Sg}(A_1, A_2, \ldots, A_\ell)$. Hence we can construct a $k+1$-chromatic solution of p in the colored numerical semigroup $\mathrm{Sg}(2A_1, 2A_2, \ldots, 2A_\ell, \{b\})$.

If $p = 2\mathsf{CF}_k(A) + b$, then $p - b = 2\mathsf{CF}_k(A)$. By the definition, $\frac{p-b}{2}$ has no k-chromatic solution in colored numerical semigroup $\mathrm{Sg}(A_1, A_2, \ldots, A_\ell)$. Hence, $p-b$ has no k-chromatic solution in the colored semigroup $\mathrm{Sg}(2A_1, 2A_2, \ldots, 2A_\ell)$. Since $b \geq 2\mathsf{CF}_k(A)$, $p - tb$ will be negative for $t \geq 2$. Overall, p has no $k+1$-chromatic solution in the colored numerical semigroup $\mathrm{Sg}(2A_1, 2A_2, \ldots, 2A_\ell, \{b\})$.

We now consider claim (b), we will prove the following statement: for any $b > \mathsf{CF}_\ell(A)$,

$$\mathsf{CF}_{\ell+1}(A_1, \dots, A_\ell, \{b\}) = \mathsf{CF}_\ell(A_1, \dots, A_\ell) + b.$$

If a number $p > \mathsf{CF}_\ell(A) + b$, then $p - b > \mathsf{CF}_\ell(A)$. By the definition of the colored Frobenius numbers, $p - b$ has a ℓ-chromatic solution in $\mathrm{Sg}(A_1, \dots, A_\ell)$, hence p has a $k+1$-chromatic solution in $\mathrm{Sg}(A_1, \dots, A_\ell, b)$.

If $p = \mathsf{CF}_\ell(A) + b$, then $p - b = \mathsf{CF}_\ell(A)$. By the definition, $p - b$ has no k-chromatic solution in $\mathrm{Sg}(A_1, \dots, A_\ell)$. Since $b \geq \mathsf{CF}_\ell(A)$, $p - tb$ will be negative for $t \geq 2$. Therefore, p has no $k+1$-chromatic solution in $\mathrm{Sg}(A_1, \dots, A_\ell, \{b\})$.

When $\ell > k$, we can choose $b \geq 2(\min A_1 + \cdots + \min A_\ell + \mathsf{F}(A))$, and when $\ell = k$, we can choose $b \geq \min A_1 + \cdots + \min A_\ell + \mathsf{F}(A)$. By Corollary 2.3.1 and the definition of colored Frobenius numbers, when $\ell > k$,

$$b \geq 2(\min A_1 + \cdots + \min A_\ell + \mathsf{F}(A)) \geq 2\mathsf{CF}_\ell(A) \geq 2\mathsf{CF}_{k+1}(A) \geq 2\mathsf{CF}_k(A);$$

when $\ell = k$,

$$b \geq \min A_1 + \cdots + \min A_\ell + \mathsf{F}(A) \geq \mathsf{CF}_\ell(A).$$

These b's satisfy the requirements.

To complete the proof, we note that since $\mathsf{F}(A)$ has some trivial bounds like the product of a_i's and there are efficient algorithms to compute the minimum of a set, b can be easily found in polynomial-time. □

2.3.2. Counting chromatic solutions. In the remainder of this chapter, we examine

$$f_k(b; A_1, \dots, A_\ell) = \# \{\text{k-chromatic solutions of } b\}.$$

for a given colored numerical semigroup $S = \mathrm{Sg}(A_1, \dots, A_\ell)$.

Recall that a function $g : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ is said to be *quasi-polynomial* of period N if

$$g(n) = p_i(n) \qquad \text{whenever} \qquad n \equiv i \bmod N,$$

for some polynomials $p_0, \ldots, p_{N-1}$. Moreover, a function $f : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ is *eventually quasi-polynomial* if there exists a quasi-polynomial function g such that $f(n) = g(n)$ for all but finitely many $n \in \mathbb{Z}_{\geq 0}$.

Fix a field $\Bbbk$ and let $R = \Bbbk[x_1, \ldots, x_n]$. A $\mathbb{Z}_{\geq 0}$-grading of R is specified by choosing $\deg(x_i) = a_i \in \mathbb{Z}_{\geq 0}$ and then defining

$$\deg(x_1^{\xi_1} x_2^{\xi_2} \cdots x_n^{\xi_n}) = \xi_1 a_1 + \xi_2 a_2 + \cdots + \xi_n a_n.$$

An element of R is *homogeneous of degree* b if all of its terms have degree b, and an ideal $I \subseteq R$ is *homogeneous* if I can be generated (as an ideal) by homogeneous elements. The *b-graded piece* of a homogeneous ideal I is

$$I_b = \operatorname{span}_{\Bbbk}\{r \in I : r \text{ is homogeneous of degree } b\},$$

and the *Hilbert function* of I is the function $h_I : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ given by $h_I(b) = \dim_{\Bbbk} I_b$. For example, if $R = \Bbbk[x, y]$, $\deg(x) = 2$, $\deg(y) = 3$, and $I = \langle x^5, y^5 \rangle$, then

$$R_{18} = \operatorname{span}_{\Bbbk}\{x^9, x^6y^2, x^3y^4, y^6\},$$

so $h_R(18) = 4$ and $h_I(18) = 3$. We direct the reader to [**67**] for background on Hilbert functions, and on the following theorem of Hilbert.

Theorem 2.3.2 (Hilbert). *Fix a $\mathbb{Z}_{\geq 0}$-graded polynomial ring R over a field $\Bbbk$ and a homogeneous ideal $I \subseteq R$. The Hilbert function of I is eventually quasi-polynomial.*

Proof of Theorem 1.2.4(3). Fix a field $\Bbbk$, let $R = \Bbbk[x_1, x_2, \ldots, x_n]$, and fix a colored numerical semigroup $S = \operatorname{Sg}(A_1, \ldots, A_\ell)$ with $A = \{a_1, \ldots, a_n\}$. The map

$$\begin{aligned} \psi : \{\text{monomials in } R\} &\longrightarrow S \\ x_1^{\xi_1} x_2^{\xi_2} \cdots x_n^{\xi_n} &\longmapsto \xi_1 a_1 + \xi_2 a_2 + \cdots + \xi_n a_n. \end{aligned}$$

induces a natural bijection between the monomials in R and representations of elements of S. The preimage of ψ induces a grading on R that sets $\deg(x_i) = a_i$ for each i, with one graded piece R_b for each $b \in S$, and the monomials in R_b each correspond to a representation of b.

Now, a monomial $x_1^{\xi_1} x_2^{\xi_2} \cdots x_n^{\xi_n} \in R$ corresponds under ψ to a k-chromatic representation precisely when the nonzero ξ_i's lie in at least k distinct color classes. As such, if $x_1^{\xi_1} x_2^{\xi_2} \cdots x_n^{\xi_n}$ corresponds to a k-chromatic representation, then so does any monomial multiple (this is essentially the proof of Theorem 2.3.1). As such, the monomials in

$$I = \langle x_1^{\xi_1} x_2^{\xi_2} \cdots x_n^{\xi_n} : \xi_1 a_1 + \xi_2 a_2 + \cdots + \xi_n a_n \text{ is } k\text{-chromatic} \rangle,$$

are precisely those that correspond to a k-chromatic representation under ψ, and thus the number of monomials in I of degree b is exactly $f_k(b; A_1, \ldots, A_\ell)$. Applying Hilbert's theorem completes the proof. $\square$

CHAPTER 3

Weighted Graded Semigroup Algebra

This chapter discusses technical details about the relations among the q-weighted Hilbert series, r-weighted Hilbert series, and s-weighted Hilbert series. Then we specialize to Ehrhart theory and extend the properties of nonnegativity and reciprocity of the Ehrhart series.

3.1. Weighted Hilbert function and series

Recall weighted Hilbert functions and series in Definition 1.1.8. We first want to know how they are related to each other.

3.1.1. What do we know in general? We prove that q-weighted Hilbert series is the most general weighted Hilbert series.

Proof of Proposition 1.3.1. Note that

$$\prod_{i=1}^{r}\left(\sum_{j=0}^{w_i(\mathbf{a})} q_i^j\right)\mathbf{t}^{\mathbf{a}}x^{\phi(\mathbf{a})} = \prod_{i=1}^{r}\left(\frac{1-q_i^{w_i(\mathbf{a})+1}}{1-q_i}\right)\mathbf{t}^{\mathbf{a}}x^{\phi(\mathbf{a})} = \sum_{I\subseteq[r]}\left(\frac{\prod_{j\in I}(-q_j)}{\prod_{i=1}^{r}(1-q_i)}\right)\left(\prod_{j\in I} q_j^{w_j(\mathbf{a})}\right)\mathbf{t}^{\mathbf{a}}x^{\phi(\mathbf{a})},$$

and for any fixed index set I,

$$\left(\prod_{j\in I} q_j^{w_j(\mathbf{a})}\right)\mathbf{t}^{\mathbf{a}}x^{\phi(\mathbf{a})} = q_1^{w_1(\mathbf{a})}\cdots q_r^{w_r(\mathbf{a})}\mathbf{t}^{\mathbf{a}}x^{\phi(\mathbf{a})}\Big|_{q_k=1,k\notin I}.$$

Therefore, $F_M^{r,w_1,\dots,w_r}(\mathbf{q},\mathbf{t},x) = \sum_{I\subseteq[r]}\left(\frac{\prod_{j\in I}(-q_j)}{\prod_{i=1}^{r}(1-q_i)}\right)\left(F_M^{q,w_1,\dots,w_r}(\mathbf{q},\mathbf{t},x)\big|_{q_k=1,k\notin I}\right)$.

Note that

$$\left[q\cdot\frac{\partial}{\partial q}\left(q^{w(\mathbf{a})}\mathbf{t}^{\mathbf{a}}x^{\phi(\mathbf{a})}\right)\right]\Big|_{q,\mathbf{t}=1} = w(\mathbf{a})q^{w(\mathbf{a})}\mathbf{t}^{\mathbf{a}}x^{\phi(\mathbf{a})}\Big|_{q,\mathbf{t}=1} = w(\mathbf{a})x^{\phi(\mathbf{a})},$$

therefore, $F_M^{s,w}(x) = \left[q\cdot\frac{\partial}{\partial q}\left(F_M^{q,w}(q,\mathbf{t},x)\right)\right]\Big|_{q,\mathbf{t}=1}$. □

Then we prove that the assumption of a linear weight is necessary.

