Stable-Limit Cherednik Theory

By

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To the two scientists in my life Sara and James Murphy

Contents

Abstract

This thesis is centered around extending families of representation theoretic objects corresponding to finite rank GL to the setting of infinite rank GL. Specifically, we study representations of the double affine Hecke algebras in type GL, the elliptic Hall algebra, and the double Dyck path algebra. Throughout this thesis we will develop new methods for constructing representation theoretic objects from families of finite rank classical objects and ways to understand these representations.

In the first chapter, we give an overview of the background information regarding Macdonald theory and Cherednik theory and of recent results in the area including the Shuffle Theorem. This chapter contains a review of the necessary algebraic, combinatorial, and representation theoretic definitions which will be used throughout the thesis.

In Chapter 2, we investigate limits of non-symmetric Macdonald polynomials and their place in the theory of almost symmetric functions. We will construct a basis of simultaneous eigenvectors for the limit Cherednik operators of Ion-Wu and investigate many of their properties. Further, we construct new operators on the space of almost symmetric functions generalizing the higher delta operators in Macdonald theory. Lastly, we explicitly compute q,t specializations of this basis to find a generalization of Schur functions to the almost symmetric functions with interesting combinatorial and representation theoretic properties.

Chapter 3 revolves around a family of modules called the Murnaghan-type representations for the elliptic Hall algebra generated using a stable-limit procedure from the vector-valued polynomial DAHA representations of Dunkl-Luque. This family of modules is indexed by partitions and generalizes the standard polynomial representation of EHA. We will construct a special family of generalized symmetric Macdonald functions as simultaneous eigenvectors for a generalized Macdonald operator and investigate their properties.

Lastly, in Chapter 4 we will construct new representations of the double Dyck path algebra built from compatible families of DAHA representations. We will use this general procedure to define Murnaghan-type representations using the EHA representations in Chapter 2.

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CHAPTER 1

Introduction

1.1. Background

1.1.1. Background. Spaces of polynomials are a meeting ground for a wealth of interesting combinatorics and representation theory. While we may first and foremost consider polynomial spaces like $\mathbb{Q}[x_1, \ldots, x_n]$ to be rings with their own algebra structures, in fact many other interesting algebras act on such spaces. For example the symmetric group algebra $\mathbb{Q}[\mathfrak{S}_n]$ will act on $\mathbb{Q}[x_1, \ldots, x_n]$ by permuting the variable indices. This action is central to the representation theory of GL_n yielding a great deal of interesting combinatorics. Another more complicated family of algebras which act on polynomials are the ***double affine Hecke algebras*** (DAHAs) of Cherednik [9]. Let us primarily focus on the DAHA corresponding to the Lie group GL_n. In this case the polynomial space is $\mathbb{Q}(q,t)[x_1, \ldots, x_n]$ and the type GL_n DAHA acts by a combination of multiplication operators X_i, Hecke operators T_i, and what are known as ***Cherednik operators*** Y_i which are related to Dunkl operators. This action generalizes to all Lie types and has an important place in modern representation theory. As it turns out, the Cherednik operators Y_i commute with each other and are simultaneously diagonalizable. Weight vectors for the Cherednik operators are known as ***non-symmetric Macdonald polynomials*** E_μ. These special polynomials satisfy many exceptional combinatorial properties and can be viewed as an orthogonal basis with respect to a natural inner product.

Often, mathematicians are most interested in the subspace of ***symmetric polynomials***, as there are fundamental links between the structure of symmetric polynomials and representation theory/combinatorics. In this case, we have that the $\mathfrak{S}_n$-invariants $\mathbb{Q}(q,t)[x_1, \ldots, x_n]^{\mathfrak{S}_n}$ are not preserved by the DAHA action, but rather by DAHA's spherical subalgebra. This algebra contains the special element $Y_1 + \ldots + Y_n$ which acts diagonally on symmetric polynomials via the finite variable ***Macdonald operator***. The normalized weight vectors for the action of $Y_1 + \ldots + Y_n$ are known as ***symmetric Macdonald polynomials*** P_λ, which generalize many other important symmetric polynomials including Jack polynomials, Hall-Littlewood polynomials, and Schur polynomials.

In many instances, interesting actions on spaces of polynomials have geometric interpretations allowing for a bridge between the purely algebraic properties of polynomials and certain properties of geometric objects. The Schur polynomials s_λ correspond both to irreducible characters of GL_n and to the cohomology classes of Schubert cells in Grassmanians. In recent decades a similar picture has been built for the symmetric Macdonald polynomials P_λ. Consider the equivariant K-theory of certain moduli spaces called the ***Hilbert schemes*** $\mathrm{Hilb}^n(\mathbb{C}^2)$. Haiman, in his groundbreaking work [23], showed that a certain transformation of the symmetric Macdonald polynomial P_λ, called the modified Macdonald polynomial $\tilde{H}_\lambda$, corresponds naturally to the torus fixed point I_λ of the Hilbert scheme $\mathrm{Hilb}^n(\mathbb{C}^2)$. This correspondence constituted a significant development in algebraic combinatorics, shedding light on both the combinatorics of the modified Macdonald polynomials and on the structure of Hilbert schemes. Later works by Schiffmann-Vasserot [34] and Carlsson-Gorsky-Mellit [7] built on this picture by directly linking the polynomials $\tilde{H}_\lambda$ to the torus-equivariant K-theory of the nested Hilbert schemes and of the parabolic flag Hilbert schemes, respectively.

In recent years there has been a new type of action on polynomial spaces which has seen an abundance of interest. The famous ***Shuffle Theorem*** of Carlsson-Mellit [8] resolved a long standing open problem in algebraic combinatorics regarding the modified Macdonald polynomials dating back to the work of Haiman and many others. The proof of Carlsson-Mellit involved the construction and study of a quiver path algebra $\mathbb{A}_{q,t}$ called the ***double Dyck path algebra*** which acts on a family of spaces $V_k = \mathbb{Q}(q,t)[z_1,\ldots,z_k]\otimes\Lambda$. Here Λ denotes the space of ***symmetric functions*** which are infinite variable versions of symmetric polynomials. At first glance, the algebra $\mathbb{A}_{q,t}$ appears to be a limit of the type GL_n DAHAs. Ion and Wu showed in [26] that there is a direct relation between the classical theory of Cherednik and $\mathbb{A}_{q,t}$. They introduced an algebra $\mathscr{H}^+$ called the positive stable-limit DAHA along with an action of $\mathscr{H}^+$ on the space of ***almost symmetric functions*** $\mathscr{P}_{as}^+ := \mathbb{Q}(q,t)[x_1,x_2,\ldots]\otimes\Lambda$. This action is generated by multiplication operators X_i, Hecke operators T_i, and what are known as the ***limit Cherednik operators*** $\mathscr{Y}_i$. These operators are the limits of certain deformations of the classical Cherednik operators Y_i, defined using a nontrivial notion of convergence for sequences of polynomials incorporating the t-adic topology of the field $\mathbb{Q}(q,t)$. It was shown by Ion-Wu that the $\mathscr{H}^+$ action on $\mathscr{P}_{as}^+$ in a sense globalizes the polynomial representation of $\mathbb{A}_{q,t}$, as you can recover the action of $\mathbb{A}_{q,t}$ on each space V_k by looking locally at the $\mathscr{H}^+$ action on $\mathbb{Q}(q,t)[x_1,\ldots,x_k]\otimes\Lambda\subset\mathscr{P}_{as}^+$.

1.2. Thesis Overview

1.2.1. Stable-Limit Non-Symmetric Macdonald Functions. In Chapter 2 of this thesis we will answer a question of Ion and Wu regarding the spectral theory of the limit Cherednik operators $\mathscr{Y}_i$. In the classical DAHA picture, the non-symmetric Macdonald polynomials are a weight basis for the Cherednik operators. Ion-Wu conjectured the existence of a similar story for their limit Cherednik operators, namely that there exists a $\mathscr{Y}$-weight basis for $\mathscr{P}_{as}^+$. By using Ion and Wu's new notion of convergence involving the t-adic topology from the field $\mathbb{Q}(q,t)$, we show that the sequences $E_{(\mu_1,\ldots,\mu_n,0^m)}$ converge to well defined elements $\widetilde{E}_\mu$ of $\mathscr{P}_{as}^+$. In the process of this convergence proof we give an explicit combinatorial formula for the $\widetilde{E}_\mu$ similar to the Haglund-Haiman-Loehr formula for the non-symmetric Macdonald polynomials. Importantly, using a continuity-like argument, it is straightforward to prove that the $\widetilde{E}_\mu$ are in fact $\mathscr{Y}$-weight vectors. However, these almost symmetric functions $\widetilde{E}_\mu$ do not span all of $\mathscr{P}_{as}^+$. To find a basis of $\mathscr{P}_{as}^+$, we will use Ion-Wu's variant of the Jing vertex operators $\partial_-^{(r)}$ to construct partial symmetrizations of the $\widetilde{E}_\mu$. We call these functions $\widetilde{E}_{(\mu|\lambda)}$ the ***stable-limit non-symmetric Macdonald functions*** as they are the analogues of the classical non-symmetric Macdonald polynomials for the setting of the stable-limit DAHA. It is perhaps also appropriate to refer to them as the ***almost symmetric Macdonald functions***. They are indexed by pairs $(\mu|\lambda)$ of (reduced) compositions μ and partitions λ.

In a significant deviation from classical Cherednik theory, the $\mathscr{Y}$-weight spaces of $\mathscr{P}_{as}^+$ are all infinite dimensional. Classically, the Macdonald operator Δ acts on symmetric functions Λ with distinct spectrum and weight vectors given by the symmetric Macdonald functions P_λ. It is thus natural to try to extend the Macdonald operator from Λ to $\mathscr{P}_{as}^+$ in a way that acts diagonally on the $\widetilde{E}_{(\mu|\lambda)}$ basis. Constructing this operator required bringing in new ideas beyond the work of Ion and Wu. We prove that there is a natural way to define an extended Macdonald operator, Ψ_{p_1}, on $\mathscr{P}_{as}^+$ which dramatically refines the $\mathscr{Y}$-weight spaces on $\mathscr{P}_{as}^+$ to be 1-dimensional. That is to say, if one considers any pair $(\mu|\lambda)$, then the weight of the operators $(\Psi_{p_1},\mathscr{Y}_1,\mathscr{Y}_2,\ldots)$ acting on $\widetilde{E}_{(\mu|\lambda)}$ is distinct from any other $(\mu'|\lambda')$. This operator Ψ_{p_1} is constructed as the limit of finite-rank DAHA operators $t^m(Y_1^{(m)} + \ldots + Y_m^{(m)})$ from a nontrivial and technical convergence argument using the stable-limit convergence of Ion and Wu. As a benefit of this limit construction one can prove that Ψ_{p_1} satisfies some interesting nontrivial algebraic relations. The refinement of the $\mathscr{Y}$-weight spaces shows that, from the perspective of this new stable-limit DAHA theory, the $\widetilde{E}_{(\mu|\lambda)}$ are the unique basis of $\mathscr{P}_{as}^+$ generalizing the classical non-symmetric Macdonald polynomials.

In classical Macdonald theory, along with the Macdonald operator Δ there are ***higher delta operators*** $F[\Delta]$ which in-part generate the action of the elliptic Hall algebra on symmetric functions Λ. Using techniques developed to show that the operator Ψ_{p_1} on $\mathscr{P}_{as}^+$ exists we show similarly that there are analogous operators Ψ_F which generalize the classical higher delta operators. The verification of this construction involves significantly more intricate calculations. This construction hints at the existence of a larger family of algebras generalizing the elliptic Hall algebra which acts naturally on $\mathscr{P}_{as}^+$.

Lastly, we will investigate the $q = \infty$, $t = 0$ specializations of the $\widetilde{E}_{(\mu|\lambda)}$. For the finite rank non-symmetric Macdonald polynomials Ion showed that this specialization yields the ***key polynomial*** basis for polynomials. The key polynomials are notable as they interpolate between finite variable Schur polynomials and monomials corresponding to partitions. They are in fact the characters of Demazure modules for the group of upper triangular matrices as is shown by the famous Demazure character formula. We will show that the $q = \infty$, $t = 0$ specializations of the $\widetilde{E}_{(\mu|\lambda)}$ give a basis for almost symmetric functions which interpolate between key polynomials and Schur functions. These ***almost symmetric Schur functions*** are limits of characters of certain parabolic subgroups of type GL and thus satisfy some interesting positivity properties. We will also give an explicit combinatorial model for the almost symmetric Schur functions using the HHL-type formula for the key polynomials.

1.2.2. Murnaghan-Type Representations for the Elliptic Hall Algebra. Dunkl and Luque introduced symmetric and non-symmetric vector-valued (vv.) Macdonald polynomials. The term vector-valued here refers to polynomial-like objects of the form $\sum_\alpha c_\alpha X^\alpha \otimes v_\alpha$ for some scalars c_α, monomials X^α, and vectors v_α lying in some $\mathbb{Q}(q,t)$-vector space. The non-symmetric vv. Macdonald polynomials are distinguished bases for certain DAHA representations built from the irreducible representations of the finite Hecke algebras in type A. These DAHA representations are indexed by Young diagrams and exhibit interesting combinatorial properties relating to periodic Young tableaux. The symmetric vv. Macdonald polynomials are distinguished bases for the spherical (i.e. Hecke-invariant) subspaces of these DAHA representations. Naturally, the spherical DAHA acts on this spherical subspace with the special element $\xi_1 + \ldots + \xi_n$ of spherical DAHA acting diagonally on the symmetric vv. Macdonald polynomials.

Dunkl and Luque (and in later work of Colmenarejo, Dunkl, and Luque) only consider the finite rank non-symmetric and symmetric vv. Macdonald polynomials. It is natural to ask if there is an infinite-rank stable-limit construction using the symmetric vv. Macdonald polynomials to give generalized symmetric Macdonald functions and an associated representation of the positive elliptic Hall algebra $\mathscr{E}^+$. In this chapter, we describe such a construction. We obtain a new family of graded $\mathscr{E}^+$-representations $\widetilde{W}_\lambda$ indexed by Young diagrams λ and a natural generalization of the symmetric Macdonald functions $\mathfrak{P}_T$ indexed by certain labellings of infinite Young diagrams built as limits of the symmetric vv. Macdonald polynomials.

For any λ we consider the increasing chains of Young diagrams $\lambda^{(n)} = (n - |\lambda|, \lambda)$ for $n \geq |\lambda| + \lambda_1$ to build the representations $\widetilde{W}_\lambda$. These special sequences of Young diagrams are central to Murnaghan's theorem regarding the reduced Kronecker coefficients. As such we refer to the $\mathscr{E}^+$-representations $\widetilde{W}_\lambda$ as Murnaghan-type. For $\lambda = \emptyset$ we recover the $\mathscr{E}^+$ action on Λ and the symmetric Macdonald functions $P_\mu[X; q, t]$.

We obtain a ***Pieri rule*** for the action of the multiplication operators $e_r^\bullet$ on the generalized symmetric Macdonald function basis $\mathfrak{P}_T$. After studying the particular case of the e_1-Pieri coefficients we will show that the modules $\widetilde{W}_\lambda$ are cyclic generated by their unique elements of minimal degree $\mathfrak{P}_{T_\lambda^{min}}$. Lastly, we show that these Murnaghan-type representations $\widetilde{W}_\lambda$ are mutually non-isomorphic. At the end of Chapter 3 we will look at a family of product-sum formulas which follow naturally using the results described thus far and a bit of simple analysis. These formulas relate certain (q, t) statistics on special infinite diagrams and appear to give rational formulas for certain sums of hyper-geometric series.

1.2.3. Double Dyck Path Algebra Representations From DAHA. In this chapter we develop a method for constructing modules for the double Dyck path algebra $\mathbb{B}_{q,t}$ directly from the representation theory of DAHA in type GL. The algebra $\mathbb{B}_{q,t}$ is a highly related geometric version of the Carlsson-Mellit algebra $\mathbb{A}_{q,t}$. We show that given any $\mathscr{D}_n^+$ module V we may construct an action of the subalgebra $\mathbb{B}_{q,t}^{(n)}$ on the space $L_{\bullet}(V)$ defined by $L_{\bullet}(V) = \bigoplus_{0\leq k\leq n} L_k(V) := \bigoplus_{0\leq k\leq n} X_1\cdots X_k\epsilon_k(V)$. Here ϵ_k are the *partial trivial idempotents* of the finite Hecke algebra. Each space may be considered as a module for the partially symmetrized positive DAHA, $\epsilon_k\,\mathscr{D}_n^+\,\epsilon_k$. We show that in the case of the polynomial representations $V_{\text{pol}}^{(n)}$ of DAHA that $L_{\bullet}(V_{\text{pol}}^{(n)})$ is a $\mathbb{B}_{q,t}^{(n)}$-module quotient of the restriction of the polynomial representation of $\mathbb{B}_{q,t}$ to $\mathbb{B}_{q,t}^{(n)}$.

Afterwards, we use stable-limits of the representations $L_{\bullet}(V)$ of $\mathbb{B}_{q,t}^{(n)}$ to build representations of $\mathbb{B}_{q,t}$. This construction requires the input of an infinite family of representations of DAHAs, $(V^{(n)})_{n\geq n_0}$, along with some connecting maps, $\Pi^{(n)} : V^{(n+1)} \to V^{(n)}$, satisfying some special assumptions. Most interestingly, we require that the following relations holds: $\Pi^{(n)}\pi^{(n+1)}T_n = \pi^{(n)}\Pi^{(n)}$. This is the same relation used by Ion-Wu in their construction of the limit Cherednik operators and is related to certain natural embeddings of the extended affine symmetric groups $\widetilde{\mathfrak{S}}_n \hookrightarrow \widetilde{\mathfrak{S}}_{n+1}$. We call such families $C = \big((V^{(n)})_{n\geq n_1}, (\Pi^{(n)})_{n\geq n_1}\big)$ *compatible* and construct spaces $\mathfrak{L}_k(C)$ given by $\mathfrak{L}_k(C) := \lim_{\leftarrow} L_k(V^{(n)})$. These are the stable-limits of the spaces $L_k(V^{(n)})$ with respect to the maps $\Pi^{(n)}$. Finally, we package together these spaces to form $\mathfrak{L}_{\bullet}(C)$ given as $\mathfrak{L}_{\bullet}(C) := \bigoplus_{k\geq 0}\mathfrak{L}_k(C)$ which may be also thought of as the stable-limit of the $\mathbb{B}_{q,t}^{(n)}$ modules $L_{\bullet}(V^{(n)})$. We show that there is a natural action of $\mathbb{B}_{q,t}$ on $\mathfrak{L}_{\bullet}(C)$ determined by the $\mathbb{B}_{q,t}^{(n)}$ module structures on $L_{\bullet}(V^{(n)})$. This construction is also functorial.

Lastly, we use this construction of the functor $C \to \mathfrak{L}_{\bullet}(C)$ to define a large family of $\mathbb{B}_{q,t}$ modules, $\mathfrak{L}_{\bullet}(\text{Ind}(C_\lambda))$, indexed by partitions λ. These representations in a sense extend the Murnaghan-type representations of the positive elliptic Hall algebra.

1.3. Polynomials and Symmetric Functions

1.3.1. Basic Combinatorics.

Definition 1.3.1. *In this paper, a* ***composition*** *will refer to a finite tuple $\mu = (\mu_1, \ldots, \mu_n)$ of non-negative integers. We allow for the empty composition $\emptyset$ with no parts. We will let* Comp *denote the set of all compositions. The length of a composition $\mu = (\mu_1, \ldots, \mu_n)$ is $\ell(\mu) = n$ and the size of the composition is $|\mu| = \mu_1 + \ldots + \mu_n$. As a convention we will set $\ell(\emptyset) = 0$ and $|\emptyset| = 0$. We say that a composition μ is* ***reduced*** *if $\mu = \emptyset$ or $\mu_{\ell(\mu)} \neq 0$. We will let* Comp^{red} *denote the set of all reduced compositions. Given two compositions $\mu = (\mu_1, \ldots, \mu_n)$ and $\beta = (\beta_1, \ldots, \beta_m)$, define*

$\mu * \beta = (\mu_1, \ldots, \mu_n, \beta_1, \ldots, \beta_m)$. *A* ***partition*** *is a composition* $\lambda = (\lambda_1, \ldots, \lambda_n)$ *with* $\lambda_1 \geq \ldots \geq \lambda_n \geq 1$. *Note that vacuously we allow for the empty partition* $\emptyset$. *We denote the set of all partitions by* $\mathbb{Y}$. *We denote by* Σ *the set of all pairs* $(\mu|\lambda)$ *with* $\mu \in \mathrm{Comp}^{red}$ *and* $\lambda \in \mathbb{Y}$.