Proof of Proposition 1.3.2. A monomial $q^m x^n$ is in $A_q^w[M]$ if and only if there is a partition λ of n, denoted $\lambda \vdash n$, such that the sum of the squares $\lambda_1^2 + \lambda_2^2 + \cdots$ is equal to m. Consider the following ideals of R''

$$I_k = (q^{i^2} x^i \mid 1 \leq i \leq k),\ k \geq 1.$$

It suffices to show that $I_{k-1} \subsetneq I_k$ for $k \geq 2$. We claim that $q^{k^2} x^k \in I_k \setminus I_{k-1}$. We argue by contradiction assuming that $q^{k^2} x^k \in I_{k-1}$. Then,

$$q^{k^2} x^k = (q^m x^n)(q^{i^2} x^i)$$

for some $q^m x^n \in A_q^w[M]$ and $1 \leq i \leq k-1$. Hence, there is a partition $(n_1, \ldots, n_\ell)$ of n such that

$$\begin{aligned}
&n = n_1 + \cdots + n_\ell,\ m = n_1^2 + \cdots + n_\ell^2,\ n_i \in \mathbb{N}, \\
&k = n + i = n_1 + \cdots + n_\ell + i,\ k^2 = m + i^2 = n_1^2 + \cdots + n_\ell^2 + i^2, \quad \therefore \\
&(n_1 + \cdots + n_\ell + i)^2 = n_1^2 + \cdots + n_\ell^2 + i^2.
\end{aligned}$$

Hence, from the last equality, we get $n_i = 0$ for $i = 1, \ldots \ell$, $n = 0, m = 0$, and consequently $q^{k^2} x^k = q^{i^2} x^i$ for some $1 \leq i \leq k-1$, a contradiction. $\square$

In what follows we will need to do monomial substitutions. We note this can be carried on in practice by the following lemma.

Lemma 3.1.1. *[**18**, Theorem 2.6] Let us fix k, the number of binomials in the denominator of a rational function. Given a rational function sum g of the form*

$$g(x) = \sum_{i \in I} \alpha_i \frac{x^{u_i}}{\prod_{j=1}^{k} (1 - x^{v_{ij}})},$$

where u_i, v_{ij} are integral d-dimensional vectors, and a monomial map $\psi : \mathbb{C}^n \longrightarrow \mathbb{C}^d$ given by the variable change $x_i \to z_1^{l_{i1}} z_2^{l_{i2}} \ldots z_n^{l_{in}}$ whose image does not lie entirely in the set of poles of $g(x)$. Then there exists a polynomial time algorithm which computes the function $g(\psi(z))$ as a sum of rational functions of the same shape as $g(z)$.

With the help of the above monomial substitution lemma, we can prove the under the linear weight assumption, all weighted Hilbert series have rational forms.

Proof of Theorem 1.3.1. The multivariate Hilbert series of the monomial algebra $\mathbb{K}[M]$ with a $\mathbb{Z}$-grading ϕ is

$$\sum_{\mathbf{a}\in M} \mathbf{t}^{\mathbf{a}} x^{\phi(\mathbf{a})}.$$

Recall that $\mathbf{t}^{\mathbf{a}}$ is the abbreviation of $t_1^{a_1}\cdots t_d^{a_d}$. By Lemma 3.1.1, we can apply the following monomial substitutions: $t_1 \mapsto q_1^{v_{1,1}}\cdots q_r^{v_{r,1}} t_1$, $\ldots$, $t_d \mapsto q_1^{v_{1,d}}\cdots q_r^{v_{r,d}} t_d$. Then by the linearity of weights w_i's,

$$\mathbf{t}^{\mathbf{a}} x^{\phi(\mathbf{a})} \mapsto q_1^{\mathbf{v}_1^{\intercal}\mathbf{a}}\cdots q_r^{\mathbf{v}_r^{\intercal}\mathbf{a}} \mathbf{t}^{\mathbf{a}} x^{\phi(\mathbf{a})}.$$

Lastly, we can just multiply the series by $q_1^{b_1}\cdots q_r^{b_r}$. Therefore, if the multivariate Hilbert series has a rational form, then the monomial substitution gives a rational form for the q-weighted multivariate Hilbert series. Hence, by Proposition 1.3.1, the rest of the Hilbert series are all rational functions. □

3.2. Weighted Ehrhart rings and series

To yield more interesting results, we have to concentrate on Ehrhart theory.

3.2.1. nonnegativity of h^* coefficients. We first use the partition technique to extend the nonnegativity of h^* coefficients.

Proof of Theorem 1.3.2. We can construct a disjoint partition of $\mathcal{P}$ using the triangulation $\mathcal{T}$, $\mathcal{P} = \sqcup_{S\in\mathcal{T}} S^*$ with S^* being possibly removing several facets of S. Note that the q-weighted Hilbert function is additive with respect to disjoint union. Therefore,

$$E_M^{q,w_1,\ldots,w_r}(\mathbf{q},\mathbf{1},n) = \sum_{S\in\mathcal{T}} E_{M_{S^*}}^{q,w_1,\ldots,w_r}(\mathbf{q},\mathbf{1},n),$$

and similarly,

$$F_M^{q,w_1,\ldots,w_r}(\mathbf{q},\mathbf{1},n) = \sum_{S\in\mathcal{T}} F_{M_{S^*}}^{q,w_1,\ldots,w_r}(\mathbf{q},\mathbf{1},n).$$

Since S^* is a simplex with several facets possibly removed, using [**20**, Theorem 3.5], it is easy to see that its q-weighted Hilbert series has the following rational form

$$F^{q,w_1,\dots,w_r}_{M_{S^*}}(\mathbf{q},\mathbf{1},n) = \frac{h_{S^*}(\mathbf{q},z)}{\prod_{i=1}^{d+1}(1-q_1^{w_1(v_i)}\cdots q_r^{w_r(v_i)}z)},$$

with v_i being the vertices of S^* and $h_{S^*}(\mathbf{q},z)\in\mathbb{N}[\mathbf{q},z]$.

Since $\mathcal{T}$ is $(w_1,\dots,w_r)$-compatible, the denominator is same for every $S\in T$, without loss of generality, we can denote it as $\prod_{i=1}^{d+1}(1-\mathbf{q}^{\alpha_i}z)$,

$$F^{q,w_1,\dots,w_r}_{M}(\mathbf{q},\mathbf{1},n) = \frac{\sum_{S\in\mathcal{T}} h_{S^*}(\mathbf{q},z)}{\prod_{i=1}^{d+1}(1-\mathbf{q}^{\alpha_i}z)}$$

.

To see that this rational form is reduced, we can simply degenerate $q=1$ and use the classical Ehrhart theory. □

3.2.2. Reciprocity for weighted Ehrhart series. Then we apply the monomial substitution technique to extend the reciprocity for q-weighted Ehrhart series.

Proof of Theorem 1.3.3. Apply Stanley's reciprocity theorem for rational cones [**20**] to the polytopal cone $\mathrm{Cone}(\mathcal{P})$ spanned by $\mathcal{P}$, which states that

$$\sum_{\mathbf{a}\in\mathrm{Cone}(\mathcal{P})\cap\mathbb{Z}^{d+1}} \left(t_1^{-1}\right)^{a_1}\cdots\left(t_d^{-1}\right)^{a_d}\left(t_{d+1}^{-1}\right)^{a_{d+1}} = (-1)^{d+1}\sum_{\mathbf{a}\in\mathrm{Cone}(\mathcal{P}^\circ)\cap\mathbb{Z}^{d+1}} t_1^{a_1}\cdots t_d^{a_d}t_{d+1}^{a_{d+1}},$$

and by Lemma 3.1.1, apply the appropriate monomial substitutions: $t_i\mapsto q_1^{v_{1,i}}\cdots q_r^{v_{r,i}}t_i$ for $i=1,\dots,d$ and $t_{d+1}\mapsto q_1^{v_{1,d+1}}\cdots q_r^{v_{r,d+1}}t_{d+1}x$

For $(a_1,\dots,a_d,n)\in\mathrm{Cone}(\mathcal{P})\cap\mathbb{Z}^{d+1}$, by the linearity of weights w_i's, we can see

$$\left(t_1^{-1}\right)^{a_1}\cdots\left(t_d^{-1}\right)^{a_d}\left(t_{d+1}^{-1}\right)^{a_{d+1}} \mapsto \left(q_1^{-1}\right)^{\mathbf{v}_1^\intercal\mathbf{a}}\cdots\left(q_r^{-1}\right)^{\mathbf{v}_r^\intercal\mathbf{a}}\left(t_1^{-1}\right)^{a_1}\cdots\left(t_d^{-1}\right)^{a_d}\left(t_{d+1}^{-1}\right)^{a_{d+1}}\left(x^{-1}\right)^{n}.$$

Similarly, for $\mathbf{a}\in\mathrm{Cone}(\mathcal{P}^\circ)\cap\mathbb{Z}^{d+1}$,

$$t_1^{a_1}\cdots t_d^{a_d}t_{d+1}^{a_{d+1}}\mapsto q_1^{\mathbf{v}_1^\intercal\mathbf{a}}\cdots q_r^{\mathbf{v}_r^\intercal\mathbf{a}}t_1^{a_1}\cdots t_d^{a_d}t_{d+1}^{a_{d+1}}x^{a_{d+1}}.$$

Lastly, we multiply the both sides by $q_1^{-b_1}\cdots q_r^{-b_r}$. □

COROLLARY 3.2.1. *$F_M^{r,w_1,\ldots,w_r}(\mathbf{q}^{-1},\mathbf{t}^{-1},x^{-1})$ can be represented by interior q-weighted multivariate Hilbert series.*

PROOF.

$$\begin{aligned}F_M^{r,w_1,\ldots,w_r}(\mathbf{q}^{-1},\mathbf{t}^{-1},x^{-1}) &= \sum_{I\subseteq[r]}\left(\frac{\prod_{j\in I}(-q_j^{-1})}{\prod_{i=1}^{r}(1-q_i^{-1})}\right)\left(F_M^{q,w_1,\ldots,w_r}(\mathbf{q}^{-1},\mathbf{t}^{-1},x^{-1})\big|_{q_k=1,k\notin I}\right)\\ &= (-1)^{d+1}\sum_{I\subseteq[r]}\left(\frac{\prod_{j\in I}(-q_j^{-1})}{\prod_{i=1}^{r}(1-q_i^{-1})}\right)\left(F_{M^\circ}^{q,w_1,\ldots,w_r}(\mathbf{q},\mathbf{t},x)\big|_{q_k=1,k\notin I}\right).\end{aligned}$$

□

Extending the reciprocity for s-weighted Ehrhart series is more complicated. We first prove a naive reciprocity lemma.

LEMMA 3.2.1 (Naive Weighted Reciprocity Lemma).