We denote by $\mathrm{sort}(\mu)$ *the partition obtained by ordering the nonzero elements of* μ *in weakly decreasing order. We define* $\mathrm{rev}(\mu)$ *to be the composition obtained by reversing the order of the elements of* μ. *The dominance ordering for partitions is defined by* $\lambda \trianglelefteq \nu$ *if for all* $i \geq 1$, $\lambda_1 + \ldots + \lambda_i \leq \nu_1 + \ldots + \nu_i$ *where we set* $\lambda_i = 0$ *whenever* $i > \ell(\lambda)$ *and similarly for* ν. *If* $\lambda \trianglelefteq \nu$ *and* $\lambda \neq \nu$. *we will write* $\lambda \triangleleft \nu$.

We will in a few instances use the notation $\mathbb{1}(p)$ *to denote the value* 1 *if the statement* p *is true and* 0 *otherwise.*

Definition 1.3.2. *The symmetric group* $\mathfrak{S}_n$ *is defined as the set of bijective maps* $\sigma : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ *with multiplication given by function composition. For* $1 \leq i \leq n-1$ *we will write* s_i *for the transposition swapping* $i, i+1$ *and fixing everything else. For any* $\mu = (\mu_1, \ldots, \mu_r)$ *with* $\mu_i \geq 1$ *and* $\mu_1 + \ldots + \mu_r = n$ *we define the* ***Young subgroup*** $\mathfrak{S}_\mu$ *to be the group generated by the* s_i *for* $i \in \{\mu_1 + \ldots + \mu_{j-1} + 1, \ldots, \mu_1 + \ldots + \mu_{j-1} + \mu_j\}$ *for some* $0 \leq j \leq r$.

We have the following alternative presentation of the symmetric group $\mathfrak{S}_n$.

Proposition 1.3.3 (Coxeter Presentation). *The symmetric group* $\mathfrak{S}_n$ *is generated by elements* $s_1, \ldots, s_{n-1}$ *subject to the relations:*

- $s_i^2 = 1$
- $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$
- $s_i s_j = s_j s_i$ *for* $|i - j| > 1$.

Definition 1.3.4. *For* $\sigma \in \mathfrak{S}_n$ *the* ***length*** *of* σ, $\ell(\sigma)$, *is defined to be the minimal number of* s_i *required to express* σ, *i.e.* $\sigma = s_{i_1} \cdots s_{i_r}$. *We will denote by* $w_0^{(n)}$ *the unique element of* $\mathfrak{S}_n$ *of maximal length* $\ell(w_0^{(n)}) = \binom{n}{2}$ *given by*

$$w_0^{(n)}(i) := n - i + 1.$$

Remark 1. *We may also express* $w_0^{(n)}$ *using the Coxeter presentation as*

$$w_0^{(n)} = (s_{n-1} \cdots s_1)(s_{n-1} \cdots s_2) \cdots (s_{n-1} s_{n-2}) s_{n-1}.$$

In line with the conventions in [19] we define the Bruhat order on the type GL_n weight lattice $\mathbb{Z}^n$ as follows.

Definition 1.3.5. *Let $e_1, ..., e_n$ be the standard basis of $\mathbb{Z}^n$ and let $\alpha \in \mathbb{Z}^n$. We define the* ***Bruhat ordering*** *on $\mathbb{Z}^n$, written simply by $<$, by first defining cover relations for the ordering and then taking their transitive closure. If $i < j$ such that $\alpha_i < \alpha_j$ then we say $\alpha > (ij)(\alpha)$ and additionally if $\alpha_j - \alpha_i > 1$ then $(ij)(\alpha) > \alpha + e_i - e_j$ where (ij) denotes the transposition swapping i and j.*

It is important to note that with respect to the Bruhat order any weakly decreasing vector $v \in \mathbb{Z}^n$ is the minimal element in its permutation orbit $\mathfrak{S}_n.v$.

1.3.2. Polynomials. Throughout this thesis the variables q and t are assumed to be commuting free variables.

Definition 1.3.6. *Define $\mathscr{P}_n := \mathbb{Q}(q,t)[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ for the space of Laurent polynomials in n variables over $\mathbb{Q}(q,t)$ and define $\mathscr{P}_n^+ := \mathbb{Q}(q,t)[x_1, \dots, x_n]$ for the subspace of polynomials. We define algebra homomorphisms $\Xi^{(n)} : \mathscr{P}_{n+1}^+ \to \mathscr{P}_n^+$ by*

$$\Xi^{(n)}(x_1^{a_1} \cdots x_n^{a_n} x_{n+1}^{a_{n+1}}) = \mathbb{1}(a_{n+1} = 0) x_1^{a_1} \cdots x_n^{a_n}.$$

The symmetric group $\mathfrak{S}_n$ acts naturally on $\mathscr{P}_n$ by algebra automorphisms via

$$\sigma(f(x_1, \dots, x_n)) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

1.3.3. Symmetric Functions.

Definition 1.3.7. *Define the* ***ring of symmetric functions*** *Λ to be the subalgebra of the inverse limit of the symmetric polynomial rings $\mathbb{Q}(q,t)[x_1, \dots, x_n]^{\mathfrak{S}_n}$ with respect to the quotient maps sending $x_n \to 0$ consisting of those elements with bounded x-degree. For $i \geq 1$ define the i-th* ***power sum symmetric function*** *by*

$$p_i = x_1^i + x_2^i + \dots.$$

It is a classical result that Λ is isomorphic to $\mathbb{Q}(q,t)[p_1, p_2, \dots]$. For any expression $G = a_1 g^{\mu_1} + a_2 g^{\mu_2} + \dots$ with rational scalars $a_i \in \mathbb{Q}$ and distinct monomials g^{μ_i} in a set of algebraically independent commuting free variables $\{g_1, g_2, \dots\}$ the ***plethystic evaluation*** *of p_i at the expression G is defined to be*

$$p_i[G] := a_1 g^{i\mu_1} + a_2 g^{i\mu_2} + \dots.$$

Note that g_i are allowed to be q or t. Here we are using the convention that $i\mu = (i\mu_1, \dots, i\mu_r)$ for $\mu = (\mu_1, \cdots, \mu_r)$. The definition of plethystic evaluation on power sum symmetric functions extends to all symmetric functions $F \in \Lambda$ by requiring $F \to F[G]$ be a $\mathbb{Q}(q,t)$-algebra homomorphism. Note that for $F \in \Lambda$, $F = F[x_1 + x_2 + \dots]$ and so we will often write $F = F[X]$ where $X := x_1 + x_2 + \dots$. For a partition λ define the ***monomial symmetric function*** *m_λ by*

$$m_\lambda := \sum_{\mu} x^\mu$$

where we range over all distinct monomials x^μ such that $\sigma(\mu) = \lambda$ for some permutation σ. For $n \geq 0$ define the ***complete homogeneous symmetric function*** *h_n by*

$$h_n := \sum_{|\lambda|=n} m_\lambda.$$

Equivalently,

$$h_n = \sum_{i_1 \leq \dots \leq i_n} x_{i_1} \cdots x_{i_n}.$$

For $n \geq 1$ define the ***elementary symmetric function*** *e_n by*

$$e_n = \sum_{i_1 < \dots < i_n} x_{i_1} \cdots x_{i_n}.$$

We can extend plethysm to $\mathbb{Q}(q,t)[[p_1, p_2, \dots]]$. The ***plethystic exponential*** *is defined to be the element of $\mathbb{Q}(q,t)[[p_1, p_2, \dots]]$ given by*

$$\mathrm{Exp}[X] := \sum_{n \geq 0} h_n[X].$$

Here we list some notable properties of the plethystic exponential which will be used later in this thesis.

- $\mathrm{Exp}[0] = 1$

- $\operatorname{Exp}[X+Y] = \operatorname{Exp}[X]\operatorname{Exp}[Y]$
- $\operatorname{Exp}[x_1 + x_2 + \dots] = \prod_{i=1}^{\infty}\left(\frac{1}{1-x_i}\right)$
- $\operatorname{Exp}[(1-t)(x_1 + x_2 + \dots)] = \prod_{i=1}^{\infty}\left(\frac{1-tx_i}{1-x_i}\right)$

Example. *Here we give a few examples of plethystic evaluation.*

- $p_3[1+5t+qt^2] = 1+5t^3+q^3t^6$
- $s_2[(1-t)X] = (\frac{p_2+p_{1,1}}{2})[(1-t)X] = \frac{(1-t^2)p_2[X]+(1-t)^2p_{1,1}[X]}{2}$
- $\operatorname{Exp}[\frac{t}{1-t}] = \prod_{n=1}^{\infty}(\frac{1}{1-t^n})$

Definition 1.3.8. *[30] Define the **(q,t)-Hall inner product** on Λ by*

$$\langle p_\lambda, p_\mu\rangle_{q,t} := \delta_{\lambda,\mu} z_\lambda \prod_{1\le i\le \ell(\lambda)} \left(\frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}\right)$$

where

$$z_\lambda := \prod_i \left(i^{m_\lambda(i)!} m_\lambda(i)!\right).$$

*Further, define the **t-Hall inner product** and **classical Hall inner product** respectively by*

- $\langle \bullet, \bullet\rangle_t := \langle \bullet, \bullet\rangle_{0,t}$
- $\langle \bullet, \bullet\rangle := \langle \bullet, \bullet\rangle_{0,0}$.

Definition 1.3.9. *[30] Define the **symmetric Macdonald functions** $P_\lambda[X;q,t]$ for $\lambda \in \mathbb{Y}$ to be the unique symmetric functions satisfying the conditions:*

- $P_\lambda[X;q,t] = m_\lambda + \sum_{\mu\lhd\lambda} c_\mu m_\mu$ *for some coefficients* $c_\mu \in \mathbb{Q}(q,t)$
- $\langle P_\lambda[X;q,t], P_\mu[X;q,t]\rangle_{q,t} = 0$ *for* $\lambda \neq \mu$.

*Define the **symmetric Hall-Littlewood functions** $P_\lambda[X;t]$ by*

$$P_\lambda[X;t] := P_\lambda[X;0,t]$$

*and the **Schur functions** $s_\lambda[X]$ by*

$$s_\lambda[X] = P_\lambda[X;0,0].$$

Proposition 1.3.10. *[30] The following sets are all $\mathbb{Q}(q,t)$-bases for Λ :*

- $\{s_\lambda[X]\}_{\lambda\in\mathbb{Y}}$

- $\{P_\lambda[X;t]\}_{\lambda\in\mathbb{Y}}$
- $\{P_\lambda[X;q,t]\}_{\lambda\in\mathbb{Y}}$.

It will be convenient in Chapter 2 to use a variant of the Hall-Littlewood functions $P_\lambda[X;t]$.

DEFINITION 1.3.11. *For $n\geq 0$ define the **Jing vertex operator** $\mathscr{B}_n\in End_{\mathbb{Q}(q,t)}(\Lambda)$ by*

$$\mathscr{B}_n[F]:=\langle z^n\rangle F[X-z^{-1}]Exp[(1-t)zX].$$

*Here $\langle z^n\rangle$ denotes the operator which extracts the coefficient of z^n of any formal series in z. For a partition $\lambda=(\lambda_1,...,\lambda_r)$ define the **dual Hall-Littlewood symmetric function**, $\mathcal{P}_\lambda$, by*

$$\mathcal{P}_\lambda:=\mathscr{B}_{\lambda_1}\cdots\mathscr{B}_{\lambda_r}(1).$$

Note that the operator $\mathscr{B}_n$ is homogeneous with degree n. As we will see later in Proposition 2.4.20 the $\mathcal{P}_\lambda[X]$ are the same as the dual Hall-Littlewood symmetric functions $Q_\lambda[X;t]$ defined by Macdonald [30]. These symmetric functions have the following useful properties.

- $\mathcal{P}_\lambda$ is homogeneous with degree $|\lambda|$
- $\mathcal{P}_{(n)}[X]=h_n[(1-t)X]$
- If $n\geq\lambda_1$ then $\mathscr{B}_n(\mathcal{P}_\lambda)=\mathcal{P}_{n*\lambda}$
- $\mathscr{B}_0(\mathcal{P}_\lambda)=t^{\ell(\lambda)}\mathcal{P}_\lambda$

1.3.4. Almost Symmetric Functions.

DEFINITION 1.3.12. *[26] Let $\mathscr{P}_\infty^+$ denote the inverse limit of the rings $\mathscr{P}_k^+$ with respect to the homomorphisms $\Xi_k:\mathscr{P}_{k+1}^+\to\mathscr{P}_k^+$ which send x_{k+1} to 0 at each step. We can naturally extend Ξ_k to a map $\mathscr{P}_\infty^+\to\mathscr{P}_k$ which will be given the same name. Let $\mathscr{P}(k)^+:=\mathbb{Q}(q,t)[x_1,\ldots,x_k]\otimes\Lambda[x_{k+1}+x_{k+2}+\ldots]$. Define the **ring of almost symmetric functions** by $\mathscr{P}_{as}^+:=\bigcup_{k\geq 0}\mathscr{P}(k)^+$. Note $\mathscr{P}_{as}^+\subset\mathscr{P}_\infty^+$. Define $\rho:\mathscr{P}_{as}^+\to x_1\mathscr{P}_{as}^+$ to be the linear map defined by $\rho(x_1^{a_1}\cdots x_n^{a_n}F[x_m+x_{m+1}+\ldots])=\mathbb{1}(a_1>0)x_1^{a_1}\cdots x_n^{a_n}F[x_m+x_{m+1}+\ldots]$ for $F\in\Lambda$. Note that ρ restricts to maps $\mathscr{P}_n\to x_1\mathscr{P}_n$ which are compatible with the quotient maps π_n.*

The ring $\mathscr{P}_{as}^+$ is a free graded Λ-module with homogeneous basis given simply by the set of monomials x^μ with μ reduced. Therefore, $\mathscr{P}_{as}^+$ has the homogeneous $\mathbb{Q}(q,t)$ basis given by all $x^\mu m_\lambda[X]$ ranging over all reduced compositions μ and partitions λ. Further, the dimension of the homogeneous degree d part of $\mathscr{P}(k)^+$ is equal to the number of pairs (μ, λ) of reduced compositions μ and partitions λ with $|\mu| + |\lambda| = d$ and $\ell(\mu) \le k$.

1.4. Hecke Algebras in Type GL

1.4.1. Finite Hecke Algebra.

DEFINITION 1.4.1. *Define the finite Hecke algebra $\mathscr{H}_n$ to be the $\mathbb{Q}(q,t)$-algebra generated by the elements $T_1, \ldots, T_{n-1}$ subject to the relations*

- $(T_i - 1)(T_i + t) = 0$ *for* $1 \le i \le n-1$
- $T_i T_{i+1} T_1 = T_{i+1} T_i T_{i+1}$ *for* $1 \le i \le n-2$
- $T_i T_j = T_j T_i$ *for* $|i-j| > 1$.

*We define the **Jucys-Murphy** elements $\overline{\theta}_1, \ldots, \overline{\theta}_n \in \mathscr{H}_n$ by $\overline{\theta}_1 := 1$ and $\overline{\theta}_{i+1} := tT_i^{-1}\overline{\theta}_i T_i^{-1}$ for $1 \le i \le n-1$. Further, define $\overline{\varphi}_1, \ldots, \overline{\varphi}_{n-1}$ by $\overline{\varphi}_i := (tT_i^{-1})\overline{\theta}_i - \overline{\theta}_i(tT_i^{-1})$. For a permutation $\sigma \in \mathfrak{S}_n$ and a reduced expression $\sigma = s_{i_1} \cdots s_{i_r}$ we write $T_\sigma := T_{i_1} \cdots T_{i_r}$.*

REMARK 2. *There are natural algebra inclusions $\mathscr{H}_n \to \mathscr{H}_{n+1}$ given by $T_i \to T_i$ for $1 \le i \le n-1$. Under this embedding $\overline{\theta}_i \to \overline{\theta}_i$ for $1 \le i \le n$ so we can naturally identify the copies of $\overline{\theta}_i$ in both $\mathscr{H}_n$ and $\mathscr{H}_{n+1}$.*

We require the following list of relations.

PROPOSITION 1.4.2. *The following relations hold:*

- $\overline{\theta}_i = t^{i-1} T_{i-1}^{-1} \cdots T_1^{-1} T_1^{-1} \cdots T_{i-1}^{-1}$ *for* $1 \le i \le n$
- $\overline{\theta}_i \overline{\theta}_j = \overline{\theta}_j \overline{\theta}_i$ *for* $1 \le i, j \le n$
- $T_i \overline{\theta}_j = \overline{\theta_j} T_i$ *for* $j \notin \{i, i+1\}$
- $\overline{\varphi}_i = tT_i^{-1}(\overline{\theta}_i - \overline{\theta}_{i+1}) + (t-1)\overline{\theta}_{i+1}$ *for* $1 \le i \le n-1$
- $\overline{\varphi}_i \overline{\varphi}_{i+1} \overline{\varphi}_i = \overline{\varphi}_{i+1} \overline{\varphi}_i \overline{\varphi}_{i+1}$ *for* $1 \le i \le n-1$
- $\overline{\varphi}_i \overline{\varphi}_j = \overline{\varphi}_j \overline{\varphi}_i$ *for* $|i-j| > 1$
- $\overline{\varphi}_i \overline{\theta}_j = \overline{\theta}_{s_i(j)} \overline{\varphi}_i$ *for* $1 \le i \le n-1$ *and* $1 \le j \le n$
- $\overline{\varphi}_i^2 = (t\overline{\theta}_i - \overline{\theta}_{i+1})(t\overline{\theta}_{i+1} - \overline{\theta}_i)$.

PROOF. This result follows directly from using the map ρ_n defined in Definition 1.4.6 and Proposition 1.4.5 which will be independently proven later. ☐

1.4.2. Affine Hecke Algebra. Throughout this thesis will use two equivalent presentations for the affine Hecke algebras in type GL_n.

Definition 1.4.3. *Define the affine Hecke algebra $\mathscr{A}_n$ to be the $\mathbb{Q}(q,t)$-algebra generated by the elements $T_1, \ldots, T_{n-1}$ and $Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}$ subject to the relations*

- $T_1, \ldots, T_{n-1}$ *generate* $\mathscr{H}_n$
- $Y_iY_j = Y_jY_i$ *for all* $1 \leq i, j \leq n$
- $Y_{i+1} = t^{-1}T_iY_iT_i$ *for* $1 \leq i \leq n-1$
- $T_iY_j = Y_jT_i$ *for* $j \notin \{i, i+1\}$

*We will refer to the Y_i as the **Cherednik elements** of $\mathscr{A}_n$. Define the special elements π_n and $\phi_1, \ldots, \phi_{n-1}$ of $\mathscr{A}_n$ by*

- $\pi_n := Y_1T_1 \cdots T_{n-1}$
- $\phi_i := T_iY_i - Y_iT_i$.

We will denote by $Y^{(n)}$ the commutative subalgebra of $\mathscr{A}_n$ generated by $Y_1^{(n)}, \ldots, Y_n^{(n)}$.

We will also use the following alternative presentation of $\mathscr{A}_n$.

Definition 1.4.4. *Define the affine Hecke algebra $\mathscr{A}_n$ to be the $\mathbb{Q}(q,t)$-algebra generated by the elements $T_1, \ldots, T_{n-1}$ and $\theta_1^{\pm 1}, \ldots, \theta_n^{\pm 1}$ subject to the relations*

- $T_1, \ldots, T_{n-1}$ *generate* $\mathscr{H}_n$
- $\theta_i\theta_j = \theta_j\theta_i$ *for all* $1 \leq i, j \leq n$
- $\theta_{i+1} = tT_i^{-1}\theta_iT_i^{-1}$ *for* $1 \leq i \leq n-1$
- $T_i\theta_j = \theta_jT_i$ *for* $j \notin \{i, i+1\}$

*We will refer to the Y_i as the **re-oriented Cherednik elements** of $\mathscr{A}_n$. Define the special elements π_n and $\varphi_1, \ldots, \varphi_{n-1}$ of $\mathscr{A}_n$ by*

- $\pi_n := t^{n-1}\theta_1T_1^{-1} \cdots T_{n-1}^{-1}$
- $\varphi_i := (tT_i^{-1})\theta_i - \theta_i(tT_i^{-1})$.

We will denote by $\theta^{(n)}$ the commutative subalgebra of $\mathscr{A}_n$ generated by $\theta_1^{(n)}, \ldots, \theta_n^{(n)}$.