- *Assume the polytope $\mathcal{P}$ is simplicial, i.e., the vertices of $\mathcal{P}$, $\{\mathbf{v}_i\}$, form a basis of $\mathbb{R}^{d+1}$.*
- *Assume $h(\mathbf{a})$ is separable and multiplicative with respect to the basis $\{\mathbf{v}_i\}$, i.e., there exist univariate functions h_i such that $h(\mathbf{a}) = h_1(\alpha_1)\cdots h_{d+1}(\alpha_{d+1})$ with $\mathbf{a} = \alpha_1\mathbf{v}_1 + \cdots + \alpha_{d+1}\mathbf{v}_{d+1}$.*
- *Assume $h_i(x) = \sum_{j=1}^{k_i} P_{ij}(x)\gamma_{ij}^x$ where P_{ij}'s are polynomials and γ_{ij} are nonzero complex numbers.*

Under these assumptions,

(1) $F_M^{s,h}(\mathbf{t},x)$ and $F_{M^\circ}^{s,h}(\mathbf{t},x)$ are rational s-weighted multivariate Hilbert series,

(2) they satisfy the reciprocity relation,

$$F_M^{s,h}\left(\mathbf{t}^{-1},x^{-1}\right) = (-1)^{d+1}F_{M^\circ}^{s,h}(\mathbf{t},x).$$

PROOF. Denote $\lozenge = \{\sum c_i\mathbf{v}_i : 0 \le c_i < 1\}$ as the fundamental parallelepiped generated by $\{\mathbf{v}_i\}$. For any $\mathbf{a}\in \mathrm{Cone}(\mathcal{P})\cap\mathbb{Z}^{d+1}$, we can represent $\mathbf{a} = \alpha_1\mathbf{v}_1+\cdots+\alpha_{d+1}\mathbf{v}_{d+1}$. Decompose it into integer parts and fractional parts, we have

$$\mathbf{a} = \underbrace{\lfloor\alpha_1\rfloor\mathbf{v}_1+\cdots+\lfloor\alpha_{d+1}\rfloor\mathbf{v}_{d+1}}_{\text{integral parts}} + \underbrace{\{\alpha_1\}\mathbf{v}_1+\cdots+\{\alpha_{d+1}\}\mathbf{v}_{d+1}}_{\text{fractional parts}}.$$

For simplicity, we denote the fractional parts as $\mathbf{r} \in \Diamond$. Then

$$\begin{aligned}
F_M^{s,h}(\mathbf{t}, x) &= \sum_{\mathbf{a} \in \mathrm{Cone}\,(\mathcal{P}) \cap \mathbb{Z}^{d+1}} h(\mathbf{a}) \mathbf{t}^{\mathbf{a}} x^{\phi(\mathbf{a})} \\
&= \sum_{\alpha_1 \mathbf{v}_1 + \cdots + \alpha_{d+1} \mathbf{v}_{d+1} \in \mathrm{Cone}\,(\mathcal{P}) \cap \mathbb{Z}^{d+1}} \prod_{i=1}^{d+1} \left(h_i(\alpha_i) \mathbf{t}^{\lfloor \alpha_i \rfloor \mathbf{v}_i} x^{\phi(\lfloor \alpha_i \rfloor \mathbf{v}_i)} \right) \cdot \mathbf{t}^{\mathbf{r}} x^{\phi(\mathbf{r})} \\
&= \sum_{\mathbf{r} \in \Diamond} \sum_{n_1=0}^{\infty} \cdots \sum_{n_{d+1}=0}^{\infty} \prod_{i=1}^{d+1} \left(h_i(n_i + \{\alpha_i\}) \mathbf{t}^{n_i \mathbf{v}_i} x^{\phi(n_i \mathbf{v}_i)} \right) \cdot \mathbf{t}^{\mathbf{r}} x^{\phi(\mathbf{r})} \\
&= \sum_{\mathbf{r} \in \Diamond} \prod_{i=1}^{d+1} \left(\sum_{n_i=0}^{\infty} h_i(n_i + \{\alpha_i\}) \mathbf{t}^{n_i \mathbf{v}_i} x^{\phi(n_i \mathbf{v}_i)} \right) \cdot \mathbf{t}^{\mathbf{r}} x^{\phi(\mathbf{r})}.
\end{aligned}$$

Similarly, for any $\mathbf{a} \in \mathrm{Cone}\,(\mathcal{P}^\circ) \cap \mathbb{Z}^{d+1}$, we can decompose it into

$$\mathbf{a} = \underbrace{\lceil \alpha_1 \rceil \mathbf{v}_1 + \cdots + \lceil \alpha_{d+1} \rceil \mathbf{v}_{d+1}}_{\text{integral parts}} - \underbrace{(\{-\alpha_1\} \mathbf{v}_1 + \cdots + \{-\alpha_{d+1}\} \mathbf{v}_{d+1})}_{\text{fractional parts}}.$$

Denote the fractional part as $\mathbf{r} \in \Diamond$. Then

$$\begin{aligned}
F_{M^\circ}^{s,h}(\mathbf{t}, x) &= \sum_{\mathbf{a} \in \mathrm{Cone}\,(\mathcal{P}^\circ) \cap \mathbb{Z}^{d+1}} h(-\mathbf{a}) \mathbf{t}^{\mathbf{a}} x^{\phi(\mathbf{a})} \\
&= \sum_{\alpha_1 \mathbf{v}_1 + \cdots + \alpha_{d+1} \mathbf{v}_{d+1} \in \mathrm{Cone}\,(\mathcal{P}^\circ) \cap \mathbb{Z}^{d+1}} \prod_{i=1}^{d+1} \left(h_i(-\alpha_i) \mathbf{t}^{\lceil \alpha_i \rceil \mathbf{v}_i} x^{\phi(\lceil \alpha_i \rceil \mathbf{v}_i)} \right) \cdot \mathbf{t}^{-\mathbf{r}} x^{\phi(-\mathbf{r})} \\
&= \sum_{\mathbf{r} \in \Diamond} \sum_{n_1=1}^{\infty} \cdots \sum_{n_{d+1}=1}^{\infty} \prod_{i=1}^{d+1} \left(h_i(-n_i + \{-\alpha_i\}) \mathbf{t}^{n_i \mathbf{v}_i} x^{\phi(n_i \mathbf{v}_i)} \right) \cdot \mathbf{t}^{-\mathbf{r}} x^{\phi(-\mathbf{r})} \\
&= \sum_{\mathbf{r} \in \Diamond} \prod_{i=1}^{d+1} \left(\sum_{n_i=1}^{\infty} h_i(-n_i + \{-\alpha_i\}) \mathbf{t}^{n_i \mathbf{v}_i} x^{\phi(n_i \mathbf{v}_i)} \right) \cdot \mathbf{t}^{-\mathbf{r}} x^{\phi(-\mathbf{r})}.
\end{aligned}$$

According to the assumption and the equivalent characterizations of rational power series. For each $\mathbf{r} \in \Diamond$, generating functions

$$G_i(\mathbf{t}, x) := \sum_{n_i=0}^{\infty} h_i(n_i + \{\alpha_i\}) \mathbf{t}^{n_i \mathbf{v}_i} x^{\phi(n_i \mathbf{v}_i)} \quad \text{and} \quad \overline{G_i}(\mathbf{t}, x) := \sum_{n_i=1}^{\infty} h_i(-n_i + \{-\alpha_i\}) \mathbf{t}^{n_i \mathbf{v}_i} x^{\phi(n_i \mathbf{v}_i)}$$

are rational. In particular, they satisfy the reciprocity property, i.e.,

$$G_i(\mathbf{t}^{-1}, x^{-1}) = (-1)\overline{G_i}(\mathbf{t}, x).$$

Therefore,

$$F_M^{s,h}(\mathbf{t}^{-1},x^{-1}) = \sum_{\mathbf{r}\in\lozenge}\prod_{i=1}^{d+1}\left(G_i(\mathbf{t}^{-1},x^{-1})\right)\cdot(\mathbf{t}^{-1})^{\mathbf{r}}(x^{-1})^{\phi(-\mathbf{r})}$$
$$= \sum_{\mathbf{r}\in\lozenge}\prod_{i=1}^{d+1}\left((-1)\overline{G_i}(\mathbf{t},x)\right)\cdot\mathbf{t}^{-\mathbf{r}}x^{\phi(-\mathbf{r})} = (-1)^{d+1}F_{M^\circ}^{s,h}(\mathbf{t},x).$$

□

Then we show that polynomials can be decomposed nicely.

Lemma 3.2.2. *Assume* $h(\mathbf{a}) = \prod_{i=1}^{d+1}\sum_{j=1}^{k_i}P_{ij}(a_i)\gamma_{ij}^{a_i}$ *with* P_{ij}*'s are polynomials and* γ_{ij} *are nonzero complex numbers, then* $h(\mathbf{a})$ *can be decomposed into finite terms where each term is separable and multiplicative with respect to any basis* $\{\mathbf{v}_i\}$ *of* $\mathbb{R}^{d+1}$*.*

Proof. Only need to prove when $h(\mathbf{a}) = a_1^{m_1}\cdots a_{d+1}^{m_{d+1}}\cdot\gamma_1^{a_1}\cdots\gamma_{d+1}^{a_{d+1}}$. Suppose $\mathbf{a} = \alpha_1\mathbf{v}_1+\cdots+\alpha_{d+1}\mathbf{v}_{d+1}$, then we can simply replace a_i by $\alpha_1 v_{1,i}+\cdots+\alpha_{d+1}v_{d+1,i}$ to get

$$h(\mathbf{a}) = \prod_{i=1}^{d+1}(\alpha_1 v_{1,i}+\cdots+\alpha_{d+1}v_{d+1,i})^{m_i}\cdot\prod_{i=1}^{d+1}\gamma_i^{\alpha_1 v_{1,i}+\cdots+\alpha_{d+1}v_{d+1,i}}$$
$$= \sum_{\mathbf{k}}c_{\mathbf{k}}\alpha^{\mathbf{k}}\prod_{i=1}^{d+1}(\gamma_i^{v_{1,1}}\cdots\gamma_d^{v_{i,d+1}})^{\alpha_i}$$
$$= \sum_{\mathbf{k}}c_{\mathbf{k}}\alpha^{\mathbf{k}}\prod_{i=1}^{d+1}\hat{\gamma}_i^{\alpha_1}.$$

□

Lastly, we can extend the reciprocity by directly applying the above two lemma.

Proof of Theorem 1.3.4. Using the fact that every rational polyhedral cone can be triangulated into simplicial cones and the Inclusion-exclusion principle, we can assume that C is simplicial. Then apply Lemma 3.2.2, we can assume $h(\mathbf{a})$ is separable and multiplicative with respect to a basis. Finally, we can apply Naive Weighted Reciprocity Lemma 3.2.1. □

CHAPTER 4

s-weighted Ehrhart Theory

This chapter discusses further the s-weighted Ehrhart functions. We construct a weight-lifting polytope and we can use it to evaluate the s-weighted Ehrhart function as a classical Ehrhart function. We present a few applications and computational experiments via the method of constructing the weight-lifting polytopes.

4.1. Proofs of Theorem 1.4.1 and other results

Here we present proofs of Theorem 1.4.1 and some variations of it.