It is important to note that when converting between the AHA conventions in this paper and those in Dunkl-Luque [12] the standard Cherednik elements Y_i of Dunkl-Luque do **not** align with the θ_i above. In particular, after the appropriate translation into our conventions we have that Y_i are given by $Y_i = t^{-i+1}T_{i-1}\cdots T_1\pi_n T_{n-1}^{-1}\cdots T_i^{-1}$ as opposed to $\theta_i = t^{-(n-i)}T_{i-1}^{-1}\cdots T_1^{-1}\pi_n T_{n-1}\cdots T_i$. The distinction between the standard Cherednik elements Y_i and the reversed orientation Cherednik elements θ_i will be important in Chapter 3 since the latter will yield operators with additional stability properties which the Y_i do not satisfy.

REMARK 3. *We will use the notation $Y_i^{(n)}$ and $Y_i^{(m)}$ to differentiate between the copies of Y_i in $\mathscr{A}_n$ and $\mathscr{A}_m$ for $n \neq m$. We will do similarly for $\theta_i^{(n)}$.*

The following proposition is standard in AHA theory and will be required at many points throughout this paper. We include the proofs of these relations for completeness and to emphasize that we may use intertwiner theory for AHA with the θ_i elements instead of the standard Y_i with only slight differences.

PROPOSITION 1.4.5. *The following relations hold:*

- $\phi_i = T_i(Y_i - Y_{i+1}) + (t-1)Y_{i+1} = (Y_{i+1} - Y_i)T_i + (1-t)Y_{i+1}$ *for* $1 \leq i \leq n-1$
- $\phi_i Y_j = Y_{s_i(j)}\phi_i$ *for* $1 \leq i \leq n-1$ *and* $1 \leq j \leq n$
- $\phi_i^2 = (Y_i - tY_{i+1})(Y_{i+1} - tY_i)$
- $\phi_i\phi_{i+1}\phi_i = \phi_{i+1}\phi_i\phi_{i+1}$ *for* $1 \leq i \leq n-2$
- $\phi_i\phi_j = \phi_j\phi_i$ *for* $|i-j| > 1$
- $\varphi_i = tT_i^{-1}(\theta_i - \theta_{i+1}) + (t-1)\theta_{i+1} = (\theta_{i+1} - \theta_i)tT_i^{-1} + (1-t)\theta_{i+1}$ *for* $1 \leq i \leq n-1$
- $\varphi_i\theta_j = \theta_{s_i(j)}\varphi_i$ *for* $1 \leq i \leq n-1$ *and* $1 \leq j \leq n$
- $\varphi_i^2 = (t\theta_i - \theta_{i+1})(t\theta_{i+1} - \theta_i)$
- $\varphi_i\varphi_{i+1}\varphi_i = \varphi_{i+1}\varphi_i\varphi_{i+1}$ *for* $1 \leq i \leq n-2$
- $\varphi_i\varphi_j = \varphi_j\varphi_i$ *for* $|i-j| > 1$.

PROOF. The proofs of the correctness of the above relations are standard but we include them for completeness. We will only give the proofs for the θ-version of the above relations since the Y-version is more standard.

We will proceed by proving each of these relations in the order in which they appear above.

Let $1 \leq i \leq n-1$. Then

$$\begin{aligned}
\varphi_i &= tT_i^{-1}\theta_i - \theta_i(tT_i^{-1}) \\
&= tT_i^{-1}\theta_i - T_i\theta_{i+1} \\
&= tT_i^{-1}\theta_i - (tT_i^{-1} + 1 - t)\theta_{i+1} \\
&= tT_i^{-1}(\theta_i - \theta_{i+1}) + (t-1)\theta_{i+1}.
\end{aligned}$$

By a similar calculation we also get

$$\varphi_i = (\theta_{i+1} - \theta_i)tT_i^{-1} + (1-t)\theta_{i+1}.$$

This can also be written as

$$\varphi_i = (\theta_{i+1} - \theta_i)T_i + (1-t)\theta_i$$

which we will need later in this proof.

Now we see

$$\begin{aligned}
\varphi_i\theta_i &= tT_i^{-1}(\theta_i - \theta_{i+1})\theta_i + (t-1)\theta_{i+1}\theta_i \\
&= tT_i^{-1}\theta_i(\theta_i - \theta_{i+1}) + (t-1)\theta_{i+1}\theta_i \\
&= \theta_{i+1}T_i(\theta_i - \theta_{i+1}) + (t-1)\theta_{i+1}\theta_i \\
&= \theta_{i+1}\left(T_i(\theta_i - \theta_{i+1}) + (t-1)\theta_i\right) \\
&= \theta_{i+1}\left((tT_i^{-1} + 1 - t)(\theta_i - \theta_{i+1}) + (t-1)\theta_i\right) \\
&= \theta_{i+1}\left(tT_i^{-1}(\theta_i - \theta_{i+1}) + (t-1)\theta_{i+1}\right) \\
&= \theta_{i+1}\varphi_i
\end{aligned}$$

and

$$\begin{aligned}
\varphi_i\theta_{i+1} &= tT_i^{-1}(\theta_i - \theta_{i+1})\theta_{i+1} + (t-1)\theta_{i+1}^2 \\
&= (T_i + t - 1)\theta_{i+1}(\theta_i - \theta_{i+1}) + (t-1)\theta_{i+1}^2 \\
&= T_i\theta_{i+1}(\theta_i - \theta_{i+1}) + (t-1)\left(\theta_{i+1}(\theta_i - \theta_{i+1}) + \theta_{i+1}^2\right) \\
&= t\theta_i T_i^{-1}(\theta_i - \theta_{i+1}) + (t-1)\theta_i\theta_{i+1} \\
&= \theta_i\left(tT_i^{-1}(\theta_i - \theta_{i+1}) + (t-1)\theta_{i+1}\right) \\
&= \theta_i\varphi_i.
\end{aligned}$$

For any $j \notin \{i, i+1\}$ it follows since θ_j commutes with both θ_i and T_i that

$$\varphi_i\theta_j = \theta_j\varphi_i.$$

Thus for any $1 \leq j \leq n$

$$\varphi_i\theta_j = \theta_{s_i(j)}\varphi_i.$$

Now we have that

$$\begin{aligned}
\varphi_i^2 &= (tT_i^{-1}\theta_i - \theta_i tT_i^{-1})^2 \\
&= t^2T_i^{-1}\theta_iT_i^{-1}\theta_i - t^2T_i^{-1}\theta_i^2T_i^{-1} - t^2\theta_iT_i^{-2}\theta_i + t^2\theta_iT_i^{-1}\theta_iT_i^{-1} \\
&= t\theta_{i+1}\theta_i - t\theta_{i+1}T_i\theta_iT_i^{-1} - t\theta_i(1 + (t-1)T_i^{-1})\theta_i + t\theta_i\theta_{i+1} \\
&= 2t\theta_i\theta_{i+1} - t\theta_{i+1}(tT_i^{-1} + 1 - t)\theta_iT_i^{-1} - t\theta_i^2 + t(1-t)\theta_iT_i^{-1}\theta_i \\
&= 2t\theta_i\theta_{i+1} - t^2\theta_{i+1}T_i^{-1}\theta_iT_i^{-1} + t(t-1)\theta_i\theta_{i+1}T_i^{-1} - t\theta_i^2 + (1-t)\theta_i\theta_{i+1}T_i \\
&= 2t\theta_i\theta_{i+1} - t\theta_{i+1}^2 - t\theta_i^2 + (1-t)\theta_i\theta_{i+1}(T_i - tT_i^{-1}) \\
&= 2t\theta_i\theta_{i+1} - t\theta_{i+1}^2 - t\theta_i^2 + (1-t)^2\theta_i\theta_{i+1} \\
&= (1+t^2)\theta_i\theta_{i+1} - t\theta_{i+1}^2 - t\theta_i^2 \\
&= (t\theta_i - \theta_{i+1})(t\theta_{i+1} - \theta_i).
\end{aligned}$$

Now suppose $1 \leq i \leq n-2$. By expanding each of the φ_j from right to left using $\varphi_j = (\theta_{j+1} - \theta_j)T_j + (1-t)\theta_j$ and repeatedly applying the relation $\varphi_j \theta_k = \theta_{s_j(k)}\varphi_j$ we find

$$\begin{aligned}\varphi_i\varphi_{i+1}\varphi_i = {} & (\theta_{i+2}-\theta_{i+1})(\theta_{i+2}-\theta_i)(\theta_{i+1}-\theta_i)T_iT_{i+1}T_i + (1-t)\theta_i(\theta_{i+2}-\theta_{i+1})(\theta_{i+2}-\theta_i)T_{i+1}T_i \\ & + (1-t)\theta_{i+1}(\theta_{i+2}-\theta_i)(\theta_{i+1}-\theta_i)T_iT_{i+1} + (1-t)^2\theta_i\theta_{i+1}(\theta_{i+2}-\theta_i)T_i \\ & + (1-t)^2\theta_i\theta_{i+1}(\theta_{i+2}-\theta_i)T_{i+1} \\ & + \left(t(1-t)\theta_i(\theta_{i+2}-\theta_{i+1})(\theta_{i+1}-\theta_i) + (1-t)^3\theta_i^2\theta_{i+1}\right).\end{aligned}$$

Using the same method we also see that

$$\begin{aligned}&\varphi_{i+1}\varphi_i\varphi_{i+1} \\ &= (\theta_{i+1}-\theta_i)(\theta_{i+2}-\theta_i)(\theta_{i+2}-\theta_{i+1})T_{i+1}T_iT_{i+1} + (1-t)\theta_{i+1}(\theta_{i+1}-\theta_i)(\theta_{i+2}-\theta_i)T_iT_{i+1} \\ &+ (1-t)\theta_i(\theta_{i+2}-\theta_i)(\theta_{i+2}-\theta_{i+1})T_{i+1}T_i + (1-t)^2\theta_i\theta_{i+1}(\theta_{i+2}-\theta_i)T_{i+1} \\ &+ (1-t)^2\theta_i\theta_{i+1}(\theta_{i+2}-\theta_i)T_i \\ &+ \left(t(1-t)\theta_i(\theta_{i+1}-\theta_i)(\theta_{i+2}-\theta_{i+1}) + (1-t)^3\theta_i^2\theta_{i+1}\right).\end{aligned}$$

From here we may use the braid relation $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$ and some rearrangement of terms to see $\varphi_i\varphi_{i+1}\varphi_i = \varphi_{i+1}\varphi_i\varphi_{i+1}$.

Lastly, consider $|i-j| > 1$. Since $T_iT_j = T_jT_i$, $T_i\theta_j = \theta_jT_i$, and $\theta_i\theta_j = \theta_j\theta_i$ we readily find that $\varphi_i\varphi_j = \varphi_j\varphi_i$. □

In Chapter 3 we will be interested in AHA modules which are *pulled back* from irreducible finite Hecke representations. To do this we need to define algebra surjections $\mathscr{A}_n \to \mathscr{H}_n$. There are many such choices for these maps but we choose the maps ρ_n defined below carefully so that the AHA modules we consider in this paper satisfy nontrivial stability conditions.

DEFINITION 1.4.6. *Define the $\mathbb{Q}(q,t)$-algebra homomorphism $\rho_n : \mathscr{A}_n \to \mathscr{H}_n$ by*

- $\rho_n(T_i) = T_i$ *for* $1 \leq i \leq n-1$

- $\rho_n(\theta_1) = 1$.

For a $\mathscr{H}_n$-module V we will denote by $\rho_n^*(V)$ the $\mathscr{A}_n$-module with action defined for $v \in V$ and $X \in \mathscr{A}_n$ by $X(v) := \rho_n(X)(v)$.

REMARK 4. *Note that $\rho_n(\pi_n) = t^{n-1}T_1^{-1} \cdots T_{n-1}^{-1}$ and for all $1 \leq i \leq n$, $\rho_n(\theta_i) = \overline{\theta}_i$.*

1.4.3. Double Affine Hecke Algebras.

DEFINITION 1.4.7. *Define the **double affine Hecke algebra** $\mathscr{D}_n$ to be the $\mathbb{Q}(q,t)$-algebra generated by $T_1, \ldots, T_{n-1}$, $X_1^{\pm 1}, \ldots, X_n^{\pm 1}$, and $Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}$ with the following relations:*

(1) $(T_i - 1)(T_i + t) = 0$,
$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$,
$T_i T_j = T_j T_i$, $|i-j| > 1$,

(2) $T_i^{-1} X_i T_i^{-1} = t^{-1} X_{i+1}$,
$T_i X_j = X_j T_i$, $j \notin \{i, i+1\}$,
$X_i X_j = X_j X_i$,

(3) $T_i Y_i T_i = t Y_{i+1}$,
$T_i Y_j = Y_j T_i$, $j \notin \{i, i+1\}$,
$Y_i Y_j = Y_j Y_i$,

(4) $Y_1 T_1 X_1 = X_2 Y_1 T_1$,

(5) $Y_1 X_1 \cdots X_n = q X_1 \cdots X_n Y_1$

Further, define the special element $\widetilde{\pi}_n$ by

$$\widetilde{\pi}_n := X_1 T_1^{-1} \cdots T_{n-1}^{-1}.$$

DEFINITION 1.4.8. *We define the **positive double affine Hecke algebra in type** GL_n, $\mathscr{D}_n^+$, to be the subalgebra of $\mathscr{D}_n$ generated by the elements $T_1, \ldots, T_{n-1}, X_1, \ldots, X_n$ and $\pi_n^{\pm 1}$.*

DEFINITION 1.4.9. *Let $\epsilon^{(n)} \in \mathscr{H}_n$ denote the (normalized) trivial idempotent given by*

$$\epsilon^{(n)} := \frac{1}{[n]_t!} \sum_{\sigma \in \mathfrak{S}_n} t^{\binom{n}{2} - \ell(\sigma)} T_\sigma$$

where $[n]_t! := \prod_{i=1}^n (\frac{1-t^i}{1-t})$. The positive spherical double affine Hecke algebra $\mathscr{D}_n^{sph}$ is the non-unital subalgebra of $\mathscr{D}_n^+$ given by $\mathscr{D}_n^{sph} := \epsilon^{(n)} \, \mathscr{D}_n^+ \, \epsilon^{(n)}$.

The element $\epsilon^{(n)} := \frac{1}{[n]_t!} \sum_{\sigma \in \mathfrak{S}_n} t^{\binom{n}{2} - \ell(\sigma)} T_\sigma \in \mathscr{H}_n$ is uniquely determined by the following properties:

- $\epsilon^{(n)} \neq 0$ (non-zero)
- $(\epsilon^{(n)})^2 = \epsilon^{(n)}$ (idempotent)
- $\epsilon^{(n)} T_i = T_i \epsilon^{(n)}$ for all $1 \leq i \leq n-1$ (central)
- $T_i \epsilon^{(n)} = \epsilon^{(n)}$ (trivial-like).

We will use without proof that $\epsilon^{(n)}$ as defined in Definition 1.4.9 satisfies these properties but it is straightforward to check this using the defining relations of $\mathscr{H}_n$. Since $(\epsilon^{(n)})^2 = \epsilon^{(n)}$ we see that $\mathscr{D}_n^{\mathrm{sph}}$ is a unital algebra with unit $\epsilon^{(n)}$. The algebra $\mathscr{D}_n^{\mathrm{sph}}$ contains both of the subalgebras $\mathbb{Q}(q,t)[X_1, \ldots, X_n]^{\mathfrak{S}_n} \epsilon^{(n)}$ and $\mathbb{Q}(q,t)[\theta_1^{\pm 1}, \ldots, \theta_n^{\pm 1}]^{\mathfrak{S}_n} \epsilon^{(n)}$.

We may use $\epsilon^{(n)}$ to generate modules for the spherical DAHA. Given any $\mathscr{D}_n$-module V the space $\epsilon^{(n)}(V)$ is naturally a $\mathscr{D}_n^{\mathrm{sph}}$-module. In the standard picture of Cherednik theory the standard polynomial representation of $\mathscr{D}_n^+$ on $\mathbb{Q}(q,t)[x_1, \ldots, x_n]$ is symmetrized using $\epsilon^{(n)}$ to yield the standard symmetric polynomial representation of $\mathscr{D}_n^{\mathrm{sph}}$ on $\mathbb{Q}(q,t)[x_1, \ldots, x_n]^{\mathfrak{S}_n}$.

Remark 5. *We will use without proof the standard result that $\mathscr{D}_n$ is a free right $\mathscr{A}_n$ module with basis $\{X^\alpha\}_{\alpha \in \mathbb{Z}^n}$. This follows from the standard PBW-type result for DAHA. Importantly, for our purposes, this implies that for any $\mathscr{A}_n$-module V with $\mathbb{Q}(q,t)$-basis $\{v_i\}_{i \in I}$, the induced module*

$$\mathrm{Ind}_{\mathscr{A}_n}^{\mathscr{D}_n} V := \mathscr{D}_n \otimes_{\mathscr{A}_n} V$$

has $\mathbb{Q}(q,t)$-basis $\{X^\alpha \otimes v_i | \alpha \in \mathbb{Z}^n, i \in I\}$. Similarly, if we consider induction from $\mathscr{A}_n$ to $\mathscr{D}_n^+$ instead then we find that

$$\mathrm{Ind}_{\mathscr{A}_n}^{\mathscr{D}_n^+} V := \mathscr{D}_n^+ \otimes_{\mathscr{A}_n} V$$

has $\mathbb{Q}(q,t)$-basis $\{X^\alpha \otimes v_i | \alpha \in \mathbb{Z}_{\geq 0}^n, i \in I\}$.

Definition 1.4.10. *The **standard representation** of $\mathscr{D}_n$ is given by the following action on $\mathscr{P}_n$:*

- $T_i f(x_1, \ldots, x_n) = s_i f(x_1, \ldots, x_n) + (1-t) x_i \frac{1-s_i}{x_i - x_{i+1}} f(x_1, \ldots, x_n)$
- $X_i f(x_1, .., x_n) = x_i f(x_1, \ldots, x_n)$
- $\pi_n f(x_1, \ldots, x_n) = f(x_2, x_3, \ldots, x_n, q x_1)$.

*Under this action the T_i operators are known as the **Demazure-Lusztig operators**. The action of the elements $Y_1, \ldots, Y_n \in \mathscr{D}_n$ are called **Cherednik operators**.*

Remark 6. *For q, t generic $\mathscr{P}_n$ is known to be a faithful representation of $\mathscr{D}_n$. It is straightforward to check that $\mathscr{P}_n^+$ is a $\mathscr{D}_n^+$-submodule of $\mathscr{P}_n$. Further, we may identify the $\mathscr{D}_n$-module $\mathscr{P}_n$ with*

$$\mathscr{P}_n \cong \mathrm{Ind}_{\mathscr{A}_n}^{\mathscr{D}_n}(1, t^{-1}, \ldots, t^{-(n-1)})$$

where $(1, t^{-1}, \ldots, t^{-(n-1)})$ is the 1-dimensional $\mathscr{A}_n$-module determined by

- $T_i \to 1$
- $Y_i \to t^{-i+1}$.

Similarly,

$$\mathscr{P}_n^+ \cong \mathrm{Ind}_{\mathscr{A}_n}^{\mathscr{D}_n^+}(1, t^{-1}, \ldots, t^{-(n-1)}).$$

As it turns out, the polynomial representation $\mathscr{P}_n$ of DAHA admits a basis of simultaneous eigenvectors for the Cherednik operators $Y_i^{(n)}$.

Definition 1.4.11. *The **non-symmetric Macdonald polynomials** (for GL_n) are a family of Laurent polynomials $E_\mu \in \mathscr{P}_n$ for $\mu \in \mathbb{Z}^n$ uniquely determined by the following:*

- *Triangularity: Each E_μ has a monomial expansion of the form $E_\mu = x^\mu + \sum_{\lambda<\mu} a_\lambda x^\lambda$*
- *Weight Vector: Each E_μ is a weight vector for the operators $Y_1^{(n)}, \ldots, Y_n^{(n)} \in \mathscr{H}_n$.*

Importantly, the set $\{E_\mu | \mu \in \mathbb{Z}^n\}$ is a basis for $\mathscr{P}_n$ with distinct $Y^{(n)}$ weights. For $\mu \in \mathbb{Z}^n$, E_μ is homogeneous with degree $\mu_1 + \ldots + \mu_n$. Further, the set of E_μ corresponding to $\mu \in \mathbb{Z}_{\geq 0}^n$ gives a basis for $\mathscr{P}_n^+$.

Remark 7. *Given a family of commuting operators $\{y_i : i \in I\}$ and a weight vector v we denote its weight by the function $\alpha : I \to \mathbb{Q}(q,t)$ such that $y_i v = \alpha(i) v$. We sometimes denote α as $(\alpha_1, \alpha_2, \ldots)$.*

For $\mu \in \mathbb{Z}^n$ we will write $\alpha_\mu := (\alpha_\mu(1), \ldots, \alpha_\mu(n))$ for the $Y^{(n)}$ weight of E_μ. We have the following explicit combinatorial description for the α_μ:

Proposition 1.4.12. *For $1 \leq i \leq n$ and $\mu \in \mathbb{Z}^n$*

$$Y_i^{(n)} E_\mu = q^{\mu_i} t^{1-\beta_\mu(i)} E_\mu$$

where

$$\beta_\mu(i) := \#\{j : 1 \leq j \leq i\ , \mu_j \leq \mu_i\} + \#\{j : i < j \leq n\ , \mu_i > \mu_j\}.$$

Proof. [19] □

In practice one may generate the non-symmetric Macdonald polynomials recursively using the ***Knop-Sahi Relations***.