Proof of Theorem 1.4.1. Note that there is a natural projection map $\pi : \mathcal{P}^* \to \mathcal{P}$ via $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x}$. It suffices to show that for any fixed $\mathbf{x} \in \mathcal{P} \cap \mathbb{Z}^d$, $w(\mathbf{x}) = |\pi^{-1}(\mathbf{x}) \cap \mathbb{Z}^{d+e}|$. Recall that $(\mathbf{x}, \mathbf{y}) \in \pi^{-1}(\mathbf{x})$ if and only if $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{C}\mathbf{y} = \sum_{i=1}^{d} x_i \mathbf{b_i} + \mathbf{e}$ where $\mathbf{x} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0}$. Given $\mathbf{x} \in \mathcal{P} \cap \mathbb{Z}^d$, we see that $(\mathbf{x}, \mathbf{y}) \in \pi^{-1}(\mathbf{x}) \cap \mathbb{Z}^{d+e}$ if and only if $\mathbf{y} \in \mathcal{Q}(x_1, \ldots, x_d) \cap \mathbb{Z}^e$. Hence, for a fixed $\mathbf{x} \in \mathcal{P} \cap \mathbb{Z}^d$, $|\pi^{-1}(\mathbf{x}) \cap \mathbb{Z}^{d+e}| = |\mathcal{Q}(\mathbf{x}) \cap \mathbb{Z}^e| = w(\mathbf{x})$.

We now consider the second part of Theorem 1.4.1. We show that $\mathcal{P}^*$ is parametric with respect to $\mathbf{b}$ in the following sense. If $\mathcal{P} = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$, then

$$\mathcal{P}^* = \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \middle| \mathbf{A}^* \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ -\mathbf{e} \end{pmatrix}, \mathbf{x} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0} \right\}.$$

Therefore, $n\mathcal{P} = \{\mathbf{x} : \mathbf{A}\mathbf{x} = n\mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ and

$$(n\mathcal{P})^* = \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \middle| \mathbf{A}^* \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} n\mathbf{b} \\ -\mathbf{e} \end{pmatrix}, \mathbf{x} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0} \right\}.$$

Given $\mathbf{e} = 0$, we can see that $(n\mathcal{P})^* = n(\mathcal{P}^*)$. By the first part of the proof we conclude,

$$E_{\mathcal{P}}^{s,w}(n) = |(n\mathcal{P})^* \cap \mathbb{Z}^{d+e}| = |n(\mathcal{P}^*) \cap \mathbb{Z}^{d+e}| = E_{\mathcal{P}^*}(n). \qquad \square$$

Now we outline more results and corollaries of Theorem 1.4.1. From now on we deal with the most general *quasi-polynomial weighted* case, i.e., $w(\mathbf{x})$ is a non-constant quasi-polynomial as in the statement of Theorem 1.4.1.

EXAMPLE 4.1.1. Consider the $(m-1)$-dimensional standard simplex

$$\Delta_{m-1} = \{\mathbf{y} \mid y_1 + \cdots + y_m = 1, y_i \geq 0\}.$$

Then $-2\Delta_{m-1} = \{\mathbf{y} \mid y_1 + \cdots + y_m = -2 \cdot 1, y_i \geq 0\}$.

DEFINITION 4.1.1. *A function $w(t)$ is a* ***late-dilated*** *Ehrhart quasi-polynomial if*

$$w(t) = |(t-c)\mathcal{Q} \cap \mathbb{Z}^e|,$$

where $c \in \mathbb{Z}$ and $\mathcal{Q}$ is a rational polytope.

EXAMPLE 4.1.2. The function $\binom{t}{m-1}$ is a late-dilated Ehrhart polynomial in the variable t, because $\binom{t}{m-1} = |(t-m+1)\Delta_{m-1} \cap \mathbb{Z}^m|$.

COROLLARY 4.1.1. *Let $w_1, w_2, \cdots, w_d$ be late-dilated Ehrhart quasi-polynomials, i.e., $w_i(t) = |(t-c_i)\mathcal{Q}_i \cap \mathbb{Z}^{e_i}|$ where $\mathcal{Q}_i = \{\mathbf{y_i} \mid \mathbf{C_i}\mathbf{y_i} = \mathbf{d_i}, \mathbf{y_i} \geq \mathbf{0}\}$ and $\mathbf{C_i} \in \mathbb{Z}^{r_i \times e_i}, \mathbf{d_i} \in \mathbb{Z}^{r_i}, c_i \in \mathbb{Z}$. Consider a rational polytope of the form $\mathcal{P} = \{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\} \subseteq \mathbb{R}^d$ where $\mathbf{A} \in \mathbb{Z}^{s \times d}, \mathbf{b} \in \mathbb{Z}^s$ and the multivariate function $w(\mathbf{x}) = \prod_{i=1}^d w_i(x_i)$. There exists a* weight-lifting polytope *$\mathcal{P}^* \subseteq \mathbb{R}^{d^*}$ of $\mathcal{P}$, where $d^* = d + e_1 + \cdots + e_d$, such that*

$$\sum_{\mathbf{x} \in \mathcal{P} \cap \mathbb{Z}^d} w(\mathbf{x}) = \left| \mathcal{P}^* \cap \mathbb{Z}^{d^*} \right|.$$

PROOF. We need only show that there is a rational polytope $Q(x_1, \ldots, x_d)$ of the form given in Theorem 1.4.1 for which $w(\mathbf{x}) = |\mathcal{Q}(x_1, \ldots, x_d) \cap \mathbb{Z}^{e_1 + \cdots + e_d}|$ and then apply Theorem 1.4.1. Let $\mathcal{Q}(x_1, \ldots, x_d) = \prod_{i=1}^d (x_i - c_i)\mathcal{Q}_i$. Specifically, $\mathcal{Q}(x_1, \ldots, x_d)$ has the form

$$\left\{ \begin{pmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_d \end{pmatrix} \middle| \begin{pmatrix} \mathbf{C}_1 & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{C}_d \end{pmatrix} \begin{pmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_d \end{pmatrix} = x_1 \begin{pmatrix} \mathbf{d}_1 \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix} + \cdots + x_d \begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{d}_d \end{pmatrix} + \mathbf{e}, \mathbf{y} \geq \mathbf{0} \right\}. \qquad \square$$

COROLLARY 4.1.2. *For every monomial $w(\mathbf{x}) = \mathbf{x}^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_d^{\alpha_d}$, there exists a weight-lifting polytope $\mathcal{P}^* \subseteq \mathbb{R}^{d^*}$ where $d^* = d + 2|\alpha| = d + 2\sum_{i=1}^d \alpha_i$ such that*

$$\sum_{\mathbf{x}\in\mathcal{P}\cap\mathbb{Z}^d} w(\mathbf{x}) = \left|\mathcal{P}^* \cap \mathbb{Z}^{d^*}\right|.$$

PROOF. By Corollary 4.1.1, we just need to show that $x_i^{\alpha_i}$ is a late-dilated Ehrhart polynomial. It is well known that $(k+1)^{\alpha_i}$ is the Ehrhart polynomial of the α_i-dimensional hypercube of length k. In particular, the hypercube has the form

$$n\mathcal{Q}_i = \left\{ \begin{pmatrix} y_1 \\ \vdots \\ y_{\alpha_i} \\ z_1 \\ \vdots \\ z_{\alpha_i} \end{pmatrix} \middle| \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_{\alpha_i} \\ z_1 \\ \vdots \\ z_{\alpha_i} \end{pmatrix} = n \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, y_i \geq 0, z_i \geq 0 \right\}. \qquad \square$$

COROLLARY 4.1.3. *For every polynomial $w(\mathbf{x}) = \sum_{\alpha\in I} c_\alpha \mathbf{x}^\alpha = \sum_{\alpha\in I} c_\alpha x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$, there exist $|I|$ weight-lifting polytopes $\mathcal{P}^*_\alpha$ indexed by the exponents of monomials such that*

$$\sum_{\mathbf{x}\in\mathcal{P}\cap\mathbb{Z}^d} w(x) = \sum_{\alpha\in I} c_\alpha \left|\mathcal{P}^*_\alpha \cap \mathbb{Z}^{d^*}\right|.$$

PROOF. This follows directly from Corollary 4.1.2 $\square$

REMARK 4.1.1. Corollary 4.1.3 implies that if $w(\mathbf{x})$ is a polynomial with $|I|$ nonzero monomials, then we can compute the sum of lattice points of $\mathcal{P}$ weighted by w by counting integral points in $|I|$ weight-lifting polytopes.

We give another two corollaries of Theorem 1.4.1.

COROLLARY 4.1.4. *Consider the polynomial $w(\mathbf{x}) = \prod_{i=1}^n \binom{x_i+\alpha_i-1}{\alpha_i-1}$. There exists a weight-lifting polytope $\mathcal{P}^* \subseteq \mathbb{R}^{d^*}$ where $d^* = d + |\alpha|$ such that*

$$\sum_{\mathbf{x}\in\mathcal{P}\cap\mathbb{Z}^d} w(\mathbf{x}) = \left|\mathcal{P}^* \cap \mathbb{Z}^{d^*}\right|.$$

Proof. Recall that $\binom{x_i+\alpha_i-1}{\alpha_i-1}$ is the Ehrhart polynomial of the standard (α_i-1)-simplex $1 = y_1+\cdots+y_{\alpha_i}$ with $y_i \geq 0$. In particular, the simplex has the form

$$n\mathcal{Q}_i = \left\{ \begin{pmatrix} y_1 \\ \vdots \\ y_{\alpha_i} \end{pmatrix} \middle| \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_{\alpha_i} \end{pmatrix} = n \cdot 1, y_i \geq 0 \right\}.$$

Applying Corollary 4.1.1 gives the weight-lifting polytope from the statement. □

Corollary 4.1.5. *Consider the polynomial* $w(\mathbf{x}) = \prod_{i=1}^{d} \binom{x_i}{\alpha_i-1}$. *There exists a weight-lifting polytope* $\mathcal{P}^*$ *of the dimension* $d^* = d+|\alpha|$ *such that* $\sum_{\mathbf{x}\in\mathcal{P}\cap\mathbb{Z}^d} w(\mathbf{x}) = \left|\mathcal{P}^* \cap \mathbb{Z}^{d^*}\right|$.

Proof. The function $\binom{x_i}{\alpha_i-1}$ is a late-dilated Ehrhart polynomial because $\binom{x_i+\alpha_i-1}{\alpha_i-1}$ is the Ehrhart polynomial of the standard (α_i-1)-simplex. Applying Corollary 4.1.1 gives the weight-lifting polytope from the statement. □

Note that $\{\binom{x+k-1}{k-1} \mid k=1,2,\ldots\}$ and $\{\binom{x}{k-1} \mid k=1,2,\ldots\}$ are two well-known bases of the vector space of polynomials in x.