Proposition 1.4.13. *For $\mu \in \mathbb{Z}^n$ we have the following relations:*

- $E_{(1+\mu_n,\mu_1,\ldots,\mu_{n-1})} = q^{-\mu_n} x_1 \pi_n E_\mu$
- *If* $s_i(\mu) > \mu$

$$E_{s_i(\mu)} = \left(T_i + \frac{(1-t)\alpha_\mu(i+1)}{\alpha_\mu(i) - \alpha_\mu(i+1)}\right) E_\mu = \left(T_i - \frac{1-t}{1 - \frac{\alpha_\mu(i)}{\alpha_\mu(i+1)}}\right) E_\mu.$$

Proof. [19] □

Example. *Beginning with $E_{(0,0,0)} = 1$ we may use the Knop-Sahi relations iteratively to construct $E_{(2,0,0)}$. We start with*

$$E_{(1,0,0)} = x_1 \pi_3 E_{(0,0,0)} = x_1.$$

Now we use the T_i operators:

- $E_{(0,1,0)} = \left(T_1 - \frac{1-t}{1-qt^{-2}}\right) x_1 = x_2 + \frac{1-t}{1-q^{-1}t^2} x_1$
- $E_{(0,0,1)} = \left(T_2 - \frac{1-t}{1-qt^{-1}}\right) \left(x_2 + \frac{1-t}{1-q^{-1}t^2} x_1\right) = x_3 + \frac{1-t}{1-q^{-1}t}(x_1 + x_2).$

Lastly, we find

$$E_{(2,0,0)} = q^{-1} x_1 \pi_3 \left(x_3 + \frac{1-t}{1-q^{-1}t}(x_1 + x_2)\right) = x_1^2 + q^{-1} \frac{1-t}{1-q^{-1}t} x_1 (x_2 + x_3).$$

The weights of these non-symmetric Macdonald polynomials are given as

- $\alpha_{(0,0,0)} = (1, t^{-1}, t^{-2})$
- $\alpha_{(1,0,0)} = (qt^{-2}, 1, t^{-1})$
- $\alpha_{(0,1,0)} = (1, qt^{-2}, t^{-1})$
- $\alpha_{(0,0,1)} = (1, t^{-1}, qt^{-2})$
- $\alpha_{(2,0,0)} = (q^2 t^{-2}, 1, t^{-1})$.

1.5. Elliptic Hall Algebra

Here we recall some basic facts about the elliptic Hall algebra which we will need in chapter 3.

Definition 1.5.1. *For $\ell \in \mathbb{Z} \setminus \{0\}$, $r > 0$ define the special elements $P_{0,\ell}^{(n)}, P_{r,0}^{(n)} \in \mathscr{D}_n^{sph}$ by*

- $P_{0,\ell}^{(n)} = \epsilon^{(n)} \left(\sum_{i=1}^n \theta_i^\ell\right) \epsilon^{(n)}$
- $P_{r,0}^{(n)} = q^r \epsilon^{(n)} \left(\sum_{i=1}^n X_i^r\right) \epsilon^{(n)}$.

Theorem 1.5.2. *[34] The elements $P_{0,\ell}^{(n)}, P_{r,0}^{(n)}$ for $\ell \in \mathbb{Z} \setminus \{0\}$, $r > 0$ generate $\mathscr{D}_n^{sph}$ as a $\mathbb{Q}(q,t)$-algebra. There is a unique $\mathbb{Z}_{\geq 0} \times \mathbb{Z}$ grading on $\mathscr{D}_n^{sph}$ determined by*

- $\deg(P_{0,\ell}^{(n)}) = (0, \ell)$
- $\deg(P_{r,0}^{(n)}) = (r, 0)$.

There is a graded algebra surjection $\mathscr{D}_{n+1}^{sph} \to \mathscr{D}_n^{sph}$ determined for $\ell \in \mathbb{Z} \setminus \{0\}$, $r > 0$ by $P_{0,\ell}^{(n+1)} \to P_{0,\ell}^{(n)}$ and $P_{r,0}^{(n+1)} \to P_{r,0}^{(n)}$.

The existence of the $\mathbb{Z}_{\geq 0} \times \mathbb{Z}$-graded algebra surjections $\mathscr{D}_{n+1}^{\text{sph}} \to \mathscr{D}_n^{\text{sph}}$ allows for the following definition.

Definition 1.5.3. *[34] The **positive elliptic Hall algebra** $\mathscr{E}^+$ is the stable limit of the $\mathbb{Z}_{\geq 0} \times \mathbb{Z}$-graded algebras $\mathscr{D}_n^{sph}$ with respect to the maps $\mathscr{D}_{n+1}^{sph} \to \mathscr{D}_n^{sph}$. For $\ell \in \mathbb{Z} \setminus \{0\}$, $r > 0$ define the special elements of $\mathscr{E}^+$, $P_{0,\ell} := \lim_n P_{0,\ell}^{(n)}$ and $P_{r,0} := \lim_n P_{r,0}^{(n)}$.*

The positive elliptic Hall algebra contains elements $P_{(a,b)}$ for $(a,b) \in \mathbb{N} \times \mathbb{Z}$ which may be defined using repeated commutators of the elements $P_{0,\ell}, P_{r,0}$. For example, $P_{(1,1)} = [P_{(0,1)}, P_{(1,0)}]$. We will not require an explicit description of these elements for the purposes of this paper. Further, we will not require knowledge of the full elliptic Hall algebra $\mathcal{E}$ which is obtained as the Drinfeld double of $\mathcal{E}^+$ with respect to a certain Hopf pairing. In the standard Macdonald theory picture, we can realize the action of the full EHA on the ring of symmetric functions Λ using multiplication operators $p_r^\bullet$, skewing operators $p_r^\perp$, and Macdonald operators $p_\ell[\Delta]$ roughly corresponding to the elements $P_{(r,0)}, P_{(-r,0)}, P_{(0,\ell)}$ respectively.

Remark 8. *We will be considering the $\mathbb{Z}_{\geq 0}$-grading on $\mathcal{E}^+$ obtained by the specialization $(a,b) \to a$ i.e. for $r > 0$ and $\ell \in \mathbb{Z} \setminus \{0\}$*

- $\deg(P_{0,\ell}) = 0$
- $\deg(P_{r,0}) = r$.

When we refer to a $\mathcal{E}^+$-module V as graded we are referring to the $\mathbb{Z}_{\geq 0}$-grading on $\mathcal{E}^+$.

1.6. Double Dyck Path Algebra

The ***Double Dyck Path Algebra*** $\mathbb{A}_{q,t}$, introduced by Carlsson and Mellit [8], is a quiver path algebra with vertices indexed by non-negative integers with the following edge operators:

- $d_+, d_+^* : k \to k+1$
- $T_1, ..., T_{k-1} : k \to k$
- $d_- : k+1 \to k$.

The full set of relations for $\mathbb{A}_{q,t}$ are omitted here because they will not be required but they can be found in [8]. In order to match the parameter conventions in Ion and Wu's work [26] we will often consider $\mathbb{A}_{t,q}$ as opposed to $\mathbb{A}_{q,t}$ formed by simply swapping q and t in the defining relations of $\mathbb{A}_{q,t}$. Here we highlight a few notable relations of $\mathbb{A}_{t,q}$ which will be required later:

- The loops $T_1, ..., T_{k-1}$ at vertex $k \geq 2$ generate a type A finite Hecke algebra $\mathscr{H}_n$
- $d_-^2 T_{k-1} = d_-^2$ starting at vertex $k \geq 2$
- $T_i d_- = d_- T_i$ at vertex k for $1 \leq i \leq k-2$
- $z_i d_- = d_- z_i$ at vertex k for $1 \leq i \leq k-1$ where $z_1 := \frac{t^k}{1-t}[d_+^*, d_-]T_{k-1}^{-1} \cdots T_1^{-1}$ and $z_{i+1} = t^{-1} T_i z_i T_i$.

Although we did not give a full description of $\mathbb{A}_{q,t}$ we will require in Chapter 4 a detailed description of the relations of the highly related algebra $\mathbb{B}_{q,t}$.

Definition 1.6.1. *[7] The algebra $\mathbb{B}_{q,t}$ is generated by a collection of orthogonal idempotents labelled by $\mathbb{Z}_{\geq 0}$, generators d_+, d_-, T_i, and z_i modulo relations:*

$(T_i - 1)(T_i + q) = 0$	$T_1 d_+^2 = d_+^2$
$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$	$d_+ T_i = T_{i+1} d_+$ *for* $1 \leq i \leq k-1$
$T_i T_j = T_j T_i$ *if* $\lvert i-j \rvert > 1$	$q\varphi d_- = d_- \varphi T_{k-1}$ *for* $k \geq 2$
$T_i^{-1} z_{i+1} T_i^{-1} = q^{-1} z_i$ *for* $1 \leq i \leq k-1$	$T_1 \varphi d_+ = q d_+ \varphi$ *for* $k \geq 1$
$z_i T_j = T_j z_i$ *if* $i \notin \{j, j+1\}$	$z_i d_- = d_- z_i$
$z_i z_j = z_j z_i$ *for* $1 \leq i, j \leq k$	$d_+ z_i = z_{i+1} d_+$
$d_-^2 T_{k-1} = d_-^2$ *for* $k \geq 2$	$z_1(q d_+ d_- - d_- d_+) = qt(d_+ d_- - d_- d_+) z_k$ *for* $k \geq 1$
$d_- T_i = T_i d_-$ *for* $1 \leq i \leq k-2$	

where $\varphi := \frac{1}{q-1}[d_+, d_-]$.

We will consider $\mathbb{B}_{q,t}$ as a graded algebra with grading determined by

- $\deg(T_i) = \deg(z_i) = \deg(d_-) = 0$
- $\deg(d_+) = 1$.

For $n \geq 0$ define $\mathbb{B}_{q,t}^{(n)}$ to be the subalgebra of $\mathbb{B}_{q,t}$ given by only considering T_i, z_i, d_-, d_+ between the idempotents labelled by $\{0, \dots, n\}$.

Remark 9. *The graded algebras $\mathbb{B}_{q,t}^{(n)}$ naturally form a directed system with*

$$\mathbb{B}_{\mathrm{q,t}} = \lim_{\to} \mathbb{B}_{q,t}^{(n)}.$$

Definition 1.6.2. *[7] Let $V_\bullet^{\mathrm{pol}} = \bigoplus_{k \geq 0} V_k^{\mathrm{pol}} := \bigoplus_{k\geq 0} \mathbb{Q}(q,t)[y_1, \dots, y_k] \otimes \Lambda$. Define an action on $V_\bullet^{\mathrm{pol}}$ by the following operators given for $F \in V_k^{\mathrm{pol}}$ by*

- $T_i F := \frac{(q-1)y_{i+1}F + (y_{i+1} - qy_i)s_i(F)}{y_{i+1} - y_i}$ *for* $1 \leq i \leq k-1$
- $d_+ F := -T_1 \cdots T_k F[X + (q-1)y_{k+1}]$
- $d_- F := F[X - (q-1)y_k] \operatorname{Exp}[-y_k^{-1}X]|_{y_k^{-1}}$
- $z_k F := T_{k-1} \cdots T_1 F[X + (1-q)ty_1 - (q-1)u, y_2, \dots, y_k, u] \operatorname{Exp}[u^{-1}ty_1 - u^{-1}X]|_{u^0}$
- $z_i := q^{-1} T_i^{-1} z_{i+1} T_i^{-1}$ *for* $1 \leq i \leq k-1$

*where $|_{y_k^{-1}}$ represents taking the coefficient of y_k^{-1}. Here we are using plethystic notation. This representation $V_\bullet^{\mathrm{pol}}$ of $\mathbb{B}_{q,t}$ is called the **polynomial representation**.*

Note that the signs ± 1 of the operators d_-, d_+ are reversed in [7]. This choice is made to align with the conventions in [26] and [31] and does not make a substantial difference in the underlying representation.

Remark 10. *Carlsson-Gorsky-Mellit also construct an action of $\mathbb{B}_{q,t}$ on the larger space $W_\bullet^{\mathrm{pol}} := \bigoplus_{k=0}^{\infty} (y_1 \cdots y_k)^{-1} V_k^{\mathrm{pol}}$. The space $V_\bullet^{\mathrm{pol}}$ is isomorphic to the equivariant K-theory of the parabolic flag Hilbert schemes of points in $\mathbb{C}^2$ and the larger space $W_\bullet^{\mathrm{pol}}$ is defined in order to relate the original $\mathbb{A}_{q,t}$ polynomial representation as defined by Carlsson-Mellit [8] to the $\mathbb{A}_{q,t}$ polynomial representation constructed in [7]. We will use the space $W_\bullet^{\mathrm{pol}}$ briefly to relate the $\mathbb{B}_{q,t}$ action on $V_\bullet^{\mathrm{pol}}$ to the work of Ion-Wu.*

1.7. Stable-Limits

1.7.1. Classical Stability. We will write $\deg(v), \deg(r)$ for the degree of either a homogeneous vector v in a graded vector space or a homogeneous element r of a graded ring. For the following definitions for a graded vector space V we will write $V(d)$ for the degree $d \geq 0$ homogeneous component of V. If R is a graded ring then we will write $R-\text{Mod}$ for the category of consisting of graded left R modules as the objects and with degree-preserving homomorphisms (homogeneous maps) as the morphisms.

We now review some formalities regarding stable-limits of spaces and modules.

DEFINITION 1.7.1. *Let $(V^{(n)})_{n\geq n_0}$ be a sequence of graded vector spaces and suppose $(\Pi^{(n)} : V^{(n+1)} \to V^{(n)})_{n\geq n_0}$ is a family of degree preserving maps. The **stable-limit** of the spaces $(V^{(n)})_{n\geq n_0}$ with respect to the maps $(\Pi^{(n)})_{n\geq n_0}$ is the graded vector space $\widetilde{V} := \lim_{\leftarrow} V^{(n)}$ constructed as follows: For each $d \geq 0$ we define*

$$\widetilde{V}(d) := \{(v_n)_{n\geq n_0} \in \prod_{n\geq n_0} V^{(n)}(d) \mid \Pi^{(n)}(v_{n+1}) = v_n\}$$

and set

$$\widetilde{V} := \bigoplus_{d\geq 0} \widetilde{V}(d).$$

LEMMA 1.7.2. *Let $(R^{(n)})_{n\geq n_0}$ be a sequence of graded rings with injective graded ring homomorphisms $(\iota^{(n)} : R^{(n)} \to R^{(n+1)})_{n\geq n_0}$. We will identify $R^{(n)}$ with its image $\iota_n(R^{(n)}) \subset R^{(n+1)}$. We write $\widetilde{R} = \lim_{\to} R^{(n)}$ for the direct limit of the rings $R^{(n)}$. Suppose $(V^{(n)})_{n\geq n_0}$ is a sequence of graded vector spaces with each $V^{(n)}$ a graded $R^{(n)}$ module and $(\Pi^{(n)} : V^{(n+1)} \to V^{(n)})_{n\geq n_0}$ a sequence of degree-preserving maps with each $\Pi^{(n)}$ a graded $R^{(n)}$ module homomorphism. Then the following defines a graded $\widetilde{R}$ module structure on $\widetilde{V} := \lim_{\leftarrow} V^{(n)}$: For $r \in \widetilde{R}$ and $v \in \widetilde{V}$ with $r \in R_N$ and $v = (v_n)_{n\geq n_0}$, define $r(v) \in \widetilde{V}$ by*

$$r(v) = \left(\Pi^{(n_0)} \cdots \Pi^{(N-1)}(r(v_N)), \ldots, \Pi^{(N-1)}(r(v_N)), r(v_N), r(v_{N+1}), r(v_{N+2}), \ldots\right).$$

REMARK 11. *It is a straightforward exercise to check that the action defined above actually yields a graded $\widetilde{R}$ module structure on $\lim_{\leftarrow} V^{(n)}$. We leave this to the reader. We call $\lim_{\leftarrow} V^{(n)}$ the **stable-limit module** corresponding to the sequence $(V^{(n)})_{n\geq n_0}$ and the maps $(\Pi^{(n)})_{n\geq n_0}$. Notice that this construction is functorial. Suppose $(W^{(n)})_{n\geq n_0}$ is another sequence of graded vector spaces with each $W^{(n)}$ a graded $R^{(n)}$ module and $(\Psi^{(n)} : W^{(n+1)} \to W^{(n)})_{n\geq n_0}$ a sequence of degree-preserving maps with each $\Psi^{(n)}$ a graded $R^{(n)}$ module homomorphism and $\phi = (\phi^{(n)})_{n\geq n_0}$ is a family of graded*

$R^{(n)}$ module maps $\phi^{(n)} : V^{(n)} \to W^{(n)}$ such that for all $n \geq n_0$

$$\phi^{(n)}\Pi^{(n)} = \Psi^{(n)}\phi^{(n+1)}.$$

Then ϕ determines a graded $\widetilde{R}$ module homomorphism $\widetilde{\phi} : \lim_{\leftarrow} V^{(n)} \to \lim_{\leftarrow} W^{(n)}$ given by

$$\widetilde{\phi}(v) := (\phi^{(n)}(v_n))_{n \geq n_0}.$$

Remark 12. *The stable-limit spaces $\widetilde{V} = \lim_{\leftarrow} V^{(n)}$ may be zero even if each $V^{(n)}$ is nonzero. However, if each $V^{(n)}$ is nonzero and the maps $\Pi^{(n)}$ are surjective then $\widetilde{V}$ is nonzero.*

1.7.2. Ion-Wu Stability. Ion-Wu define the following generalization of classical stable-limits by utilizing the t-adic topology on $\mathbb{Q}(q,t)$.

Definition 1.7.3. *[26] Let $(f_m)_{m \geq 1}$ be a sequence of polynomials with $f_m \in \mathscr{P}_m^+$. Then the sequence $(f_m)_{m \geq 1}$ is **convergent** if there exist some N and auxiliary sequences $(h_m)_{m \geq 1}$, $(g_m^{(i)})_{m \geq 1}$, and $(a_m^{(i)})_{m \geq 1}$ for $1 \leq i \leq N$ with $h_m, g_m^{(i)} \in \mathscr{P}_m^+$, $a_m^{(i)} \in \mathbb{Q}(q,t)$ with the following properties:*

- *For all m, $f_m = h_m + \sum_{i=1}^{N} a_m^{(i)} g_m^{(i)}$.*
- *The sequences $(h_m)_{m \geq 1}$, $(g_m^{(i)})_{m \geq 1}$ for $1 \leq i \leq N$ converge in $\mathscr{P}_\infty^+$ with limits $h, g^{(i)}$ respectively. That is to say, $\Xi_m(h_{m+1}) = h_m$ and $\Xi_m(g_{m+1}^{(i)}) = g_m^{(i)}$ for all $1 \leq i \leq N$ and $m \geq 1$. Further, we require $g^{(i)} \in \mathscr{P}_{as}^+$.*
- *The sequences $a_m^{(i)}$ for $1 \leq i \leq N$ converge with respect to the t-adic topology on $\mathbb{Q}(q,t)$ with limits $a^{(i)}$ which are required to be in $\mathbb{Q}(q,t)$.*

The sequence is said to have a limit given by $\lim_m f_m = h + \sum_{i=1}^{N} a^{(i)} g^{(i)}$.

This definition of convergence is a mix of both the stronger topology arising from the inverse system given by the maps Ξ_m and the t-adic topology arising from the ring $\mathbb{Q}(q,t)$. It is important to note that part of the above definition requires convergent sequences to always be written as a finite sum of fixed length with terms that converge independently.

Here we list a few instructive examples of convergent sequences and their limits:

- $\lim_m t^m = 0$
- $\lim_m 1 + \ldots + t^m = \frac{1}{1-t}$
- $\lim_m \frac{1}{q^2 - t^m}(x_3^2 + \ldots + x_m^2) = q^{-2} p_2[x_3 + \ldots]$.

Remark 13. *In this thesis we will be entirely concerned with convergent sequences* $(f_m)_{m\geq 1}$ *with almost symmetric limits* $\lim_m f_m \in \mathscr{P}^+_{as}$*. In this case it follows readily from definition that each of these convergent sequences necessarily will have the form*

$$f_m(x_1, \ldots, x_m) = \sum_{i=1}^{N} c_i^{(m)} x^{\mu^{(i)}} F_i[x_1 + \ldots + x_m]$$

where $N \geq 1$ *is fixed,* $c_i^{(m)}$ *are convergent sequences of scalars with* $\lim_m c_i^{(m)} \in \mathbb{Q}(q,t)$*,* F_i *are symmetric functions, and* $\mu^{(i)}$ *are compositions. Here we will consider* $x^{\mu^{(i)}} = 0$ *in* $\mathscr{P}_m$ *whenever* $\ell(\mu^{(i)}) > m$*.*

Definition 1.7.4. *[26] For* $m \geq 1$ *suppose* A_m *is an operator on* $\mathscr{P}^+_m$*. The sequence* $(A_m)_{m\geq 1}$ *of operators is said to* ***converge*** *if for every* $f \in \mathscr{P}^+_{as}$ *the sequence* $(A_m(\Xi_m(f)))_{m\geq 1}$ *converges to an element of* $\mathscr{P}^+_{as}$*. From [26] the corresponding operator on* $\mathscr{P}^+_{as}$ *given by* $A(f) := \lim_m A_m(\Xi_m(f))$ *is well defined and said to be the limit of the sequence* $(A_m)_{m\geq 1}$*. In this case we will simply write* $A = \lim_m A_m$*.*

There are two important examples of convergent operator sequences which will be relevant for the rest of this paper. For all $i \geq 1$ and $m \geq 1$ let $X_i^{(m)}$ denote the operator on $\mathscr{P}^+_m$ given by 0 if $m < i$ and by $X_i^{(m)} f = x_i f$ if $i \leq m$. Similarly for $i \geq 1$ and $m \geq 1$ let $T_i^{(m)}$ denote the operator on $\mathscr{P}^+_m$ given by 0 if $m-1 < i$ and by $T_i f = s_i f + (1-t)x_i \frac{f - s_i f}{x_i - x_{i+1}}$ if $i \leq m-1$. Then for all $i \geq 1$ it is immediate from definition that the sequences $(X_i^{(m)})_{m\geq 1}$ and $(T_i^{(m)})_{m\geq 1}$ converge to operators X_i and T_i respectively on $\mathscr{P}^+_{as}$. Further, their corresponding actions are given for $f \in \mathscr{P}^+_{as}$ simply by

- $X_i(f) = x_i f$
- $T_i(f) = s_i f + (1-t)x_i \frac{f - s_i f}{x_i - x_{i+1}}$.

The following important technical proposition of Ion and Wu will be used repeatedly in this paper.

Proposition 1.7.5 (Prop. 6.21 [26]). *If* $A = \lim_m A_m$ *and* $f = \lim_m f_m$ *are limit operators and limit functions respectively then* $A(f) = \lim_m A_m(f_m)$*.*

This is a sort of continuity statement for convergent sequences of operators. The utility of the above proposition is that for an operator arising as the limit of finite variable operators, $A = \lim_m A_m$ say, we can use ***any*** sequence $(f_m)_{m\geq 1}$ converging to $f \in \mathscr{P}^+_{as}$ in order to calculate $A(f)$.

It is easy to verify the following proposition using Proposition 1.7.5.

Proposition 1.7.6. *[26] If $A = \lim_m A_m$ and $B = \lim_m B_m$ then $AB = \lim_m A_m B_m$.*

Proof. Let $f \in \mathscr{P}_{as}^{+}$. Then $(B_m(\Xi_m(f)))_{m \geq 1}$ converges to $B(f)$ and thus

$$
\begin{aligned}
AB(f) &= A(B(f)) \\
&= \lim_m A_m(B_m(\Xi_m(f))) \\
&= \lim_m (A_m B_m)(\Xi_m(f))).
\end{aligned}
$$

Therefore, $AB = \lim_m A_m B_m$. $\square$

CHAPTER 2

Stable-Limit Non-Symmetric Macdonald Functions

2.1. Introduction

The Shuffle Conjecture [20], now the Shuffle Theorem [8], is a combinatorial statement regarding the Frobenius character, $\mathcal{F}_{R_n}$, of the diagonal coinvariant algebra R_n which generalizes the coinvariant algebra arising from the geometry of flag varieties. The conjecture built on the work of many people during the 1990s, including but not limited to Bergeron, Garsia, Haiman, and Tesler [4] [16] [5]. The following explicit formula is due to Haiman [24]

$$\mathcal{F}_{R_n}(X;q,t) = (-1)^n \nabla e_n[X]$$

where the operator ∇ is a diagonalizable operator on symmetric functions prescribed by its action on the modified Macdonald symmetric functions $\widetilde{H}_\mu$ as

$$\nabla \widetilde{H}_\mu = \widetilde{H}_\mu[-1] \cdot \widetilde{H}_\mu.$$

The original conjecture of Haglund, Haiman, Loehr, Remmel, and Ulyanov [20] states the following:

Theorem 2.1.1 (Shuffle Theorem). *[8]*

$$(-1)^n \nabla e_n[X] = \sum_{\pi} \sum_{w \in \mathrm{WP}_\pi} t^{\mathrm{area}(\pi)} q^{\mathrm{dinv}(\pi,w)} x_w.$$

In the above, π ranges over the set of Dyck paths of length n and WP_π is the set of word parking functions corresponding to π. The values $\mathrm{area}(\pi)$ and $\mathrm{dinv}(\pi,w)$ are certain statistics corresponding to π and $w \in \mathrm{WP}_\pi$.

In [8], Carlsson and Mellit prove the Compositional Shuffle Conjecture of Haglund, Morse, and Zabrocki [21], a generalization of the original Shuffle Conjecture. Carlsson and Mellit construct and investigate a quiver path algebra called the Double Dyck Path algebra $\mathbb{A}_{q,t}$. They construct a representation of $\mathbb{A}_{q,t}$, called the standard representation, built on certain mixed symmetric and non-symmetric polynomial algebras with actions from Demazure-Lusztig operators, Hall-Littlewood creation operators, and plethysms. The Compositional Shuffle Theorem falls out after a rich understanding of the standard representation is developed. Later analysis done by Carlsson, Gorsky, and Mellit [7] showed that in fact $\mathbb{A}_{q,t}$ occurs naturally in the context of equivariant cohomology of Hilbert schemes.

Recent work by Ion and Wu [26] has solidified the links between the work of Carlsson and Mellit on $\mathbb{A}_{q,t}$ and the representation theory of double affine Hecke algebras. Ion and Wu introduce the $^{+}$stable-limit double affine Hecke algebra $\mathscr{H}^{+}$ along with a representation of $\mathscr{H}^{+}$ on the space of almost-symmetric functions, $\mathscr{P}_{as}^{+}$, from which one can recover the standard $\mathbb{A}_{q,t}$ representation. The main obstruction in making a stable-limit theory for the double affine Hecke algebras is the lack of an inverse/directed-limit system of the double affine Hecke algebras in the traditional sense. Ion and Wu get around this obstruction by introducing a new notion of convergence (Defn. 1.3.12) for sequences of polynomials with increasing numbers of variables along with limit versions of the standard Cherednik operators defined by this convergence.

Central to the study of the standard Cherednik operators are the non-symmetric Macdonald polynomials. The non-symmetric Macdonald polynomials in full generality were introduced first by Cherednik [9] in the context of proving the Macdonald constant-term conjecture. The introduction of the double affine Hecke algebra, along with the non-symmetric Macdonald polynomials by Cherednik, constituted a significant development in representation theory. They serve as a non-symmetric counterpart to the symmetric Macdonald polynomials introduced by Macdonald as a q,t-analog of Schur functions. Further, they give an orthogonal basis of the polynomial representation consisting of weight vectors for the Cherednik operators. The spectral theory of non-symmetric Macdonald polynomials is well understood using the combinatorics of affine Weyl groups. The correct choice of symmetrization applied to a non-symmetric Macdonald polynomial will yield their symmetric counterpart. The type A symmetric Macdonald polynomials are a remarkable basis for symmetric polynomials simultaneously generalizing many other well studied bases which can be recovered by appropriate specializations of values for q and t. The aforementioned modified Macdonald functions $\widetilde{H}_{\mu}$ can be obtained via a plethystic transformation from the symmetric Macdonald polynomials in sufficiently many variables.

It is natural to seek a stable-limit extension for the non-symmetric Macdonald polynomials following the methods of Ion and Wu. In particular, does the standard $\mathscr{H}^+$ representation $\mathscr{P}_{as}^+$ have a basis of weight vectors for the limit Cherednik operators $\mathscr{Y}_i$? The first main theorem of this chapter (Theorem 4.2.12) answers this question in the affirmative. In the second main theorem of this chapter (Theorem 2.6.5) we use a new operator Ψ_{p_1}, which commutes with the limit Cherednik operators, to distinguish between $\mathscr{Y}$-weight vectors with the same $\mathscr{Y}$-weight. The operator Ψ_{p_1} is up to a change of variables an extension of Haiman's operator Δ' [22] from Λ to $\mathscr{P}_{as}^+$ (Remark 14). The operator Ψ_{p_1} is a limit of operators from finite variable DAHAs.

At the end of this chapter we will investigate further properties of the stable-limit non-symmetric Macdonald functions. We will construct higher delta operators generalizing Ψ_{p_1} which act diagonally on the stable-limit non-symmetric Macdonald function basis and satisfy many other interesting properties. Lastly, we will give a detailed analysis of the $q = \infty$, $t = 0$ specialization of the stable-limit non-symmetric Macdonald functions which give an almost symmetric analogue of the Schur functions. We will find an explicit combinatorial expansion of these almost symmetric Schur functions and prove some positivity properties. In the process in proving these positivity results we will develop a representation theoretic interpretation of the almost symmetric Schur functions realizing them as limits of characters of representations certain parabolic subgroups of GL_n.

2.1.1. Stable-Limit DAHA of Ion and Wu. As the index n varies, the standard $\mathscr{H}_n$ representations, $\mathscr{P}_n$, fail to form a direct/inverse system of compatible $\mathscr{H}_n$ representations. However, as the authors Ion and Wu investigate in [26], this sequence of representations is compatible enough to allow for the construction of a limiting representation for a new algebra resembling a direct limit of the double affine Hecke algebras of type GL. We will start by giving the definition of this algebra.

Definition 2.1.2. *[26] The* $^+$***stable-limit double affine Hecke algebra*** *of Ion and Wu,* $\mathscr{H}^+$*, is the algebra generated over* $\mathbb{Q}(q,t)$ *by the elements* T_i, X_i, Y_i *for* $i \geq 1$ *satisfying the following relations:*

- *The generators* T_i, X_i *for* $i \in \mathbb{N}$ *satisfy* (1) *and* (2) *of Defn. 1.4.7.*
- *The generators* T_i, Y_i *for* $i \in \mathbb{N}$ *satisfy* (1) *and* (3) *of Defn. 1.4.7.*
- $Y_1T_1X_1 = X_2Y_1T_1$.

Ion and Wu begin their construction of the standard representation of $\mathscr{H}^+$ by noting the following key fact.

Proposition 2.1.3. *[26] For $n \geq 1$*

$$\pi_{n-1} t^n Y_1^{(n)} X_1 = t^{n-1} Y_1^{(n-1)} X_1 \pi_{n-1}.$$

In other words, the action of the operators $t^n Y_1^{(n)}$ and $t^{n-1} Y_1^{(n-1)}$ are compatible on $x_1 \mathscr{P}_n$. As such there exists a limit operator $Y_1^{(\infty)} : x_1 \mathscr{P}_\infty^+ \to x_1 \mathscr{P}_\infty^+$ such that $\pi_n Y_1^{(\infty)} = t^n Y_1^{(n)}$. A crucial idea of Ion and Wu is to extend the action of the operators $t^n Y_1^{(n)}$ on $x_1 \mathscr{P}_n$ to all of $\mathscr{P}_n$ using the previously defined projection $\rho : \mathscr{P}_n \to x_1 \mathscr{P}_n$.

Definition 2.1.4. *[26] Define the operator $\widetilde{Y}_1^{(n)} := \rho \circ t^n Y_1^{(n)}$. For $2 \leq i \leq n$ define $\widetilde{Y}_i^{(n)}$ by requiring $\widetilde{Y}_i^{(n)} = t^{-1} T_{i-1} \widetilde{Y}_{i-1}^{(n)} T_{i-1}$.*

A direct check shows that $\widetilde{Y}_1^{(n)} X_1 = t^n Y_1^{(n)} X_1$ so that $\widetilde{Y}_1^{(n)}$ extends the action of $t^n Y_1^{(n)}$ on $x_1 \mathscr{P}_n$ as desired. The main utility of this specific choice of definition is the following theorem.

Theorem 2.1.5. *[26] The sequence $(\widetilde{Y}_1^{(m)})_{m \geq 1}$ converges to an operator $\mathscr{Y}_1$ on $\mathscr{P}_{as}^+$. Define the operators $\mathscr{Y}_i$ for $i \geq 2$ by $\mathscr{Y}_i := t^{-1} T_{i-1} \mathscr{Y}_{i-1} T_{i-1}$. The operators $\mathscr{Y}_i$ along with the Demazure-Lusztig action of the T_i's and multiplication by the X_i's generate an $\mathscr{H}^+$ action on $\mathscr{P}_{as}^+$.*

In particular, the authors Ion and Wu show that despite the fact that for $1 \leq i \neq j \leq n$, $\widetilde{Y}_i^{(n)} \widetilde{Y}_j^{(n)} \neq \widetilde{Y}_j^{(n)} \widetilde{Y}_i^{(n)}$ the limit Cherednik operators commute:

$$\mathscr{Y}_i \mathscr{Y}_j = \mathscr{Y}_j \mathscr{Y}_i.$$

The action of the $\mathscr{Y}_i$ operators respect the canonical filtration of $\mathscr{P}_{as}^+ = \bigcup_{k \geq 0} \mathscr{P}(k)^+$. For all $n \geq 0$, the operators $\{\mathscr{Y}_1, ..., \mathscr{Y}_n\}$ restrict to operators on the space $\mathscr{P}(n)^+$ whereas the operators $\{\mathscr{Y}_{n+1}, \mathscr{Y}_{n+2}, ...\}$ annihilate $\mathscr{P}(n)^+$. Note that for $n = 0$, $\mathscr{P}(0)^+ = \Lambda$ so all of the operators $\mathscr{Y}_i$ annihilate Λ.

2.1.2. Double Dyck Path Algebra. The ***Double Dyck Path Algebra*** $\mathbb{A}_{q,t}$, introduced by Carlsson and Mellit [8], is a quiver path algebra with vertices indexed by non-negative integers with the following edge operators:

- $d_+, d_+^* : k \to k+1$
- $T_1, ..., T_{k-1} : k \to k$
- $d_- : k+1 \to k$.

The full set of relations for $\mathbb{A}_{q,t}$ are omitted here but can be found in [8]. In order to match the parameter conventions in Ion and Wu's work [26] we will consider $\mathbb{A}_{t,q}$ as opposed to $\mathbb{A}_{q,t}$ formed by simply swapping q and t in the defining relations of $\mathbb{A}_{q,t}$. Here we highlight a few notable relations of $\mathbb{A}_{t,q}$ which will be required later:

- The loops $T_1, ..., T_{k-1}$ at vertex $k \geq 2$ generate a type A finite Hecke algebra
- $d_-^2 T_{k-1} = d_-^2$ starting at vertex $k \geq 2$
- $T_i d_- = d_- T_i$ at vertex k for $1 \leq i \leq k-2$
- $z_i d_- = d_- z_i$ at vertex k for $1 \leq i \leq k-1$ where $z_1 := \frac{t^k}{1-t}[d_+^*, d_-]T_{k-1}^{-1} \cdots T_1^{-1}$ and $z_{i+1} = t^{-1} T_i z_i T_i$.

2.1.2.1. *The Standard $\mathbb{A}_{t,q}$ Representation and the +Stable-Limit DAHA.* Vital to the proof of the Compositional Shuffle Conjecture by Carlsson and Mellit [8] is their construction of a particular representation of $\mathbb{A}_{t,q}$.

DEFINITION 2.1.6. *[8] For $k \geq 0$ let $V_k = \mathbb{Q}(q,t)[y_1, \ldots, y_k] \otimes \Lambda$ be associated to the vertex k and denote by $V_\bullet$ be the system of spaces V_k. Let ζ_k denote the algebra homomorphism*

$$\zeta_k f(y_1, \ldots, .y_{k-1}, y_k) = f(y_2, \ldots, y_k, qy_1).$$

If f is a formal series with respect to the variable y with coefficients in some ring R denote by $\mathfrak{c}_y(f) \in R$ the constant term of f i.e. the coefficient of y^0 in f. Note that each $\mathfrak{S}_k$ acts on V_k by permuting the variables $y_1, ..., y_k$. Define the following operators:

- $T_i F = s_i F + (1-t) y_i \frac{F - s_i F}{y_i - y_{i+1}}$
- $d_- F = \mathfrak{c}_{y_k}(F[X - (t-1)y_k] \operatorname{Exp}[-y_k^{-1} X])$
- $d_+ F = -T_1 \cdots T_k (y_{k+1} F[X + (t-1)y_{k+1}])$
- $d_+^* F = \zeta_k F[X + (t-1)y_{k+1}]$.

THEOREM 2.1.7. *[8] The above operators define a representation of $\mathbb{A}_{t,q}$ on $V_\bullet$.*

Ion and Wu use their construction of the standard $\mathscr{H}^+$ representation $\mathscr{P}_{as}^+$ to recover the standard $\mathbb{A}_{t,q}$ representation $V_\bullet$.

THEOREM 2.1.8. *[26] There exists an $\mathbb{A}_{t,q}$ representation structure on $\mathscr{P}_\bullet = (\mathscr{P}(k)^+)_{k\geq 0}$ isomorphic to the standard representation $V_\bullet$ such that at each vertex k, z_i acts by $\mathscr{Y}_i$ and y_i acts by X_i. Further, according to this isomorphism $\mathscr{P}(k)^+$ is identified with V_k via the map*

$$x_1^{a_1}\cdots x_k^{a_k}F[x_{k+1}+\ldots] \to y_1^{a_1}\cdots y_k^{a_k}F[\frac{X}{t-1}].$$

2.2. Combinatorial Formula for Non-symmetric Macdonald Polynomials

Note that the q,t conventions in [19] differ from those appearing in this thesis. In the below theorem the appropriate translation $q \to q^{-1}$ has been made.

In [19], Haglund, Haiman, and Loehr give an explicit monomial expansion formula for the non-symmetric Macdonald polynomials in terms of the combinatorics of ***non-attacking labellings*** of certain box diagrams corresponding to compositions which we will now review.

DEFINITION 2.2.1. *[19] For a composition $\mu = (\mu_1, \ldots, \mu_n)$ define the column diagram of μ as*

$$dg'(\mu) := \{(i,j) \in \mathbb{N}^2 : 1 \leq i \leq n, 1 \leq j \leq \mu_i\}.$$

This is represented by a collection of boxes in positions given by $dg'(\mu)$. The augmented diagram of μ is given by

$$\widehat{dg}(\mu) := dg'(\mu) \cup \{(i,0) : 1 \leq i \leq n\}.$$

Visually, to get $\widehat{dg}(\mu)$ we are adding a bottom row of boxes on length n below the diagram $dg'(\mu)$. Given $u = (i,j) \in dg'(\mu)$ define the following:

- $\mathrm{leg}(u) := \{(i,j') \in dg'(\mu) : j' > j\}$
- $\mathrm{arm}^{\mathrm{left}}(u) := \{(i',j) \in dg'(\mu) : i' < i, \mu_{i'} \leq \mu_i\}$
- $\mathrm{arm}^{\mathrm{right}}(u) := \{(i',j-1) \in \widehat{dg}(\mu) : i' > i, \mu_{i'} < \mu_i\}$
- $\mathrm{arm}(u) := \mathrm{arm}^{\mathrm{left}}(u) \cup \mathrm{arm}^{\mathrm{right}}(u)$
- $\mathrm{lg}(u) := |\mathrm{leg}(u)| = \mu_i - j$
- $a(u) := |\mathrm{arm}(u)|$.