Corollary 4.1.6. *For every monomial* $w(\mathbf{x}) = \mathbf{x}^\alpha = x_1^{\alpha_1}\cdots x_d^{\alpha_d}$, *there exist at most* $(\alpha_1+1)\cdots(\alpha_d+1)$ *weight-lifting polytopes* $\mathcal{P}^*_\beta$ *indexed by the vector* β *and* $\mathcal{P}^*_\beta \subset \mathbb{R}^{d^*}$ *where* $d^* = d+|\beta|$ *such that*

$$\sum_{\mathbf{x}\in\mathcal{P}\cap\mathbb{Z}^d} w(\mathbf{x}) = \sum_{\beta\leq\alpha} c_\beta \left|\mathcal{P}^*_\beta \cap \mathbb{Z}^{d^*}\right|.$$

Proof. Let $v_k(x)$ be one of the two binomial bases described above. We can transform the monomial basis $\{x^k \mid k=0,1,2,\ldots\}$ into the binomial basis,

$$x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_d^{\alpha_d} = \sum_{\beta\leq\alpha} c(\alpha,\beta)\cdot v_{\beta_1}(x_1)v_{\beta_2}(x_2)\cdots v_{\beta_d}(x_d).$$

By Corollaries 4.1.4 and 4.1.5, for each β and each polynomial $v_{\beta_1}(x_1)v_{\beta_2}(x_2)\cdots v_{\beta_d}(x_d)$, there exists a corresponding weight-lifting polytope $\mathcal{P}^*_\beta \subset \mathbb{R}^{d+|\beta|}$. □

Remark 4.1.2. In Corollary 4.1.2 we express the weighted sum of lattice points of $\mathcal{P}$ using a single $\mathcal{P}^* \subset \mathbb{R}^{d+2|\alpha|}$, but in Corollary 4.1.6 we express this sum using at most $(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_d+1)$ polytopes of lower dimension $\mathcal{P}^*_\beta \subset \mathbb{R}^{d+|\beta|}$.

Proof of Corollary 1.4.1. Applying Theorem 1.4.1 to $\mathcal{P}$ gives a weight-lifting polytope $\mathcal{P}^*$ for which $E_{\mathcal{P}}^{s,w}(n) = E_{\mathcal{P}^*}(n)$. Applying a classical result relating the volume and lead coefficient of the Ehrhart quasi-polynomial of $\mathcal{P}^*$ completes the proof. Both $E_{\mathcal{P}}^{s,w}(n)$ and $E_{\mathcal{P}^*}(n)$ are quasi-polynomial functions of n, and concretely, this equality implies that their leading coefficients are the same.

We can then replace integration of $w(\mathbf{x})$ over $\mathcal{P}$ with computation of the leading coefficient of $E_{\mathcal{P}^*}(n)$, which is equivalent to computing the volume of $\mathcal{P}^*$. Note that this transformation can be carried out in several steps whose complexity is polynomial in the size of the input describing $\mathcal{P}^*$.

For the second claim, we start by recalling an elementary fact. Let $S = \{s_1, \ldots, s_r\}$ be a set of non-negative real numbers. Then $\max\{s_i \mid s_i \in S\} = \lim_{k\to\infty} \sqrt[k]{\sum_{j=1}^{r} s_j^k}$. The arithmetic mean of S is at most its maximum value, which in turn is at most as big as $\sum_i s_i$. We apply these ideas to the set $S = \{w(\alpha) \mid \alpha \in \mathcal{P} \cap \mathbb{Z}^d\}$. This gives upper and lower bounds for each positive integer k:

$$L_k = \sqrt[k]{\frac{\sum\limits_{\alpha\in\mathcal{P}\cap\mathbb{Z}^d} w(\alpha)^k}{|\mathcal{P}\cap\mathbb{Z}^d|}} \leq \max\{w(\alpha) : \alpha \in \mathcal{P}\cap\mathbb{Z}^d\} \leq \sqrt[k]{\sum_{\alpha\in\mathcal{P}\cap\mathbb{Z}^d} w(\alpha)^k} = U_k.$$

As $k \to \infty$, L_k and U_k approach this maximum value monotonically (from below and above, respectively). Trivially, if the difference between the (rounded) upper and lower bounds becomes strictly less than 1, we have determined $\max\{w(\mathbf{x}) \mid \mathbf{x} \in \mathcal{P}\cap\mathbb{Z}^d\} = \lceil L_k \rceil$. Thus the process terminates with the correct value. Finally, the key value in the sequences L_k and U_k is the term $E_{\mathcal{P}}^{s,w^k}(n) = \sum\limits_{\alpha\in n\mathcal{P}\cap\mathbb{Z}^d} w(\alpha)^k$. Corollary 4.1.1 describes how to construct the weight-lifting polytope $\mathcal{P}^*$ corresponding to the pair $\mathcal{P}$ and $w(\alpha)^k$. $\square$

4.2. Applications

Theorem 1.4.1 has applications beyond integration and maximization of Ehrhart quasi-polynomials. In this section, we discuss how to use it to find new algebraic combinatorial identities by carefully choosing the polytope $\mathcal{P}$ and reinterpreting the weight function w in terms of Ehrhart quasi-polynomials of some polytopes $\mathcal{Q}_i$.

4.2.1. Weighted Ehrhart in number theory. Simultaneous Core Partitions. We first describe an area in which weighted Ehrhart machinery has already been applied to prove a significant result. Let λ be a partition and $\mathcal{H}(\lambda)$ denote its multiset of hook lengths. The partition λ is called an *a-core partition* if no element of $\mathcal{H}(\lambda)$ is divisible by a. If λ is both an a-core partition and a b-core partition, then we say that it is an (a, b)-core partition. There is extensive literature about statistical properties of sizes of simultaneous core partitions [**32**, **68**]. Anderson proved that if a and b are relatively prime positive integers then the number of (a, b)-core partitions is $\frac{1}{a+b}\binom{a+b}{a}$ [**4**]. Johnson proved a conjecture of Armstrong, showing that the average size of an (a, b)-core partition is $(a + b + 1)(a - 1)(b - 1)/24$ [**53**]. Johnson's proof fits into the framework of weighted Ehrhart theory.

Suppose that a and b are relatively prime positive integers. It is not hard to show that a-core partitions are in bijection with elements of $\Lambda_a = \{(c_0, \dots, c_{a-1}) \in \mathbb{Z}^a \colon \sum_i c_i = 0\}$. Let $r_a(x)$ be the remainder when x is divided by a. We use cyclic indexing for elements $\mathbf{c} \in \Lambda_a$, that is, for $k \in \mathbb{Z}$ we set $c_k = c_{r_a(k)}$. Simultaneous (a, b)-core partitions are in bijection with the elements of Λ_a satisfying the inequalities $c_{i+b} - c_i \leq \left\lfloor \frac{b+i}{a} \right\rfloor$ for each $i \in \{0, 1, \dots, a-1\}$ [**53**, Lemma 23]. In this way, we see that (a, b)-core partitions are in bijection with integer points in a rational polytope $\mathrm{SC}_a(b)$. The size of the a-core partition corresponding to $\mathbf{c} = (c_0, \dots, c_{a-1})$ is $h_a(\mathbf{c}) = \frac{a}{2} \sum_{i=0}^{a-1} \left(c_i^2 + ic_i\right)$ [**53**, Theorem 22]. Therefore, Anderson's theorem is equivalent to computing the number of integer points in $\mathrm{SC}_a(b)$, and Johnson's theorem is equivalent to computing $\sum_{\mathbf{c} \in \mathrm{SC}_a(b)} h_a(\mathbf{c})$.

Johnson computes this weighted sum of lattice points by relating it to a sum over the subset of integer points $(z_0, \dots, z_{a-1})$ of the dilation of the standard simplex $b\Delta_{a-1}$ that satisfy $\sum iz_i \equiv 0 \pmod{a}$. Johnson then shows that the sum he needs to compute is equal to $1/a$ times the sum of a quadratic function w taken over all integer points of $b\Delta_{a-1}$. In order to conclude, he applies a result from Euler–Maclaurin theory, which is a version of the first part of Corollary 1.4.1, and also applies a version of weighted Ehrhart reciprocity that appears in [**7**].

By Corollary 4.1.3, there exists a family of weight-lifting polytopes $\mathcal{P}^*_\alpha \subset \mathbb{R}^{d^*}$ such that

$$\sum_{x \in b\triangle_{a-1} \cap \mathbb{Z}^a} w(x) = \sum_{\alpha \in I} c_\alpha \left| \mathcal{P}^*_\alpha \cap \mathbb{Z}^{d^*} \right|.$$

It seems likely that further study of these kinds of weight-lifting polytopes can lead to new techniques in the study of simultaneous core partitions.

Numerical Semigroups. A numerical semigroup S is an additive submonoid of $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ with finite complement. The elements of $\mathbb{N}_0 \setminus S$ are the *gaps* of S, denoted $G(S) = \{h_1, \ldots, h_g\}$. The *weight* of S is defined by $w(S) = (h_1 + \cdots + h_g) - (1 + 2 + \cdots + g)$. The motivation for studying $w(S)$ comes from the theory of Weierstrass semigroups of algebraic curves [**6**, Chapter 1, Appendix E].

Numerical semigroups containing m are in bijection with integer points $(x_1, \ldots, x_{m-1})$ in the *Kunz polyhedron* $\mathcal{P}_m \subset \mathbb{R}^{m-1}$, which is defined via bounding inequalities

$$x_i + x_j \geq x_{i+j} \text{ if } i + j < m, \quad x_i + x_j + 1 \geq x_{i+j-m} \text{ if } i + j > m.$$

Let $NS(m, g)$ be the set of numerical semigroups containing m with genus g. These semigroups are in bijection with the integer points of $\mathcal{P}_{m,g}$, the polytope we get from $\mathcal{P}_m$ by adding the additional constraint $\sum x_i = g$. For a more extensive discussion of the connection between numerical semigroups containing m and integer points in the Kunz polyhedron, see [**56**, Section 4]. If $(k_1, \ldots, k_{m-1})$ corresponds to a semigroup S, then $w(S) = \frac{m}{2} \sum_{i=1}^{m-1} k_i(k_i - 1) + \sum_{i=1}^{m-1} ik_i - \frac{1}{2}\left(\sum_{i=1}^{m-1} k_i\right)\left(1 + \sum_{i=1}^{m-1} k_i\right)$. There has been recent interest in the statistical properties of weights of semigroups, see [**58**, Section 5] and [**57**].

By Corollary 4.1.3, there exists a family of weight-lifting polytopes $\mathcal{P}^*_\alpha \subset \mathbb{R}^{d^*}$ such that

$$\sum_{S \in NS(g,m)} w(S) = \sum_{S \in \mathcal{P}_{m,g} \cap \mathbb{Z}^{m-1}} w(S) = \sum_{\alpha \in I} c_\alpha \left| \mathcal{P}^*_\alpha \cap \mathbb{Z}^{d^*} \right|.$$

Studying this family of polytopes and applying a version of Corollary 1.4.1 suggests an approach to the following two questions:

1. What is the maximum of $w(S)$ for $S \in NS(g, m)$?
2. For fixed m, what is the main term in the expression for $\sum_{S \in NS(g,m)} w(S)$ as $g \to \infty$?

4.2.2. Weighted Ehrhart in combinatorial representation theory. There is a long tradition of using lattice points of polytopes in representation theory (see [**40**] and the references there). Here, as an application of Theorem 1.4.1, we provide new connections.