A filling of μ is a function $\sigma : dg'(\mu) \to \{1,...,n\}$ and given a filling there is an associated augmented filling $\widehat{\sigma} : \widehat{dg}(\mu) \to \{1,...,n\}$ extending σ with the additional bottom row boxes filled according to $\widehat{\sigma}((j,0)) = j$ for $j = 1,\ldots,n$. Distinct lattice squares $u, v \in \mathbb{N}^2$ are said to attack each other if one of the following is true:

- *u and v are in the same row*
- *u and v are in consecutive rows and the box in the lower row is to the right of the box in the upper row.*

A filling $\sigma : dg'(\mu) \to \{1, \dots, n\}$ is non-attacking if $\widehat{\sigma}(u) \neq \widehat{\sigma}(v)$ for every pair of attacking boxes $u, v \in \widehat{dg}(\mu)$. For a box $u = (i, j)$ let $d(u) = (i, j-1)$ denote the box just below u. Given a filling $\sigma : dg'(\mu) \to \{1, \dots, n\}$, a descent of σ is a box $u \in dg'(\mu)$ such that $\widehat{\sigma}(u) > \widehat{\sigma}(d(u))$. Set $\mathrm{Des}(\widehat{\sigma})$ to be the set of descents of $\widehat{\sigma}$ and define

$$\mathrm{maj}(\widehat{\sigma}) := \sum_{u \in \mathrm{Des}(\widehat{\sigma})} (\mathrm{lg}(u) + 1).$$

The reading order on the diagram $\widehat{dg}(\mu)$ is the total ordering on the boxes of $\widehat{dg}(\mu)$ row by row, from top to bottom, and from right to left within each row. If $\sigma : dg'(\mu) \to \{1, \dots, n\}$ is a filling, an inversion of $\widehat{\sigma}$ is a pair of attacking boxes $u, v \in \widehat{dg}(\mu)$ such that $u < v$ in reading order and $\widehat{\sigma}(u) > \widehat{\sigma}(v)$. Set $\mathrm{Inv}(\widehat{\sigma})$ to be the set of inversions of $\widehat{\sigma}$. Define the statistics

- $\mathrm{inv}(\widehat{\sigma}) := |\,\mathrm{Inv}(\widehat{\sigma})| - |\{i < j : \mu_i \leq \mu_j\}| - \sum_{u \in \mathrm{Des}(\widehat{\sigma})} a(u)$
- $\mathrm{coinv}(\widehat{\sigma}) := \left(\sum_{u \in dg'(\mu)} a(u)\right) - \mathrm{inv}(\widehat{\sigma})$.

Lastly, for a filling $\sigma : dg'(\mu) \to \{1, \dots, n\}$ set

$$x^{\sigma} := x_1^{|\sigma^{-1}(1)|} \cdots x_n^{|\sigma^{-1}(n)|}.$$

The combinatorial formula for non-symmetric Macdonald polynomials can now be stated.

THEOREM 2.2.2. *[19] For a composition μ with $\ell(\mu) = n$ the following holds:*

$$E_{\mu} = \sum_{\substack{\sigma : \mu \to [n] \\ \textit{non-attacking}}} x^{\sigma} q^{-\mathrm{maj}(\widehat{\sigma})} t^{\mathrm{coinv}(\widehat{\sigma})} \prod_{\substack{u \in dg'(\mu) \\ \widehat{\sigma}(u) \neq \widehat{\sigma}(d(u))}} \left(\frac{1 - t}{1 - q^{-(\mathrm{lg}(u)+1)} t^{(a(u)+1)}} \right).$$

We may better understand the statistic coinv through the next definition.

DEFINITION 2.2.3. *[19] Let $\sigma : \mu \to [n]$ be a non-attacking labelling. A **co-inversion triple** is a triple of boxes (u, v, w) in the diagram $\widehat{dg}(\mu)$ of one of the following two types*

$$\textit{Type 1: } \begin{array}{|c|cc} \hline u & & \\ \hline w & & \multicolumn{1}{|c|}{v} \\ \hline \end{array} \qquad \textit{Type 2: } \begin{array}{|c|c|c|} \hline v & & u \\ \hline & & w \\ \hline \end{array}$$

that satisfy the following criteria:

- *in Type 1 the column containing u and w is strictly taller than the column containing v*
- *in Type 2 the column containing u and w is weakly taller than the column containing v*
- *in either Type 1 or Type 2 $\widehat{\sigma}(u) < \widehat{\sigma}(v) < \widehat{\sigma}(w)$ or $\widehat{\sigma}(v) < \widehat{\sigma}(w) < \widehat{\sigma}(u)$ or $\widehat{\sigma}(w) < \widehat{\sigma}(u) < \widehat{\sigma}(v)$.*

Informally, in Type 1 we require the entries to strictly increase clockwise and in Type 2 we require the entries to strictly increase counterclockwise.

Co-inversion triples are important because they have the same count as the complicated coinv statistic from Definition 2.2.1.

Lemma 2.2.4. *[19] For a non-attacking labelling $\sigma : \mu \to [n]$, $\operatorname{coinv}(\widehat{\sigma})$ equals the number of co-inversion triples of $\widehat{\sigma}$.*

Example. *We finish this subsection with a visual example of a non-attacking filling and its associated statistics. Below is the augmented filling $\widehat{\sigma}$ of a non-attacking filling $\sigma : (3, 2, 0, 1, 0, 0) \to [6]$ pictured as labels inside the boxes of $\widehat{dg}(3, 2, 0, 1, 0, 0)$.*

6					
4	1				
1	2		3		
1	2	3	4	5	6

Let u be the column 1 box of $\widehat{dg}(3, 2, 0, 1, 0, 0)$ filled with a 4 in the above diagram. Notice that u is a descent box of $\widehat{\sigma}$ as 4 is larger than the label 1 of the box $d(u)$ just below u. Further, we see that $a(u) = 2$ and $\lg(u) = 1$. Considering the diagram as a whole now we see that $x^\sigma = x_1^2 x_2 x_3 x_4 x_6$, $\operatorname{maj}(\widehat{\sigma}) = 3$, $|\operatorname{Inv}(\widehat{\sigma})| = 21$, $\operatorname{inv}(\widehat{\sigma}) = 14$, and $\operatorname{coinv}(\widehat{\sigma}) = 1$. The contribution of this non-attacking labelling to the HHL formula for $E_{(3,2,0,1,0,0)} \in \mathscr{P}_6^+$ is

$$x_1^2 x_2 x_3 x_4 x_6 q^{-3} t^1 \left(\frac{1-t}{1-q^{-1}t^3}\right)\left(\frac{1-t}{1-q^{-1}t^2}\right)\left(\frac{1-t}{1-q^{-2}t^3}\right)\left(\frac{1-t}{1-q^{-1}t^2}\right).$$

2.3. Stable-Limits of Non-symmetric Macdonald Polynomials

We start by investigating the properties of certain sequences of non-symmetric Macdonald polynomials. We will find that if we fix any composition μ and consider the sequence of compositions $(\mu * 0^m)_{m\geq 0}$ the corresponding sequence of non-symmetric Macdonald polynomials $(E_{\mu*0^m})_{m\geq 0}$

will converge in the sense of Definition 1.7.3. It is important to note that in most cases the sequence $(E_{\mu*0^m})_{m\geq 0}$ will not converge with respect to the inverse system $(\Xi_k : \mathscr{P}_{k+1} \to \mathscr{P}_k)_{k\geq 1}$. This should be expected because the spectra of the Cherednik operators acting on $\mathscr{P}_{k+1}$ are incompatible with the spectra from the Cherednik operators acting on $\mathscr{P}_k$. However, by using the HHL explicit combinatorial formula for the non-symmetric Macdonald polynomials we show that the combinatorics of non-attacking labellings underlying the sequence $(E_{\mu*0^m})_{m\geq 0}$ converge in a certain sense. The weaker convergence notion introduced by Ion and Wu is consistent with these combinatorics. For our purposes later in this chapter we will heavily rely on the convergence of these sequences as a bridge between the limit Cherednik operators $\mathscr{Y}_i$ and their classical counterparts.

We now show the convergence of the sequence $(E_{\mu*0^m})_{m\geq 0}$. First, we describe a convenient rearrangement of the monomials in each $E_{\mu*0^m}$.

THEOREM 2.3.1. *Let μ be a composition with $\ell(\mu) = n$ and $m \geq 0$. Then $E_{\mu*0^m}$ has the explicit expression given by*

$$E_{\mu*0^m} = \sum_{\substack{\lambda \text{ partition} \\ |\lambda|\leq|\mu|}} m_\lambda[x_{n+1} + \ldots + x_{n+m}] \sum_{\substack{\sigma:\mu*0^{\ell(\lambda)}\to[n+\ell(\lambda)] \\ \text{non-attacking} \\ \forall i=1,\ldots,\ell(\lambda) \\ \lambda_i = |\sigma^{-1}(n+i)|}} x_1^{|\sigma^{-1}(1)|} \cdots x_n^{|\sigma^{-1}(n)|} \Gamma^{(m)}(\widehat{\sigma})$$

where

$$\Gamma^{(m)}(\widehat{\sigma}) :=$$
$$q^{-\operatorname{maj}(\widehat{\sigma})} t^{\operatorname{coinv}(\widehat{\sigma})} \prod_{\substack{u\in dg'(\mu*0^{\ell(\lambda)}) \\ \widehat{\sigma}(u)\neq\widehat{\sigma}(d(u)) \\ u \text{ not in row } 1}} \left(\frac{1-t}{1-q^{-(\operatorname{lg}(u)+1)}t^{(a(u)+1)}}\right) \prod_{\substack{u\in dg'(\mu*0^{\ell(\lambda)}) \\ \widehat{\sigma}(u)\neq\widehat{\sigma}(d(u)) \\ u \text{ in row } 1}} \left(\frac{1-t}{1-q^{-(\operatorname{lg}(u)+1)}t^{(a(u)+m+1)}}\right).$$

PROOF. First, start with directly applying the HHL formula (2.2.2):

$$E_{\mu*0^m} = \sum_{\substack{\sigma:\mu*0^m\to[n+m] \\ \text{non-attacking}}} x^\sigma q^{-\operatorname{maj}(\widehat{\sigma})} t^{\operatorname{coinv}(\widehat{\sigma})} \prod_{\substack{u\in dg'(\mu*0^m) \\ \widehat{\sigma}(u)\neq\widehat{\sigma}(d(u))}} \left(\frac{1-t}{1-q^{-(\operatorname{lg}(u)+1)}t^{(a(u)+1)}}\right).$$

We know that $E_{\mu*0^m}$ is symmetric in the variables $x_{n+1}, ..., x_{n+m}$ [9] so it follows that the coefficient (as a polynomial in $\mathbb{Q}(q,t)[x_1, ..., x_n]$) of each monomial in $x_{n+1}, ..., x_{n+m}$ is independent of the ordering of the latter variables. Hence, we find that by grouping these monomials by symmetry

$$E_{\mu*0^m} = \sum_{\lambda} m_\lambda[x_{n+1} + ... + x_{n+m}] \sum_{\substack{\sigma:\mu*0^m \to [n+m] \\ \text{non-attacking} \\ \forall i \ \lambda_i = |\sigma^{-1}(n+i)|}} x_1^{|\sigma^{-1}(1)|} \cdots x_n^{|\sigma^{-1}(n)|} q^{-\operatorname{maj}(\widehat{\sigma})} t^{\operatorname{coinv}(\widehat{\sigma})} \times$$

$$\prod_{\substack{u \in dg'(\mu) \\ \widehat{\sigma}(u) \neq \widehat{\sigma}(d(u))}} \left(\frac{1-t}{1 - q^{-(\operatorname{lg}(u)+1)} t^{(a(u)+1)}} \right).$$

Note that by degree considerations the only possible partitions λ that have a nonzero contribution to the above sum have $|\lambda| \leq |\mu|$ and hence we can rewrite the above sums as

$$\sum_{\lambda} \sum_{\substack{\sigma:\mu*0^m \to [n+m] \\ \text{non-attacking} \\ \forall i \ \lambda_i = |\sigma^{-1}(n+i)|}} = \sum_{\substack{\lambda \text{ partition} \\ |\lambda| \leq |\mu|}} \sum_{\substack{\sigma:\mu*0^m \to [n+\ell(\lambda)] \\ \text{non-attacking} \\ \forall i \ \lambda_i = |\sigma^{-1}(n+i)|}}.$$

In the latter sum above we have written each σ as a non-attacking labelling $\sigma : \mu * 0^m \to [n + \ell(\lambda)]$ to emphasize that the numbers occurring in this labelling are contained in the set $[n + \ell(\lambda)]$ which is independent of m. However, these are still considered labellings of the diagram corresponding to $\mu * 0^m$ and hence we calculate the corresponding q, t coefficients in the HHL formula accordingly.

We must now understand the dependence on m of the statistics maj, coinv, lg, and a in each of the non-attacking labellings $\sigma : \mu * 0^m \to [n + \ell(\lambda)]$ as m varies. Fix a non-attacking labelling $\sigma : \mu * 0^k \to [n + k]$ for some $k \leq m$ and let σ_m be the associated labelling of $\mu * 0^m$. Recall that

$$\operatorname{maj}(\widehat{\sigma}) = \sum_{u \in \operatorname{Des}(\widehat{\sigma})} (\operatorname{lg}(u) + 1)$$

and similarly for $\operatorname{maj}(\widehat{\sigma}_m)$. The only descent boxes of $\widehat{\sigma}_m$ occur in the diagram $dg'(\mu)$ itself and $\operatorname{lg}(u)$ for these boxes will not depend on m. Therefore, $\operatorname{maj}(\widehat{\sigma}_m) = \operatorname{maj}(\widehat{\sigma})$. For $u \in dg'(\mu * 0^m)$ clearly $u \in dg'(\mu)$ and by direct computation we see that when u is not in row 1 then $a(u)$ does not depend on m. However, for u in row 1 $a(u)$ when calculated in the diagram $\widehat{dg}(\mu)$ increases to $a(u) + m$ when calculated in the diagram $\widehat{dg}(\mu * 0^m)$. This comes from counting the extra row 0 boxes for each box in row 1. Also note that in any non-attacking labelling there cannot be descent boxes in row 1. Now from careful counting we get the following:

- $|\operatorname{Inv}(\widehat{\sigma}_m)| = |\operatorname{Inv}(\widehat{\sigma})| + (n+k)(m-k) + \binom{m-k}{2}$
- $|\{i<j : (\mu * 0^m)_i \le (\mu * 0^m)_j\}|$

$$= |\{i<j : (\mu * 0^k)_i \le (\mu * 0^k)_j\}| + (\#\{i : \mu_i = 0\} + k)(m-k) + \binom{m-k}{2}$$

- $\sum_{u \in \operatorname{Des}(\widehat{\sigma}_m)} a(u) = \sum_{u \in \operatorname{Des}(\widehat{\sigma})} a(u)$.

By using the above calculations and cancelling out terms we get

$$\begin{aligned}
\operatorname{inv}(\widehat{\sigma}_m) &= |\operatorname{Inv}(\widehat{\sigma}_m)| - |\{i<j : (\mu * 0^m)_i \le (\mu * 0^m)_j\}| - \sum_{u \in \operatorname{Des}(\widehat{\sigma}_m)} a(u) \\
&= |\operatorname{Inv}(\widehat{\sigma})| - |\{i<j : (\mu * 0^k)_i \le (\mu * 0^k)_j\}| - \sum_{u \in \operatorname{Des}(\widehat{\sigma})} a(u) + (n - \#\{i : \mu_i = 0\})(m-k) \\
&= \operatorname{inv}(\widehat{\sigma}) + \#\{i : \mu_i \neq 0\}(m-k).
\end{aligned}$$

Further, from the prior observation about how arm, $a(u)$, changes with m we see that

$$\sum_{u \in dg'(\mu * 0^m)} a(u) = \#\{i : \mu_i \neq 0\}(m-k) + \sum_{u \in dg'(\mu * 0^k)} a(u)$$

where arm has been calculated in the corresponding diagrams.

We then have

$$\begin{aligned}
\operatorname{coinv}(\widehat{\sigma}_m) &= \left(\sum_{u\in dg'(\mu*0^m)} a(u)\right) - \operatorname{inv}(\widehat{\sigma}_m)\\
&= \left(\#\{i:\mu_i\neq 0\}(m-k) + \sum_{u\in dg'(\mu*0^k)} a(u)\right) - (\operatorname{inv}(\widehat{\sigma}) + \#\{i:\mu_i\neq 0\}(m-k))\\
&= \left(\sum_{u\in dg'(\mu*0^k)} a(u)\right) - \operatorname{inv}(\widehat{\sigma})\\
&= \operatorname{coinv}(\widehat{\sigma}).
\end{aligned}$$

Thus $\operatorname{maj}(\widehat{\sigma}_m) = \operatorname{maj}(\widehat{\sigma})$ and $\operatorname{coinv}(\widehat{\sigma}_m) = \operatorname{coinv}(\widehat{\sigma})$.

Lastly, we return to the expansion of $E_{\mu*0^m}$ we found above. For each partition λ with $|\lambda| \leq |\mu|$ we now see that

$$\begin{aligned}
&\sum_{\substack{\sigma:\mu*0^m\to[n+\ell(\lambda)]\\ \text{non-attacking}\\ \forall i\ \lambda_i=|\sigma^{-1}(n+i)|}} x_1^{|\sigma^{-1}(1)|}\cdots x_n^{|\sigma^{-1}(n)|} q^{-\operatorname{maj}(\widehat{\sigma})} t^{\operatorname{coinv}(\widehat{\sigma})} \prod_{\substack{u\in dg'(\mu)\\ \widehat{\sigma}(u)\neq\widehat{\sigma}(d(u))}} \left(\frac{1-t}{1-q^{-(\operatorname{lg}(u)+1)}t^{(a(u)+1)}}\right)\\
&= \sum_{\substack{\sigma:\mu*0^{\ell(\lambda)}\to[n+\ell(\lambda)]\\ \text{non-attacking}\\ \forall i\ \lambda_i=|\sigma^{-1}(n+i)|}} x_1^{|\sigma^{-1}(1)|}\cdots x_n^{|\sigma^{-1}(n)|}\Gamma^{(m)}(\widehat{\sigma}).
\end{aligned}$$

where

$$\Gamma^{(m)}(\widehat{\sigma}) :=$$

$$q^{-\operatorname{maj}(\widehat{\sigma})} t^{\operatorname{coinv}(\widehat{\sigma})} \prod_{\substack{u\in dg'(\mu*0^{\ell(\lambda)})\\ \widehat{\sigma}(u)\neq\widehat{\sigma}(d(u))\\ u \text{ not in row } 1}} \left(\frac{1-t}{1-q^{-(\operatorname{lg}(u)+1)}t^{(a(u)+1)}}\right) \prod_{\substack{u\in dg'(\mu*0^{\ell(\lambda)})\\ \widehat{\sigma}(u)\neq\widehat{\sigma}(d(u))\\ u \text{ in row } 1}} \left(\frac{1-t}{1-q^{-(\operatorname{lg}(u)+1)}t^{(a(u)+m+1)}}\right).$$

and we calculate all of the associated statistics in their respective diagrams.

□

Now that we have conveniently rearranged the monomial terms of each $E_{\mu*0^m}$ and identified the dependence of the coefficients on the parameter m we can give a simple proof that the sequence $(E_{\mu*0^m})_{m\geq 0}$ converges.

Corollary/Definition 2.3.2. *Let μ be a composition with $\ell(\mu)=n$. The sequence $(E_{\mu*0^m})_{m\geq 1}$ converges to an almost symmetric function $\widetilde{E}_\mu := \lim_m E_{\mu*0^m} \in \mathscr{P}_{as}^+$ given explicitly by*

$$\widetilde{E}_\mu = \sum_{\substack{\lambda \text{ partition} \\ |\lambda|\leq|\mu|}} m_\lambda[x_{n+1}+\ldots] \sum_{\substack{\sigma:\mu*0^{\ell(\lambda)}\to[n+\ell(\lambda)] \\ \text{non-attacking} \\ \forall i=1,\ldots,\ell(\lambda) \\ \lambda_i=|\sigma^{-1}(n+i)|}} x_1^{|\sigma^{-1}(1)|}\cdots x_n^{|\sigma^{-1}(n)|}\widetilde{\Gamma}(\widehat{\sigma})$$

where

$$\widetilde{\Gamma}(\widehat{\sigma}) := \lim_m \Gamma^{(m)}(\widehat{\sigma}) = q^{-\operatorname{maj}(\widehat{\sigma})}t^{\operatorname{coinv}(\widehat{\sigma})} \prod_{\substack{u\in dg'(\mu*0^{\ell(\lambda)}) \\ \widehat{\sigma}(u)\neq\widehat{\sigma}(d(u)) \\ u \text{ not in row } 1}} \left(\frac{1-t}{1-q^{-(\operatorname{lg}(u)+1)}t^{(a(u)+1)}}\right) \prod_{\substack{u\in dg'(\mu*0^{\ell(\lambda)}) \\ \widehat{\sigma}(u)\neq\widehat{\sigma}(d(u)) \\ u \text{ in row } 1}} (1-t)\,.$$

Proof. Note that the formula in Theorem 2.3.1 is a fixed size finite sum where the only dependence on m is in the m_λ symmetric function terms and the t^m occurring in the $\Gamma^{(m)}$ terms. Thus in the sense of Ion and Wu, see Definition 1.7.3, this sequence converges to a well defined element of $\mathscr{P}_{as}^+$. In particular, each $m_\lambda[x_{n+1}+\ldots+x_{n+m}]$ converges to $m_\lambda[x_{n+1}+\ldots]$ and t^m converges to 0 in the $\widetilde{\Gamma}$-term. Simplifying gives the formula above.

□

It follows from Corollary 2.3.2 that the almost symmetric functions $\widetilde{E}_\mu$ are homogeneous of degree $|\mu|$ and $\widetilde{E}_\mu \in \mathscr{P}(\ell(\mu))^+$. Note importantly, that for any composition μ (not necessarily reduced) and any $n\geq 0$, by shifting the terms of the sequence $(E_{\mu*0^m})_{m\geq 0}$ we see that $\widetilde{E}_{\mu*0^n} = \widetilde{E}_\mu$.

Corollary 2.3.3. *Let λ be a partition with $\ell(\lambda)=n$ and $|\lambda|=N$. Then $\widetilde{E}_\lambda$ is determined by $E_{\lambda*0^N}\in\mathscr{P}_{n+N}^+$. That is to say, if*

$$E_{\lambda*0^N}(x_1,...,x_{n+N}) = c_1x^{\mu^{(1)}}m_{\nu^{(1)}}[x_{n+1}+...+x_{n+N}]+...+c_kx^{\mu^{(k)}}m_{\nu^{(k)}}[x_{n+1}+...+x_{n+N}]$$

then

$$\widetilde{E}_\lambda = c_1x^{\mu^{(1)}}m_{\nu^{(1)}}[x_{n+1}+...]+...+c_kx^{\mu^{(k)}}m_{\nu^{(k)}}[x_{n+1}+...].$$

Proof. As λ is a partition, row 1 of any non-attacking labelling of λ must be 1,2,,...,$\ell(\lambda)$. Thus no boxes of $dg'(\lambda)$ in row 1 will have $\widehat{\sigma}(u) \neq \widehat{\sigma}(d(u))$ and so there will be no contributions from any of the terms of the form

$$\prod_{\substack{u \in dg'(\lambda) \\ \widehat{\sigma}(u) \neq \widehat{\sigma}(d(u)) \\ u \text{ row } 1}} \left(\frac{1-t}{1-q^{-(\lg(u)+1)} t^{(a(u)+m+1)}} \right).$$

Further, from Corollary 2.3.2 it is clear that these are the only coefficients that depend on m in the limit. Also it follows that each term of the form $x^{\mu} m_{\nu}[x_{n+1} + \ldots]$ that occurs in the expansion of $\widetilde{E}_{\lambda}$ appears at least by the $m = N$ step of the limit. From these two facts it follows that the expansion of $\widetilde{E}_{\lambda}$ will match that of $E_{\lambda * 0^N}(x_1, ..., x_{n+N})$ up to truncating each $m_{\nu}[x_{n+1} + \ldots]$ to $m_{\nu}[x_{n+1} + \ldots + x_{n+N}]$ using Ξ_{n+N}.

□

2.4. $\mathscr{Y}$-Weight Basis of $\mathscr{P}_{as}^{+}$

2.4.1. The $\widetilde{E}_{\mu}$ are $\mathscr{Y}$-Weight Vectors. In what follows, the classical spectral theory for non-symmetric Macdonald polynomials is used to demonstrate that the limit functions $\widetilde{E}_{\mu}$ are $\mathscr{Y}$-weight vectors. The below lemma is a simple application of this classical theory and basic properties of the t-adic topology on $\mathbb{Q}(q,t)$.

Lemma 2.4.1. *For a composition μ with $\ell(\mu) = n$ define $\alpha_{\mu}^{(m)}$ to be the $Y^{(n+m)}$-weight of $E_{\mu * 0^m}$. Then in the t-adic topology on $\mathbb{Q}(q,t)$ the sequence $(t^{n+m} \alpha_{\mu}^{(m)}(i))_{m \geq 0}$ converges in m to some $\widetilde{\alpha}_{\mu}(i) \in \mathbb{Q}(q,t)$. In particular, $\widetilde{\alpha}_{\mu}(i) = 0$ for $i > n$ and for $1 \leq i \leq n$ we have that $\widetilde{\alpha}_{\mu}(i) = 0$ exactly when $\mu_i = 0$.*

Proof. Take $\mu = (\mu_1, \ldots, \mu_n)$. From classical double affine Hecke algebra theory [9] we have $\alpha_{\mu}^{(0)}(i) = q^{\mu_i} t^{1-\beta_{\mu}(i)}$ where

$$\beta_{\mu}(i) := \#\{j : 1 \leq j \leq i \ , \mu_j \leq \mu_i\} + \#\{j : i < j \leq n \ , \mu_i > \mu_j\}.$$

If we calculate $\beta_{\mu * 0^m}(i)$ directly it follows then that

$$t^{n+m}\alpha_\mu^{(m)}(i) = \begin{cases} q^{\mu_i} t^{n+m+1-(\beta_\mu(i)+m\mathbb{1}(\mu_i \neq 0))} = t^n \alpha_\mu^{(0)}(i) & i \leq n, \mu_i \neq 0 \\ q^{\mu_i} t^{n+m+1-(\beta_\mu(i)+m\mathbb{1}(\mu_i \neq 0))} = t^{n+m} \alpha_\mu^{(0)}(i) & i \leq n, \mu_i = 0 \\ t^{n+m+1-(\#\{j:\mu_j = 0\}+i-n)} = t^{\#\{j:\mu_j \neq 0\}} t^{m+1-(i-n)} & i > n. \end{cases}$$

Lastly, by taking the limit $m \to \infty$ we get the result. □

For a composition μ define the weight $\widetilde{\alpha}_\mu$ using the formula in Lemma 2.4.1 for the list of scalars $\widetilde{\alpha}_\mu(i)$ for $i \in \mathbb{N}$.

LEMMA 2.4.2. *For a composition $\mu = (\mu_1, \ldots, \mu_n)$ with $\mu_i \neq 0$ for $1 \leq i \leq n$, $\widetilde{E}_\mu$ is a $\mathscr{Y}$-weight vector with weight $\widetilde{\alpha}_\mu$.*

PROOF. Fix any $r \in \mathbb{N}$. We start by rewriting the operator $\mathscr{Y}_r$ explicitly in terms of the limit definition of $\mathscr{Y}_1$.

$$\begin{aligned}
\mathscr{Y}_r &= t^{-(r-1)} T_{r-1} \cdots T_1 \mathscr{Y}_1 T_1 \cdots T_{r-1} \\
&= t^{-(r-1)} T_{r-1} \cdots T_1 \lim_k t^k \rho \pi_k T_{k-1}^{-1} \cdots T_1^{-1} T_1 \cdots T_{r-1} \Xi_k \\
&= \lim_k t^k T_{r-1} \cdots T_1 \rho t^{-(r-1)} \pi_k T_{k-1}^{-1} \cdots T_r^{-1} \Xi_k \\
&= \lim_k t^k T_{r-1} \cdots T_1 \rho T_1^{-1} \cdots T_{r-1}^{-1} t^{-(r-1)} T_{r-1} \cdots T_1 \pi_k T_{k-1}^{-1} \cdots T_r^{-1} \Xi_k \\
&= \lim_k t^k T_{r-1} \cdots T_1 \rho T_1^{-1} \cdots T_{r-1}^{-1} Y_r^{(k)} \Xi_k.
\end{aligned}$$

Applying $\mathscr{Y}_r$ to $\widetilde{E}_\mu$ we see by taking $k = n+m \geq n$ and shifting the indices that

$$\begin{aligned}
\mathscr{Y}_r(\widetilde{E}_\mu) &= \lim_m t^{n+m} T_{r-1} \cdots T_1 \rho T_1^{-1} \cdots T_{r-1}^{-1} Y_r^{(n+m)} (E_{\mu * 0^m}) \\
&= \lim_m T_{r-1} \cdots T_1 \rho T_1^{-1} \cdots T_{r-1}^{-1} t^{n+m} \alpha_\mu^{(m)}(r) E_{\mu * 0^m}
\end{aligned}$$

and by Lemma 2.4.1 this converges to

$$\mathscr{Y}_r(\widetilde{E}_\mu) = \widetilde{\alpha}_\mu(r)(T_{r-1}\cdots T_1\rho T_1^{-1}\cdots T_{r-1}^{-1})\widetilde{E}_\mu.$$

Importantly, we have implicitly used the fact that both of the sequences $(E_{\mu*0^m})_m$ and $(\alpha_\mu^{(m)}(r))_m$ converge, that the operator $T_{r-1}\cdots T_1\rho T_1^{-1}\cdots T_{r-1}^{-1}$ commutes with the quotient maps $\Xi_k : \mathscr{P}_{k+1} \to \mathscr{P}_k$ for $k > r$, and Proposition 6.21 in [26]. We will show that the right side is $\widetilde{\alpha}_\mu(r)\widetilde{E}_\mu$. As $\widetilde{\alpha}_\mu(r) = 0$ for $r > n$ by Lemma 2.4.1 we reduce to the sub case $r \leq n$. Fix $r \leq n$. If we could show that x_1 divides $T_1^{-1}\cdots T_{r-1}^{-1}\widetilde{E}_\mu$ then we would have

$$\rho(T_1^{-1}\cdots T_{r-1}^{-1}\widetilde{E}_\mu) = T_1^{-1}\cdots T_{r-1}^{-1}\widetilde{E}_\mu$$

implying that

$$\begin{aligned}\mathscr{Y}_r(\widetilde{E}_\mu) &= \widetilde{\alpha}_\mu(r)(T_{r-1}\cdots T_1\rho T_1^{-1}\cdots T_{r-1}^{-1})\widetilde{E}_\mu \\ &= \widetilde{\alpha}_\mu(r)T_{r-1}\cdots T_1 T_1^{-1}\cdots T_{r-1}^{-1})\widetilde{E}_\mu \\ &= \widetilde{\alpha}_\mu(r)\widetilde{E}_\mu\end{aligned}$$

as desired. To show that $x_1|T_1^{-1}\cdots T_{r-1}^{-1}\widetilde{E}_\mu$ it suffices to show that for all $m \geq 0$, x_1 divides $T_1^{-1}\cdots T_{r-1}^{-1}E_{\mu*0^m}$. To this end fix $m \geq 0$. We have that

$$\begin{aligned}\alpha_\mu^{(m)}(r)E_{\mu*0^m} &= Y_r^{(n+m)}(E_{\mu*0^m}) \\ &= t^{n+m-r+1}T_{r-1}\cdots T_1\pi_{n+m}T_{n+m-1}^{-1}\cdots T_r^{-1}E_{\mu*0^m}.\end{aligned}$$

Since $\alpha_\mu^{(m)}(r) \neq 0$ we can have $\frac{1}{\alpha_\mu^{(m)}(r)}T_1^{-1}\cdots T_{r-1}^{-1}$ act on both sides of the above to get

$$T_1^{-1}\cdots T_{r-1}^{-1}E_{\mu*0^m} = \frac{t^{n+m-r+1}}{\alpha_\mu^{(m)}(r)}\pi_{n+m}T_{n+m-1}^{-1}\cdots T_r^{-1}E_{\mu*0^m}.$$

By HHL any non-attacking labelling of $\mu * 0^m$ will have row 1 diagram labels given by $\{1, 2, \ldots, n\}$ so in particular x_r divides $E_{\mu * 0^m}$ for all $m > 0$. Lastly,

$$\begin{aligned}\pi_{n+m} T_{n+m-1}^{-1} \cdots T_r^{-1} X_r &= \pi_{n+m} t^{-(n+m-r)} X_{n+m} T_{n+m-1} \cdots T_r \\ &= q t^{-(n+m-r)} X_1 \pi_{n+m} T_{n+m-1} \cdots T_r.\end{aligned}$$

Thus x_1 divides $T_1^{-1} \cdots T_{r-1}^{-1} E_{\mu * 0^m}$ for all $m \geq 0$ showing the result.

□

Now we consider the general situation where the composition μ can have some parts which are 0. We can extend the above result, Lemma 2.4.2, by a straight-forward argument using intertwiner theory from the study of affine Hecke algebras.

Theorem 2.4.3. *For all compositions μ, $\widetilde{E}_\mu$ is a $\mathscr{Y}$-weight vector with weight $\widetilde{\alpha}_\mu$.*

Proof. Lemma 2.4.2 shows that this statement holds for any composition with all parts nonzero. Fix a composition μ with length n. We know that by sorting in decreasing order that μ can be written as a permutation of a composition of the form $\nu * 0^m$ for a partition ν and some $m \geq 0$. From the definition of Bruhat order it follows that $\nu * 0^m$ will be the minimal element out of all of its distinct permutations, including μ. Necessarily, this finite subposet generated by the permutations of $\nu * 0^m$ is isomorphic to the Bruhat ordering on the coset space $\mathfrak{S}_n / \mathfrak{S}_\kappa$ where $\mathfrak{S}_\kappa$ is the Young subgroup of $\mathfrak{S}_n$ corresponding to the stabilizer of $\nu * 0^m$. Hence, it suffices to show inductively that for any composition β with $\nu * 0^m \leq \beta < s_i(\beta) \leq \mu$, if $\widetilde{E}_\beta$ satisfies the theorem then so will $\widetilde{E}_{s_i(\beta)}$. As μ is finitely many covering elements away in Bruhat from $\nu * 0^m$ this induction will indeed terminate after finitely many steps.

Using the intertwiner operators from affine Hecke algebra theory, given by $\phi_i = T_i \mathscr{Y}_i - \mathscr{Y}_i T_i$ in this context, we only need to show that for any composition β with $\nu * 0^m \leq \beta < s_i(\beta) \leq \mu$,

$$\phi_i \widetilde{E}_\beta = (\widetilde{\alpha}_\beta(i) - \widetilde{\alpha}_\beta(i+1)) \widetilde{E}_{s_i(\beta)}.$$

Suppose the theorem holds for some β with $\nu * 0^m \leq \beta < s_i(\beta) \leq \mu$. Then we have the following:

$$\begin{aligned}
\phi_i \widetilde{E}_\beta &= (T_i(\mathscr{Y}_i - \mathscr{Y}_{i+1}) + (1-t)\mathscr{Y}_{i+1})\widetilde{E}_\beta \\
&= (\widetilde{\alpha}_\beta(i) - \widetilde{\alpha}_\beta(i+1))T_i\widetilde{E}_\beta + (1-t)\widetilde{\alpha}_\beta(i+1)\widetilde{E}_\beta \\
&= \lim_m (t^{n+m}\alpha_\beta^{(m)}(i) - t^{n+m}\alpha_\beta^{(m)}(i+1))T_i E_{\beta*0^m} + (1-t)t^{n+m}\alpha_\beta^{(m)}(i+1)E_{\beta*0^m} \\
&= \lim_m (t^{n+m}\alpha_\beta^{(m)}(i) - t^{n+m}\alpha_\beta^{(m)}(i+1))E_{s_i(\beta)*0^m} \\
&= (\widetilde{\alpha}_\beta(i) - \widetilde{\alpha}_\beta(i+1))\widetilde{E}_{s_i(\beta)}.
\end{aligned}$$

□

As an immediate consequence of the proof of Theorem 2.4.3 we have the following.

Corollary 2.4.4. *Let μ be a composition and $i \geq 1$ such that $s_i(\mu) > \mu$. Then*

$$\widetilde{E}_{s_i(\mu)} = \left(T_i + \frac{(1-t)\widetilde{\alpha}_\mu(i+1)}{\widetilde{\alpha}_\mu(i) - \widetilde{\alpha}_\mu(i+1)} \right) \widetilde{E}_\mu.$$

We have shown in Theorem 2.4.3 there is an explicit collection of $\mathscr{Y}$-weight vectors $\widetilde{E}_\mu$ in $\mathscr{P}_{as}^+$ arising as the limits of non-symmetric Macdonald polynomials $E_{\mu*0^m}$. Unfortunately, these $\widetilde{E}_\mu$ do not span $\mathscr{P}_{as}^+$. To see this note that one cannot write a non-constant symmetric function as a linear combination of the $\widetilde{E}_\mu$. However, in the below work we build a full $\mathscr{Y}$-weight basis of $\mathscr{P}_{as}^+$.

2.4.2. Constructing a Full $\mathscr{Y}$-Weight Basis.

2.4.2.1. *Defining the Stable-Limit Non-symmetric Macdonald Functions.* To complete our construction of a full weight basis of $\mathscr{P}_{as}^+$ we will need the $\partial_-^{(k)}$ operators from Ion and Wu. These operators are, up to a change of variables and plethysm, the d_- operators from Carlsson and Mellit's standard $\mathbb{A}_{t,q}$ representation.

Definition 2.4.5. *[26] Define the operator $\partial_-^{(k)} : \mathscr{P}(k)^+ \to \mathscr{P}(k-1)^+$ to be the $\mathscr{P}_{k-1}^+$-linear map which acts on elements of the form $x_k^n F[x_{k+1} + x_{k+2} + \ldots]$ for $F \in \Lambda$ and $n \geq 0$ as*

$$\partial_-^{(k)}(x_k^n F[x_{k+1} + x_{k+2} + \ldots]) = \mathscr{B}_n(F)[x_k + x_{k+1} + \ldots].$$

Here the $\mathscr{B}_n$ are the Jing operators which serve as creation operators for Hall-Littlewood symmetric functions $\mathcal{P}_\lambda$ given explicitly by the following plethystic formula:

$$\mathscr{B}_n(F)[X] = \langle z^n \rangle F[X - z^{-1}] \operatorname{Exp}[(1-t)zX].$$

Importantly, the $\partial_-^{(k)}$ operators do not come from the $\mathscr{H}^+$ action itself. Note that the $\partial_-^{(k)}$ operators are homogeneous by construction.

We will require the useful alternative expression for the $\partial_-^{(k)}$ operators which can be found in [26]. Recall the notation $\mathfrak{c}_y$ from Definition 2.1.6.

LEMMA 2.4.6. *Let τ_k denote the alphabet shift $\mathfrak{X}_k \to \mathfrak{X}_{k-1}$ acting on symmetric functions where $\mathfrak{X}_i := x_{i+1} + x_{i+2} + \ldots$. Then for $f \in \mathscr{P}_k$ and $F \in \Lambda$*

$$\partial_-^{(k)}(f(x_1, \ldots x_k) F[\mathfrak{X}_k]) = \tau_k \mathfrak{c}_{x_k} f(x_1, ..., x_k) F[\mathfrak{X}_k - x_k] \operatorname{Exp}[-(t-1)x_k^{-1}\mathfrak{X}_k].$$

PROOF. [26]. □

As an immediate consequence of this explicit description of the action of the $\partial_-^{(k)}$ operator we get the following required lemmas.

LEMMA 2.4.7. *[26] The map $\partial_-^{(k)} : \mathscr{P}(k)^+ \to \mathscr{P}(k-1)^+$ is a projection onto $\mathscr{P}(k-1)^+$ i.e. for $f \in \mathscr{P}(k-1)^+ \subset \mathscr{P}(k)^+$ we have that $\partial_-^{(k)}(f) = f$.*

PROOF. Fix $F \in \Lambda$. It suffices to show that $\partial_-^{(k)}(F[\mathfrak{X}_{k-1}]) = F[\mathfrak{X}_{k-1}]$. By using the coproduct on Λ we can expand $F[\mathfrak{X}_{k-1}] = F[x_k + \mathfrak{X}_k]$ in powers of x_k^i with some coefficients $F_i \in \Lambda$ as

$F[x_k + \mathfrak{X}_k] = \sum_{i\geq 0} x_k^i F_i[\mathfrak{X}_k]$. From Lemma 2.4.6 we have

$$\begin{aligned}
\partial_-^{(k)}(F[\mathfrak{X}_{k-1}]) &= \partial_-^{(k)}(F[x_k + \mathfrak{X}_k]) \\
&= \partial_-^{(k)}(\sum_{i\geq 0} x_k^i F_i[\mathfrak{X}_k]) \\
&= \tau_k \mathfrak{c}_{x_k} \left(\sum_{i\geq 0} x_k^i F_i[\mathfrak{X}_k - x_k] \operatorname{Exp}[-(t-1)x_k^{-1}\mathfrak{X}_k] \right) \\
&= \tau_k \mathfrak{c}_{x_k} F[\mathfrak{X}_k - x_k + x_k] \operatorname{Exp}[-(t-1)x_k^{-1}\mathfrak{X}_k] \\
&= \tau_k \mathfrak{c}_{x_k} F[\mathfrak{X}_k] \operatorname{Exp}[-(t-1)x_k^{-1}\mathfrak{X}_k] \\
&= \tau_k F[\mathfrak{X}_k] \mathfrak{c}_{x_k} \operatorname{Exp}[-(t-1)x_k^{-1}\mathfrak{X}_k] \\
&= \tau_k F[\mathfrak{X}_k] \\
&= F[\mathfrak{X}_{k-1}].
\end{aligned}$$

□

We will need the following lemma showing that the maps $\partial_-^{(k)}$ are $\Lambda[x_k + x_{k+1} + \ldots]$-module maps.