Maximizing Kostka numbers. Fix a partition $\lambda \vdash n$ and let $SSYT(\lambda)$ denote the set of semi-standard Young tableaux of shape λ. The Schur function s_λ is

$$s_\lambda(x) = \sum_{T \in SSYT(\lambda)} x^T = \sum_{\alpha \in \text{comp}(n)} K_{\lambda\alpha} x^\alpha,$$

where comp(n) is the set of weak compositions n and $K_{\lambda\alpha}$ is the *Kostka number* that counts the number of tableaux in $SSYT(\lambda)$ with content α. Evaluating s_λ at $x_1 = 1, x_2 = 1, \ldots, x_N = 1, x_{N+1} = 0, x_{N+2} = 0, \ldots$ yields

$$|SSYT(\lambda, N)| = \sum_{\alpha \in N\text{-comp}(n)} K_{\lambda\alpha},$$

where $SSYT(\lambda, N)$ is the set of semi-standard Young tableaux of shape λ and entries bounded by N and N-comp(n) is the set of weak composition of n with N parts.

A weak composition of n with N parts is a lattice point in the scaled standard $(N-1)$-simplex $n\triangle_{N-1}$. The Kostka number $K_{\lambda\alpha}$ equals the number of lattice points in the Gelfand–Tsetlin polytope $GT(\lambda, \alpha)$ (see e.g., [**40**]), so $w(\alpha) = K_{\lambda\alpha}$ is a weight function. There have been contributions to understanding the behavior of $K_{\lambda\alpha}$ as (λ, α) vary and an example is [**52**] in which it is shown that they are log-concave. Applying the method in Corollary 1.4.1 one can use the weight-lifting polytope given by Theorem 1.4.1 to compute $max_{\alpha \in N\text{-comp(n)}} K_{\lambda\alpha}$.

Robinson–Schensted–Knuth (RSK) identity. Fix partitions $\mu, \nu \vdash n$ and recall the famous *RSK identity* (for details see e.g., [**70**]):

$$\sum_{\lambda \vdash n} K_{\lambda\mu} K_{\lambda\nu} = N_{\mu,\nu}.$$

The left sum is over partitions of n and the summands are products of Kostka numbers. In fact, the left side of the identity is a weighted sum over the lattice points of

$$P = \{\mathbf{x} \in \mathbb{R}^n \mid x_1 + \cdots + x_n = n, x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}. \tag{4.1}$$

This is because the weight function $w(\lambda) = K_{\lambda\mu} K_{\lambda\nu}$ is the number of lattice points in the Cartesian product $GT(\lambda, \mu) \times GT(\lambda, \nu)$ of two Gelfand–Tsetlin polytopes. The right-hand side of RSK, $N_{\mu\nu}$, is the number of lattice points in the transportation polytope

$$Mat_{n,n}(\mu,\nu) = \left\{ (z_{ij})_{1\le i,j\le n} \mid \sum_j z_{ij} = \mu_i, \sum_i z_{ij} = \nu_j, z_{ij} \ge 0 \right\}.$$

While RSK provides more information (e.g., a bijection), Theorem 1.4.1 gives a new polytope whose number of lattice points is the sum $\sum_{\lambda\vdash n} K_{\lambda\mu}K_{\lambda\nu}$.

Corollary 4.2.1 (A new RSK-like identity). *There exists a weight-lifting polytope $\mathcal{P}^*(\mu,\nu) \subseteq \mathbb{R}^{n^2+2n}$ which is combinatorially different from $Mat_{n,n}(\mu,\nu)$ such that*

$$\sum_{\lambda\vdash n} K_{\lambda\mu}K_{\lambda\nu} = |\mathcal{P}^*(\mu,\nu) \cap \mathbb{Z}^{n^2+2n}|.$$

Littlewood–Richardson Coefficients. Schur functions are central objects in representation theory and combinatorics. The skew Schur function for partitions $\lambda,\mu \vdash n$ is

$$s_{\lambda/\mu}(x) = \sum_{\alpha\in \mathrm{comp}(n)} K_{\lambda/\nu,\alpha} x^\alpha,$$

where the sum is over all compositions of n and $K_{\lambda/\nu,\alpha}$ counts the number of skew semi-standard Young tableaux of shape λ/ν and weight α. The Littlewood–Richardson rule (see e.g., [**74**]) expresses the skew Schur functions in terms of Schur functions,

$$s_{\lambda/\mu}(x) = \sum_{\nu\vdash n} c^\lambda_{\mu\nu} s_\nu(x).$$

Comparing the expression of the coefficient of the monomial x^α yields

$$K_{\lambda/\nu,\alpha} = \sum_{\nu\vdash n} c^\lambda_{\mu\nu} K_{\nu\alpha}.$$

The Littlewood–Richardson coefficient $c^\lambda_{\mu\nu}$ counts the number of lattice points in the *hive polytope* $H^\lambda_{\mu\nu}$ (see e.g., [**28**]). Applying Theorem 1.4.1 to the simplex in (4.1) and $w(\nu) = c^\lambda_{\mu\nu}K_{\nu\alpha}$, which counts the number of lattice points in $H^\lambda_{\mu\nu} \times GT(\nu,\alpha)$, we obtain the following corollary.

Corollary 4.2.2. *There exists a weight-lifting polytope $\mathcal{P}^*(\lambda/\mu,\alpha) \subseteq \mathbb{R}^{n^2+2n}$ such that*

$$\sum_{\lambda\vdash n} c^\lambda_{\mu\nu} K_{\nu\alpha} = |\mathcal{P}^*(\lambda/\mu,\alpha) \cap \mathbb{Z}^{n^2+2n}|.$$

4.3. Experiments

4.3.1. Integration over polytopes. We present an experiment related to symbolic computing. Let $\mathcal{P}$ be a d-dimensional rational convex polyhedron inside $\mathbb{R}^n$ and let $w \in \mathbb{Q}[x_1, \dots, x_n]$ be a (homogeneous) polynomial with rational coefficients. We consider the problem of efficiently computing the *exact* value of the integral of the polynomial w over $\mathcal{P}$, denoted $\int_{\mathcal{P}} w\mathrm{d}m$, where $\mathrm{d}m$ is the *integral Lebesgue measure* on the affine hull of the polytope $\mathcal{P}$. For rational input, the output will always be a rational number $\int_{\mathcal{P}} f\mathrm{d}m$ (Integration over polytopes was studied extensively in [**16**], [**17**] and more recently at [**8**, **36**].).

Integration over polytopes is in general an important but difficult problem. (see [**8**, **49**] and the references therein). Our contribution starts from an old observation: It is known the computation of the leading coefficient of $E_{\mathcal{P}}^{s,w}(n)$ is the same as computing the integral of w over the polytope $\mathcal{P}$ (see [**8**]). But the new Theorem 1.4.1 provides a new avenue of compute that leading coefficient because $E_{\mathcal{P}}^{s,w}(n) = E_{\mathcal{P}^*}(n)$. We can replace integration with computation of the leading coefficient of a usual Ehrhart leading coefficient or direct volume computation.

In the papers [**36**] the authors released an integration software implemented in LattE (code is available from `https://www.math.ucdavis.edu/~latte/`) several fast algorithms for integration. They depend on two main facts. The first key fact is integrals of arbitrary *powers of linear forms* can be computed in polynomial time Therefore, to integrate an input polynomial LattE decomposes it into a finite sum of *powers of linear forms*, $\sum_{\ell} c_{\ell} \langle \ell, x \rangle^M$, using the well-known identity shown by Waring's theorem. The second fact is we can triangulate any polytope and just do the integral over simplices, then add the pieces. For all details of the LattE Integration algorithms and implementation see [**2**, **8**, **16**, **17**, **36**].

We implemented our new algorithm in SAGE, we call it the *WLPvolume* algorithm. We used Theorem 1.4.1, to find the weight-lifting polytope and then we computed its volume using the existing LattE code. We compared this with LattE *Integration* as implemented in [**36**] and available in the latest LattE release. Table 4.1, Table 4.2 present a comparison of the LattE Integration method and the WLPvolume method. The columns represent the dimension of the polytope and the rows mean the degree of the integrand. Each cell has two running times, the integration method is the top one, and the WLPvolume method is the bottom one.

Table 4.1 is the case when we integrate a monomial over the standard simplex. The LattE integration method is extremely slow when both dimension and degree are high. LattE's algorithm needs to turn one monomial into a sum of powers of linear forms and often thousands of linear forms. In the last case, it did not even finish. In contrast, the largest computation of the WLPvolume method was 3 seconds. Table 4.2 is the case when we integrate the power of a linear form over the standard simplex. The performance reverses and the WLPvolume method is extremely inefficient when both dimension and degree are high. The WLPvolume algorithm often needs to decompose the constructed weight-lifting polytope into thousands of simplicial cones to compute the volume.

	Dimension of the simplex									
Deg	1	2	3	4	5	6	7	8	9	10
1	0.01	0.00	0.00	0.01	0.01	0.02	0.01	0.02	0.02	0.06
	0.00	0.01	0.01	0.02	0.02	0.02	0.03	0.05	0.06	0.08
2	0.00	0.01	0.01	0.01	0.01	0.04	0.12	0.41	1.43	5.13
	0.01	0.01	0.01	0.03	0.04	0.06	0.10	0.13	0.21	0.31
3	0.01	0.00	0.00	0.02	0.04	0.21	1.01	5.12	25.87	123.82
	0.00	0.01	0.03	0.03	0.09	0.13	0.21	0.35	0.55	0.80
4	0.00	0.00	0.01	0.02	0.11	0.80	5.38	35.49	222.07	1345.95
	0.01	0.01	0.03	0.07	0.13	0.27	0.45	0.74	1.18	1.73
5	0.00	0.01	0.01	0.04	0.29	2.55	21.53	166.20	1282.92	-
	0.01	0.02	0.06	0.12	0.25	0.48	0.87	1.39	2.21	3.32

TABLE 4.1. Integration(upper) v.s. WLPvolume(lower): one monomial $\prod x_i^{row}$ over the standard *col*-simplex.

4.3.2. Weight of numerical semigroups. As we described in Section 4.2.1, numerical semigroups containing integer m with genus g are in bijection with the lattice points inside the Kunz polytope $\mathcal{P}_{m,g}$ and the weight of a numerical semigroup becomes a quadratic polynomial on the same polytope. Therefore, we can study the average weight of these finitely many numerical semigroups for each m and g.

Note that the weight of a numerical semigroup with genus g is the sum of elements in the gaps minus the sum from 1 to g and elements in the gaps are distinct g integers ranging from 1 to $2g-1$. So roughly speaking, the weight of a numerical semigroup has a quadratic growth with respect to the genus g. Specifically, we are interested in the average weight growth among the numerical semigroups containing integer m with genus g. Hence, we implemented the weight-lifting method via the Latte software to collect some data for $3 \leq m \leq 8$ and $g \leq 200$.

Deg	Dimension of the simplex 1	2	3	4	5	6	7	8	9	10
2	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02
	0.01	0.01	0.01	0.02	0.01	0.01	0.02	0.01	0.03	0.03
3	0.01	0.00	0.01	0.00	0.01	0.02	0.01	0.01	0.01	0.01
	0.01	0.01	0.02	0.02	0.01	0.02	0.03	0.03	0.04	0.05
4	0.00	0.01	0.01	0.01	0.01	0.00	0.00	0.01	0.02	0.02
	0.01	0.01	0.01	0.02	0.02	0.03	0.03	0.07	0.11	0.15
5	0.00	0.01	0.01	0.00	0.02	0.01	0.01	0.01	0.01	0.01
	0.01	0.01	0.02	0.02	0.03	0.05	0.10	0.17	0.32	0.54
6	0.00	0.01	0.00	0.01	0.00	0.01	0.01	0.01	0.01	0.01
	0.00	0.01	0.02	0.03	0.04	0.08	0.21	0.41	1.48	3.43
7	0.01	0.00	0.00	0.00	0.02	0.02	0.01	0.01	0.01	0.02
	0.01	0.02	0.02	0.03	0.06	0.19	0.53	1.61	6.42	14.95
8	0.00	0.01	0.01	0.01	0.00	0.01	0.01	0.01	0.01	0.01
	0.01	0.02	0.03	0.04	0.10	0.33	1.13	5.49	20.54	72.00
9	0.00	0.00	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.01
	0.02	0.03	0.03	0.04	0.15	0.76	2.85	16.05	76.25	236.78
10	0.00	0.01	0.01	0.01	0.01	0.00	0.01	0.01	0.02	0.01
	0.02	0.02	0.02	0.06	0.30	1.43	6.68	38.55	231.76	1694.71

Table 4.2. Integration(upper) v.s. WLPvolume(lower): a power of a linear form $(\sum c_i x_i)^{row}$ over the standard *col*-simplex.

The results are presented in Figure 4.1. For each m and g, we first calculate the sum of weight over numerical semigroups containing integer m with genus g, then calculate the number of these numerical semigroups. Dividing these two numbers, we get the average weight of numerical semigroups containing m with genus g. Lastly, we divide the average weight by genus square. We can observe that all quotients have convergence behavior.

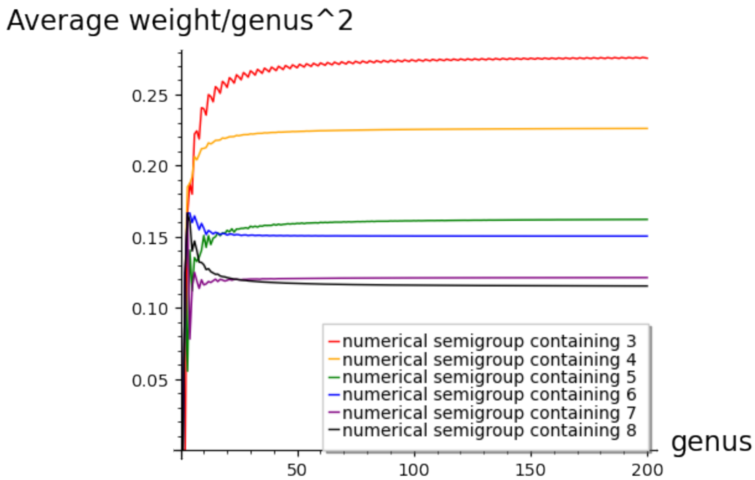

FIGURE 4.1. Curve plot of the quotient of average weight and genus square.

CHAPTER 5

Kakeya's Conjecture

This final chapter discusses Kakeya's generating function method and amends his proof with the help of Galois theory. Then we discuss two possible ways to tackle Kakeya's conjecture.

5.1. Kakeya's criterion

In this section, let t be an indeterminate variable other than x_i's and let $\mathbb{E}$ be a field containing the ring of symmetric polynomials in n variables, Λ_n, then we consider the collection of all degree n polynomials in variable t with the coefficients in $\mathbb{E}$ and the constant term 1, i.e.,

$$\left\{1 + \sum_{i=1}^{n} \alpha_i t^i : \alpha_i \in \mathbb{E}\right\}.$$

Next, we can consider the formal Taylor series for $\operatorname{Log}\left(1 + \sum_{i=1}^{n} \alpha_i t^i\right)$ with respect to variable t,

$$\operatorname{Log}\left(1 + \sum_{i=1}^{n} \alpha_i t^i\right) = \sum_{k=1}^{\infty} F_k(\alpha_1, \alpha_2, \dots, \alpha_n) t^k,$$

where F_k's are polynomial functions in α_i's. In particular, the closed form is

$$F_k(\alpha_1, \alpha_2, \dots, \alpha_n) = \sum_{\lambda \vdash k} (-1)^{l(\lambda)} \frac{C_\lambda \alpha_\lambda}{l(\lambda)},$$

where α_λ is a shorthand notation for the monomial $\alpha_{\lambda_1} \alpha_{\lambda_2} \dots \alpha_{\lambda_n}$, $l(\lambda)$ is the length of the partition λ and C_λ is the coefficient of the term α_λ in the $(\alpha_1 + \alpha_2 + \dots + \alpha_n)^{l(\lambda)}$.

EXAMPLE 5.1.1. When $n = 2$,

$$\begin{aligned}
F_1(\alpha_1, \alpha_2) &= -\frac{\alpha_1}{1}, \\
F_2(\alpha_1, \alpha_2) &= \frac{\alpha_1^2}{2} - \frac{\alpha_2}{1}, \\
F_3(\alpha_1, \alpha_2) &= -\frac{\alpha_1^3}{3} + \frac{2\alpha_1\alpha_2}{2}, \\
F_4(\alpha_1, \alpha_2) &= \frac{\alpha_1^4}{4} - \frac{3\alpha_1^2\alpha_2}{3} + \frac{\alpha_2^2}{2}, \\
&\vdots
\end{aligned}$$

In summary, the polynomials F_k's determine all the coefficients in the formal Taylor series expansion of $\operatorname{Log}\left(1 + \sum_{i=1}^{n} \alpha_i t^i\right)$.

On the other hand, by considering the algebraic closure of the field $\mathbb{E}$, the polynomial can be completely factorized into linear terms,

$$1 + \sum_{i=1}^{n} \alpha_i t^i = \prod_{j=1}^{n} (1 + r_j t).$$

In particular, the coefficient α_i is the i-th elementary symmetric polynomial of r_j's, i.e., $\alpha_i = e_i(r_1, r_2, \ldots, r_n)$. Then take the formal logarithm with respect to variable t, we yield

$$\operatorname{Log}\left(1 + \sum_{i=1}^{n} \alpha_i t^i\right) = \operatorname{Log}\left(\prod_{j=1}^{n} (1 + r_j t)\right) = \sum_{j=1}^{n} \operatorname{Log}(1 + r_j t).$$

Recall the formal Taylor series expansion of $\operatorname{Log}(1 + r_j t)$ with respect to variable t,

$$\operatorname{Log}(1 + r_j t) = \sum_{k=1}^{\infty} \left((-1)^k \frac{r_j^k}{k}\right) t^k.$$

Therefore,

$$\operatorname{Log}\left(1 + \sum_{i=1}^{n} e_i(r_1, r_2, \ldots, r_n) t^i\right) = \sum_{k=1}^{\infty} \left((-1)^k \frac{p_k(r_1, r_2, \ldots, r_n)}{k}\right) t^k.$$

Hence, these yield well-known algebraic equations, the *Newton's identities*, in terms of e_i's and p_i's.

$$F_k(e_1, e_2, \ldots, e_n) = (-1)^k \frac{p_k}{k}.$$

Given n positive integers $c_1 < c_2 < \ldots < c_n$ and suppose that the set $\{p_{c_1}, p_{c_2}, \ldots, p_{c_n}\}$ forms a fundamental system, then e_i's can be expressed as a rational function of p_{c_i}'s. This implies that if the coefficients of t^{c_i}'s in the series expansion of $\operatorname{Log}\left(1 + \sum_{i=1}^{n} \alpha_i t^i\right)$ are determined, then the coefficients, α_i's, of the polynomial are uniquely determined. In other words, polynomials can be distinguished by their coefficients at the term t^{c_i}'s under the formal logarithm map.

Kakeya took a very similar approach as above. Instead of considering $\operatorname{Log}\left(1 + \sum_{i=1}^{n} \alpha_i t^i\right)$, Kakeya considered formal Taylor series expansion of

$$\operatorname{Log}\left(\frac{1 + \sum_{i=1}^{n} \alpha_i t^i}{1 + \sum_{i=1}^{n} e_i t^i}\right) = \operatorname{Log}\left(1 + \sum_{i=1}^{n} \alpha_i t^i\right) - \operatorname{Log}\left(1 + \sum_{i=1}^{n} e_i t^i\right).$$

And he concluded that this series can be identified as 0 by checking only coefficients of n terms $t^{c_1}, t^{c_2}, \ldots$, and t^{c_n}. Moreover, Kakeya also utilized the following decomposition of rational polynomials

$$\begin{aligned}\frac{1 + \sum_{i=1}^{n} \alpha_i t^i}{1 + \sum_{i=1}^{n} e_i t^i} &= 1 - \left(\frac{A_1 x_1 t}{1 + x_1 t} + \frac{A_2 x_2 t}{1 + x_2 t} + \ldots + \frac{A_n x_n t}{1 + x_n t}\right) \\ &= 1 + \sum_{k=1}^{\infty} (-1)^k (A_1 x_1^k + A_2 x_2^k + \ldots + A_n x_n^k) t^k,\end{aligned}$$

where $A_i \in \mathbb{E}$ and is uniquely determined by α_i's. Therefore,

$$\operatorname{Log}\left(\frac{1 + \sum_{i=1}^{n} \alpha_i t^i}{1 + \sum_{i=1}^{n} e_i t^i}\right) = \sum_{k=1}^{\infty} G_k(A_1, A_2, \ldots, A_n) t^k.$$

Specifically,

$$G_k(A_1, A_2, \ldots, A_n) = \sum_{\lambda \vdash k} (-1)^{l(\lambda)} \frac{C_\lambda \zeta_\lambda}{l(\lambda)},$$

where ζ_λ is a shorthand notation for the monomial $\zeta_{\lambda_1} \zeta_{\lambda_2} \ldots \zeta_{\lambda_n}$ and ζ_i is a conventional notation for $(-1)^i (A_1 x_1^i + A_2 x_2^i + \ldots + A_n x_n^i)$.

Proof of Theorem 1.5.1. We will only prove the equivalence between item 1 and item 4, since item 5 (resp. item 3) is just the algebraic translation of item 4 (resp. item 2) and item 2 is a reformulation of item 4.

The "item 1 $\Longrightarrow$ item 4" part has been proved by the above argument. We will prove the "item 5 $\Longrightarrow$ item 1" part by contrapositive. Suppose the set $\{p_{c_1}, p_{c_2}, \dots, p_{c_n}\}$ does not form a fundamental system, then $e_i \notin \mathbb{K}(p_{c_1}, p_{c_2}, \dots, p_{c_n})$ for some i.

According to the determinant test (Theorem 10.83 [**64**]), the set $\{p_{c_1}, p_{c_2}, \dots, p_{c_n}\}$ are algebraically independent over $\mathbb{K}$, therefore, $\mathbb{K}(p_{c_1}, p_{c_2}, \dots, p_{c_n})/\mathbb{K}$ is a transcendental field extension of degree n. Recall that $\mathbb{K}(e_1, e_2, \dots, e_n)/\mathbb{K}$ is also a transcendental field extension of degree n, hence, $\mathbb{K}(e_1, e_2, \dots, e_n)/\mathbb{K}(p_{c_1}, p_{c_2}, \dots, p_{c_n})$ is an algebraic field extension. So if $e_i \notin \mathbb{K}(p_{c_1}, p_{c_2}, \dots, p_{c_n})$ for some i, then there must exist a minimal polynomial $P(x) \in \mathbb{K}(p_{c_1}, p_{c_2}, \dots, p_{c_n})[x]$ and $\deg P > 1$ such that $P(e_i) = 0$.

Since $\mathbb{E}$ is algebraically closed and $\mathbb{E} \supseteq \mathrm{Frac}(\Lambda_n) \supseteq \mathbb{K}(e_1, e_2, \dots, e_n) \supseteq \mathbb{K}(p_{c_1}, p_{c_2}, \dots, p_{c_n})$, there exists an intermediate field $\mathbb{E} \supseteq \mathbb{L} \supseteq \mathbb{K}(p_{c_1}, p_{c_2}, \dots, p_{c_n})$ such that $\mathbb{L}$ is a splitting field of the polynomial $P(x)$. $\mathbb{L}$ is a Galois extension since every minimal polynomial in characteristics 0 is irreducible and hence separable. Since the degree of $P(x)$ is strictly greater than 1, then e_i has a Galois conjugate $\overline{e_i}$ in $\mathbb{L}$. Since the Galois group acts transitively on the roots of polynomials, there exists a field automorphism $\sigma_{\mathbb{L}}$ of $\mathbb{L}/\mathbb{K}(p_{c_1}, p_{c_2}, \dots, p_{c_n})$ such that $e_i \overset{\sigma_{\mathbb{L}}}{\mapsto} \overline{e_i}$. Since $\mathbb{E}$ is algebraically closed and hence algebraic over $\mathbb{L}$, then by the Isomorphism Extension Theorem, the field automorphism $\sigma_{\mathbb{L}}$ of $\mathbb{L}$ can be extended to a field automorphism $\sigma_{\mathbb{E}}$ of $\mathbb{E}$.

It is easy to verify that $(\sigma_{\mathbb{E}}(e_1), \sigma_{\mathbb{E}}(e_2), \dots, \sigma_{\mathbb{E}}(e_n))$ is another solution of the polynomial equations in $\mathbb{E}^n$ other than $(e_1, e_2, \dots, e_n)$, since $\sigma_{\mathbb{E}}$ fixes the element in $\mathbb{K}(p_{c_1}, p_{c_2}, \dots, p_{c_n})$. $\square$

5.2. Kakeya's conjecture

In this section, we focus on presenting several possible ways of cracking Conjecture 1.5.1.

5.2.1. Searching for nontrivial solutions of Equation (1.2) or Equation (1.3). Using the idea we discussed in Theorem 1.5.1, Kakeya managed to prove that if an index set $\{c_1, c_2, \dots, c_n\}$ forms a gap of a numerical semigroup, then Equation (1.2) has the only trivial solution, which implies that $\{p_{c_1}, p_{c_2}, \dots, p_{c_n}\}$ forms a fundamental system. It is natural to try what we can derive from the condition that an index set $\{c_1, c_2, \dots, c_n\}$ does not form a gap of a numerical semigroup. This means that we need to search for nontrivial solutions of polynomial equations, however, in general, there is no known criterion that can guarantee the existence of nontrivial solutions.

THEOREM 5.2.1. *If an index set $\{c_1 < c_2 < \ldots < c_n\}$ does not form a gap of numerical semigroup but the index subset $\{c_1 < c_2 < \ldots < c_{n-1}\}$ forms a gap of numerical semigroup, then $\{p_{c_1}, p_{c_2}, \ldots, p_{c_n}\}$ does not form a fundamental system.*

PROOF. We will use the criterion Item 3 of Theorem 1.5.1 to prove the statement.

Since the subset of the first $n-1$ indices $\{c_1 < c_2 < \ldots < c_{n-1}\}$ forms a gap of numerical semigroup, then by induction, we can prove that the first $n-1$ polynomial equations of Equation (1.2) implies that $\zeta_{c_1} = 0, \zeta_{c_2} = 0, \ldots, \zeta_{c_{n-1}} = 0$. This can simplify the last polynomial equations of Equation (1.2) to

$$\sum_{\substack{\lambda \vdash c_n \\ \lambda_i \neq c_j \forall j=1,\ldots,n-1}} (-1)^{l(\lambda)} \frac{C_\lambda \zeta_\lambda}{l(\lambda)} = 0. \tag{5.1}$$

Recall that $\{c_1 < c_2 < \ldots < c_n\}$ does not form a gap of numerical semigroup, so c_n can be represented as the sum of at least two integers in $\mathbb{N} \setminus \{c_1 < c_2 < \ldots < c_{n-1}\}$, namely, the above polynomial equation contains terms other than ζ_{c_n}.

Recall that $\zeta_i = (-1)^i (A_1 x_1^i + A_2 x_2^i + \ldots + A_n x_n^i)$, if we can prove that $\zeta_{c_n} \neq 0$. then we find a nonzero solution of $(A_1, A_2, \ldots, A_n)$. So we can denote ζ_{c_n} by z, then

$$\begin{pmatrix} (-x_1)^{c_1} & (-x_2)^{c_1} & \ldots & (-x_n)^{c_1} \\ (-x_1)^{c_2} & (-x_2)^{c_2} & \ldots & (-x_n)^{c_2} \\ \vdots & \vdots & \ddots & \vdots \\ (-x_1)^{c_n} & (-x_2)^{c_n} & \ldots & (-x_n)^{c_n} \end{pmatrix} \cdot \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ z \end{bmatrix}.$$

Note that the matrix is an invertible generalized Vandermonde matrix, $V_{c_1, c_2, \ldots, c_n}$ and we denote its determinant as D_{c_n} for simplicity, we can see that A_i's are linear with respect to the variable z, hence, ζ_i's are linear with respect to the variable z. Specifically,

$$\zeta_i = \begin{bmatrix}(-x_1)^i & (-x_2)^i & \dots & (-x_n)^i\end{bmatrix} \cdot \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} = \begin{bmatrix}(-x_1)^i & (-x_2)^i & \dots & (-x_n)^i\end{bmatrix} \cdot V^{-1}_{c_1,c_2,\dots,c_n} \cdot \begin{bmatrix} 0 \\ 0 \\ \vdots \\ z \end{bmatrix}$$

$$= \frac{\det\left(V_{c_1,c_2,\dots,c_{n-1},i}\right)}{\det\left(V_{c_1,c_2,\dots,c_{n-1},c_n}\right)} z = \frac{D_i}{D_{c_n}} z.$$

So Equation (5.1) becomes a univariate polynomial of z. The coefficient of the linear term z is -1 which is nonzero. The coefficient of the quadratic term z^2 consists of sum of $\frac{D_{\lambda_1}D_{\lambda_2}}{D^2_{c_n}}$ where $\lambda_1, \lambda_2 \neq c_1, c_2, \dots, c_{n-1}$ and $\lambda_1 + \lambda_2 = c_n$. We will show that the coefficient contains a monomial of x_i's.

Among all λ_1, λ_2 appeared in the coefficient of z^2, we consider the one with λ_1 being maximal. We can order $(c_1, \dots, c_{n-1}, \lambda_1)$ and $(c_1, \dots, c_{n-1}, \lambda_2)$,

$$c_1 < \dots < c_{i-1} < c_i < c_{i+1} < \dots < c_j < \lambda_1 < c_{j+1} < \dots < c_{n-1}$$

$$c_1 < \dots < c_{i-1} < \lambda_2 < c_i < \dots < c_{j-1} < c_j < c_{j+1} < \dots < c_{n-1}.$$

Then $x_1^{2c_1} \cdots x_{i-1}^{2c_{i-1}} x_i^{c_i+\lambda_2} x_{i+1}^{c_{i+1}+c_i} \cdots x_j^{c_j+c_{j-1}} x_{j+1}^{\lambda_1+c_j} x_{j+2}^{2c_{j+1}} \cdots x_n^{2c_{n-1}}$ is a monomial appeared in the $D_{\lambda_1}D_{\lambda_2}$. And by the choice of maximal λ_1, we can see this monomial doesn't appear in other determinant products. So the coefficient of z^2 is nonzero.

Hence, the polynomial equation of z has a nonzero solution. □

5.2.2. Showing directly Frac$(\Lambda_n) \neq \mathbb{K}(p_{c_1}, p_{c_2}, \dots, p_{c_n})$.

PROPOSITION 5.2.1. *For any integers $1 < c_1 < c_2$, $p_1 = x_1 + x_2$ cannot be represented as a rational function of $p_{c_1} = x_1^{c_1} + x_2^{c_1}$ and $p_{c_2} = x_1^{c_2} + x_2^{c_2}$.*

PROOF. Prove by contradiction.

Suppose $p_1 = \frac{g(p_{c_1},p_{c_2})}{h(p_{c_1},p_{c_2})}$ with $gcd(g,h) = 1$. Since the rational form is reduced and all the power sum polynomials are homogeneous functions, the rational form contains exactly one power of p_{c_1}

and one power of p_{c_2}. Since $1 < c_1 < c_2$, we can always find a root γ of $1 + t^{c_2} = 0$ such that $1+\gamma^{c_1} \neq 0$ and $1+\gamma \neq 0$. Then we can let $x_1 = 1$ and $x_2 = \gamma$, namely, $p_1 \neq 0, p_{c_1} \neq 0$ and $p_{c_2} = 0$.

If the power of p_{c_1} and the power of p_{c_2} appear in the numerator, then we will observe an invalid relation $1 = \frac{1}{0}$.

If the power of p_{c_1} and the power of p_{c_2} appear in the denominator, then we will observe an invalid relation $1 = \frac{0}{1}$.

If the power of p_{c_1} and the power of p_{c_2} appear separately, then we will observe either $1 = \frac{0}{1}$ or $1 = \frac{1}{0}$. $\square$

Corollary 5.2.1. *Kakeya's conjecture is true when $n = 2$.*

Proof. By Proposition 5.2.1 and Theorem 5.2.1. $\square$

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