Lemma 2.4.8. *For all $G \in \Lambda$ and $g(x) \in \mathscr{P}(k)^+$*

$$\partial_-^{(k)}(G[x_k + x_{k+1} + \ldots]g(x)) = G[x_k + x_{k+1} + \ldots]\partial_-^{(k)}(g(x)).$$

Proof. It suffices to take $g(x) \in \mathscr{P}(k)^+$ to be of the form $g(x) = f(x_1, \ldots, x_k)F[\mathfrak{X}_k]$ with $f \in \mathscr{P}_k^+$ and $F \in \Lambda$. From Lemma 2.4.6 we get the following:

$$
\begin{aligned}
\partial_-^{(k)}(G[x_k + x_{k+1} + \ldots]g(x)) &= \partial_-^{(k)}(G[\mathfrak{X}_{k-1}]g(x)) \\
&= \tau_k \mathfrak{c}_{x_k} G[\mathfrak{X}_{k-1} - x_k] f(x_1, \ldots, x_k) F[\mathfrak{X}_k - x_k] \operatorname{Exp}[-(t-1)x_k^{-1}\mathfrak{X}_k] \\
&= \tau_k \mathfrak{c}_{x_k} G[\mathfrak{X}_k] f(x_1, \ldots, x_k) F[\mathfrak{X}_k - x_k] \operatorname{Exp}[-(t-1)x_k^{-1}\mathfrak{X}_k] \\
&= \tau_k G[\mathfrak{X}_k] \mathfrak{c}_{x_k} f(x_1, \ldots, x_k) F[\mathfrak{X}_k - x_k] \operatorname{Exp}[-(t-1)x_k^{-1}\mathfrak{X}_k] \\
&= G[\mathfrak{X}_{k-1}] \tau_k \mathfrak{c}_{x_k} f(x_1, \ldots, x_k) F[\mathfrak{X}_k - x_k] \operatorname{Exp}[-(t-1)x_k^{-1}\mathfrak{X}_k] \\
&= G[\mathfrak{X}_{k-1}] \partial_-^{(k)}(f(x_1, \ldots, x_k) F[\mathfrak{X}_k]) \\
&= G[\mathfrak{X}_{k-1}] \partial_-^{(k)}(g(x)).
\end{aligned}
$$

□

Corollary 2.4.9. *For $G \in \Lambda$ and $g(x) \in \mathscr{P}(k)^+$*

$$\partial_-^{(k)}(G[X]g(x)) = G[X]\partial_-^{(k)}(g(x)).$$

Proof. Take $G \in \Lambda$ and $g(x) \in \mathscr{P}(k)^+$. Expand $G[X]$ as a finite sum of terms of the form $f_i(x_1, \ldots, x_{k-1})F_i[x_k + \ldots]$, where $f_i \in \mathscr{P}_{k-1}$ and $F_i \in \Lambda$ so

$$G[X] = \sum_i f_i(x_1, \ldots, x_{k-1})F_i[x_k + \ldots].$$

By Lemma 2.4.8 and the fact that $\partial_-^{(k)}$ is a $\mathscr{P}_{k-1}^+$-linear map from Definition 2.4.5 we now see that

$$
\begin{aligned}
\partial_-^{(k)}(G[X]g(x)) &= \sum_i \partial_-^{(k)}(f_i(x_1, \ldots, x_{k-1})F_i[x_k + \ldots]g(x)) \\
&= \sum_i f_i(x_1, \ldots, x_{k-1})F_i[x_k + \ldots]\partial_-^{(k)}(g(x)) \\
&= G[X]\partial_-^{(k)}(g(x)).
\end{aligned}
$$

□

We can now construct a full $\mathscr{Y}$-weight basis of $\mathscr{P}_{as}^{+}$. We parameterize this basis by pairs $(\mu|\lambda) \in \Sigma$. Combinatorially, this is reasonable because, as already mentioned, the monomial basis for $\mathscr{P}_{as}^{+}$, $\{x^{\mu} m_{\lambda} \mid (\mu|\lambda) \in \Sigma\}$, is indexed by Σ.

Definition 2.4.10. *For $(\mu|\lambda) \in \Sigma$ define the **stable-limit non-symmetric Macdonald function** corresponding to $(\mu|\lambda)$ as*

$$\widetilde{E}_{(\mu|\lambda)} := \partial_{-}^{(\ell(\mu)+1)} \cdots \partial_{-}^{(\ell(\mu)+\ell(\lambda))} \widetilde{E}_{\mu * \lambda}.$$

For a partition λ define

$$\mathcal{A}_{\lambda} := \widetilde{E}_{(\emptyset|\lambda)} \in \Lambda. \tag{2.1}$$

Later in Theorem 4.2.12, we will show that the collection $\{\widetilde{E}_{(\mu|\lambda)} \mid (\mu|\lambda) \in \Sigma\}$ is a $\mathscr{Y}$-weight basis for $\mathscr{P}_{as}^{+}$.

Remark. *Note importantly that $\widetilde{E}_{(\mu|\lambda)} \in \mathscr{P}(\ell(\mu))^{+}$ and $\widetilde{E}_{(\mu|\lambda)}$ is homogeneous of degree $|\mu|+|\lambda|$. Further, we have $\widetilde{E}_{(\mu|\emptyset)} = \widetilde{E}_{\mu}$ and $\widetilde{E}_{(\emptyset|\lambda)} = \mathcal{A}_{\lambda}$. Notice that in Definition 2.4.10 it makes sense to consider $\widetilde{E}_{(\mu|\lambda)}$ when μ is not necessarily reduced. However, it is a nontrivial consequence of Theorem 2.6.5 that an analogously defined $\widetilde{E}_{(\mu*0|\lambda)}$ is a nonzero scalar multiple of $\widetilde{E}_{(\mu|\lambda)}$. Thus there is no need to consider the case of μ non-reduced when building a basis of $\mathscr{P}_{as}^{+}$.*

*There is another basis of $\mathscr{P}_{as}^{+}$ given by Ion and Wu in their unpublished work [**27**] which is equipped with a natural ordering with respect to which the limit Cherednik operators are triangular. It follows then that after we show in Corollary 2.4.12 that the $\widetilde{E}_{(\mu|\lambda)}$ are $\mathscr{Y}$-weight vectors that each $\widetilde{E}_{(\mu|\lambda)}$ has a triangular expansion in Ion and Wu's basis.*

Remark. *The stable-limit non-symmetric Macdonald functions $\widetilde{E}_{(\mu|\lambda)}$ as defined in this chapter are distinct from the stable-limits of non-symmetric Macdonald polynomials occurring in [**19**]. In their paper Haglund, Haiman, and Loehr investigate stable-limits of the form $(E_{0^m * \mu})_{m \geq 0}$ where μ is a composition. Their analysis does not require the convergence definition of Ion and Wu as the sequences $(E_{0^m * \mu})_{m \geq 0}$ have stable limits in the traditional sense. Further, the limits of the $(E_{0^m * \mu})_{m \geq 0}$ sequences are symmetric functions whereas, as we will see soon, the $\widetilde{E}_{(\mu|\lambda)}$ are not fully symmetric in general.*

The following simple lemma will be used to show that since the $\widetilde{E}_{\mu*\lambda}$ are $\mathscr{Y}$-weight vectors the stable-limit non-symmetric Macdonald functions $\widetilde{E}_{(\mu|\lambda)}$ are $\mathscr{Y}$-weight vectors as well. We describe their weights in Corollary 2.4.12.

LEMMA 2.4.11. *Suppose $f \in \mathcal{P}(k)^+$ is a $\mathscr{Y}$-weight vector with weight $(\alpha_1, \ldots, \alpha_k, 0, 0, \ldots)$. Then $\partial_-^{(k)} f \in \mathcal{P}(k-1)^+$ is a $\mathscr{Y}$-weight vector with weight $(\alpha_1, \ldots, \alpha_{k-1}, 0, 0, \ldots)$.*

PROOF. We know that from [26] for $g \in \mathcal{P}(k)^+$ and $1 \leq i \leq k-1$, $\mathscr{Y}_i \partial_-^{(k)} g = \partial_-^{(k)} \mathscr{Y}_i g$ so $\mathscr{Y}_i \partial_-^{(k)} f = \partial_-^{(k)} \mathscr{Y}_i f = \alpha_i \partial_-^{(k)} f$. From [26] we have that if $i \geq k$ then $\mathscr{Y}_i$ annihilates $\mathscr{P}(k-1)$. Since $\partial_-^{(k)} f \in \mathcal{P}(k-1)^+$ for all $i \geq k$, $\mathscr{Y}_i \partial_-^{(k)} f = 0$.

$\square$

EXAMPLE. *Here we give a few basic examples of stable-limit non-symmetric Macdonald functions expanded in the Hall-Littlewood basis $\mathcal{P}_\lambda$ and their corresponding weights.*

- $\widetilde{E}_{(\emptyset|2)} = \mathcal{P}_2[x_1 + \ldots] + \dfrac{q^{-1}}{1 - q^{-1}t} \mathcal{P}_{1,1}[x_1 + \ldots]$; *weight* $\widetilde{\alpha}_{(\emptyset|2)} = (0, 0, \ldots)$
- $\widetilde{E}_{(2|\emptyset)} = x_1^2 + \dfrac{q^{-1}}{1 - q^{-1}t} x_1 \mathcal{P}_1[x_2 + \ldots]$; *weight* $\widetilde{\alpha}_{(2|\emptyset)} = (q^2 t, 0, \ldots)$
- $\widetilde{E}_{(1,1,1|\emptyset)} = x_1 x_2 x_3$; *weight* $\widetilde{\alpha}_{(1,1,1|\emptyset)} = (qt^3, qt^2, qt, 0, \ldots)$
- $\widetilde{E}_{(1,1|1)} = x_1 x_2 \mathcal{P}_1[x_3 + \ldots]$; *weight* $\widetilde{\alpha}_{(1,1|1)} = (qt^3, qt^2, 0, \ldots)$
- $\widetilde{E}_{(1|1,1)} = x_1 \mathcal{P}_{1,1}[x_2 + \cdots]$; *weight* $\widetilde{\alpha}_{(1|1,1)} = (qt^3, 0, \ldots)$

As an immediate result of Lemma 2.4.11 we have the following:

COROLLARY 2.4.12. *For $(\mu|\lambda) \in \Sigma$, $\widetilde{E}_{(\mu|\lambda)} \in \mathscr{P}_{as}^+$ is a $\mathscr{Y}$-weight vector with weight $\widetilde{\alpha}_{(\mu|\lambda)}$ given explicitly by*

$$\widetilde{\alpha}_{(\mu|\lambda)}(i) = \begin{cases} \widetilde{\alpha}_{\mu*\lambda}(i) = q^{\mu_i} t^{\ell(\mu)+\ell(\lambda)+1-\beta_{\mu*\lambda}(i)} & i \leq \ell(\mu), \mu_i \neq 0 \\ 0 & \textit{otherwise.} \end{cases}$$

PROOF. By Definition 2.4.10 we have that

$$\widetilde{E}_{(\mu|\lambda)} := \partial_-^{(\ell(\mu)+1)} \cdots \partial_-^{(\ell(\mu)+\ell(\lambda))} \widetilde{E}_{\mu*\lambda}.$$

From Theorem 2.4.3 we know that $\widetilde{E}_{\mu*\lambda}$ is a $\mathscr{Y}$-weight vector with weight $\widetilde{\alpha}_{\mu*\lambda}$. Recall that from Lemma 2.4.1 that $\widetilde{\alpha}_{\mu*\lambda}(i) = q^{(\mu*\lambda)_i} t^{\ell(\mu*\lambda)+1-\beta_{\mu*\lambda}(i)}$ for $i \leq \ell(\mu*\lambda)$ and equals 0 for $i > \ell(\mu*\lambda)$. Using Lemma 2.4.11 inductively now shows that $\widetilde{E}_{(\mu|\lambda)}$ is a $\mathscr{Y}$-weight vector with weight $\widetilde{\alpha}_{(\mu|\lambda)}$ given by the expression given in the statement of this corollary. $\square$

By using the HHL-type formula we proved for the functions $\widetilde{E}_\mu$ in Corollary 2.3.2, we readily find a similar formula for the full set of stable-limit non-symmetric Macdonald functions.

Corollary 2.4.13. *For $(\mu|\lambda) \in \Sigma$ we have that*

$$\widetilde{E}_{(\mu|\lambda)} = \sum_{\substack{\nu \text{ partition} \\ |\nu| \leq |\mu|+|\lambda|}} \sum_{\substack{\sigma:\mu*\lambda*0^{\ell(\nu)} \to [\ell(\mu)+\ell(\lambda)+\ell(\nu)] \\ \text{non-attacking} \\ \forall i=1,\dots,\ell(\nu) \\ \nu_i = |\sigma^{-1}(\ell(\mu)+\ell(\lambda)+i)|}} \widetilde{\Gamma}(\widehat{\sigma}) x_1^{|\sigma^{-1}(1)|} \cdots x_{\ell(\mu)}^{|\sigma^{-1}(\ell(\mu))|} \times$$
$$\mathscr{B}_{|\sigma^{-1}(\ell(\mu)+1)|} \cdots \mathscr{B}_{|\sigma^{-1}(\ell(\mu)+\ell(\lambda))|}(m_\nu)[\mathfrak{X}_{\ell(\mu)+\ell(\lambda)}]$$

where

$$\widetilde{\Gamma}(\widehat{\sigma}) := q^{-\operatorname{maj}(\widehat{\sigma})} t^{\operatorname{coinv}(\widehat{\sigma})} \prod_{\substack{u \in dg'(\mu*\lambda*0^{\ell(\nu)}) \\ \widehat{\sigma}(u) \neq \widehat{\sigma}(d(u)) \\ u \text{ not in row } 1}} \left(\frac{1-t}{1-q^{-(\operatorname{lg}(u)+1)} t^{(a(u)+1)}} \right) \prod_{\substack{u \in dg'(\mu*\lambda*0^{\ell(\nu)}) \\ \widehat{\sigma}(u) \neq \widehat{\sigma}(d(u)) \\ u \text{ in row } 1}} (1-t)\,.$$

Unfortunately, this formula is not nearly as elegant or useful as the HHL formula (2.2.2). The main obstruction comes from not having a full understanding of the action of the Jing operators $\mathscr{B}_a$ on the monomial symmetric functions. If one were to find an explicit expansion of elements like $\mathscr{B}_{a_1} \cdots \mathscr{B}_{a_r}(m_\lambda)$ into another suitable basis of Λ (possibly the $\mathcal{P}_\nu$ basis) one would be able to give a much more elegant description of these functions. Likely there is a nice way to do this that has eluded this author.

2.4.3. $\mathcal{A}_\lambda$ Basis for Λ and Symmetrization via the Trivial Hecke Idempotent. Lemma 2.4.7 shows that the following operator is well defined on $\mathscr{P}_{as}^+$ i.e. independent of k.

Definition 2.4.14. *For $f \in \mathscr{P}(k)^+ \subset \mathscr{P}_{as}^+$ define*

$$\widetilde{\sigma}(f) := \partial_-^{(1)} \cdots \partial_-^{(k)} f. \tag{2.2}$$

Then $\widetilde{\sigma}$ defines an operator $\mathscr{P}_{as}^{+} \to \Lambda$ which we call the ***stable-limit symmetrization operator****.*

REMARK. *Note that $\widetilde{\sigma}(\widetilde{E}_{\lambda}) = \mathcal{A}_{\lambda}$ and $\widetilde{\sigma}(\widetilde{E}_{(\mu|\lambda)}) = \widetilde{\sigma}(\widetilde{E}_{\mu*\lambda})$.*

DEFINITION 2.4.15. *For all $0 \leq k < n$ define the operator $\epsilon_k^{(n)} : \mathscr{P}_n^{+} \to \mathscr{P}_n^{+}$ as*

$$\epsilon_k^{(n)}(f) := \frac{1}{[n-k]_t!} \sum_{\sigma \in \mathfrak{S}_{(1^k, n-k)}} t^{\binom{n-k}{2} - \ell(\sigma)} T_{\sigma}(f). \tag{2.3}$$

Here $\mathfrak{S}_{(1^k,n-k)}$ is the Young subgroup of $\mathfrak{S}_n$ corresponding to the composition $(1^k, n-k)$, $T_\sigma = T_{s_{i_1}} \cdots T_{s_{i_r}}$ whenever $\sigma = s_{i_1} \cdots s_{i_r}$ is a reduced word representing σ, and $[m]_t! := \prod_{i=1}^{m} (\frac{1-t^i}{1-t})$ is the t-factorial. We will simply write $\epsilon^{(n)}$ for $\epsilon_0^{(n)}$.

For $n \geq 1$ define the rational function

$$\Omega_n(x) = \Omega_n(x_1, \ldots, x_n; t) := \prod_{1 \leq i < j \leq n} (\frac{x_i - t x_j}{x_i - x_j}). \tag{2.4}$$

We will need the following technical result relating the action of $\epsilon^{(n)}$ on polynomials to a Weyl character type sum involving Ω_n.

PROPOSITION 2.4.16. *For $f(x) \in \mathscr{P}_n^{+}$*

$$\epsilon^{(n)}(f(x)) = \frac{1}{[n]_t!} \sum_{\sigma \in \mathfrak{S}_n} \sigma(f(x)\Omega_n(x)). \tag{2.5}$$

PROOF. See Remark 4.17 in [33]. After translating the finite Hecke algebra quadratic relations in [33] to match those occurring in this chapter the formula matches. □

From the formula above in Proposition 2.4.16 we can show that the sequence of trivial idempotents $(\epsilon^{(n)})_{n \geq 1}$ converges in the sense of [26].

PROPOSITION 2.4.17. *The sequence of operators $(\epsilon^{(n)})_{n \geq 1}$ converges to an idempotent operator $\epsilon : \mathscr{P}_{as}^{+} \to \Lambda$ such that for all $i \geq 1$, $\epsilon T_i = \epsilon$.*

PROOF. From [30] in Chapter 3 and Proposition 2.4.16 we see that for all partitions λ with $\ell(\lambda) = k$ and $n \geq k$ that

$$\epsilon^{(n)}(x^{\lambda}) = \frac{[n-k]_t!}{[n]_t!} v_{\lambda}(t) P_{\lambda}[x_1 + \ldots + x_n; t] \tag{2.6}$$

where $P_\lambda[X;t]$ is the Hall-Littlewood symmetric function defined by Macdonald (not to be confused with $\mathcal{P}_\lambda[X]$ seen previously in this thesis) and $v_\lambda(t) := \prod_{i\geq 1}([m_i(\lambda)]_t!)$ where $m_i(\lambda)$ is the number of i 's in $\lambda = 1^{m_1(\lambda)}2^{m_2(\lambda)}\cdots$. Now we note that with respect to the t-adic topology,

$$\lim_{n\to\infty} \frac{[n-k]_t!}{[n]_t!} = (1-t)^k$$

so that

$$\lim_n \epsilon^{(n)}(x^\lambda) = v_\lambda(t)(1-t)^{\ell(\lambda)} P_\lambda[X;t]$$

and hence $(\epsilon^{(n)}(x^\lambda))_{n\geq 1}$ converges. Note that following Macdonald's definitions,

$$v_\lambda(t)(1-t)^{\ell(\lambda)} P_\lambda[X;t] = Q_\lambda[X;t].$$

Since $\epsilon^{(n)}T_i = \epsilon^{(n)}$ for $1 \leq i \leq n-1$ it follows that for all compositions μ, the sequence $(\epsilon^{(n)}(x^\mu))_{n\geq 1}$ is convergent. Clearly from definition we have that for all symmetric functions $F \in \Lambda$ and $f(x) \in \mathscr{P}_n^+$

$$\epsilon^{(n)}(F[x_1+\ldots+x_n]f(x)) = F[x_1+\ldots+x_n]\epsilon^{(n)}(f(x)).$$

It follows now from a straightforward convergence argument using Remark 13 that for all $g \in \mathscr{P}_{as}^+$ the sequence $(\epsilon^{(n)}(\Xi_n(g)))_{n\geq 1}$ converges. The resulting operator $\epsilon := \lim_n \epsilon^{(n)} \circ \Xi_n$ is evidently idempotent as its codomain is Λ and certainly ϵ acts as the identity on symmetric functions. Further, for all $i \in \mathbb{N}$ we have

$$\epsilon T_i = \lim_n \epsilon^{(n)} \circ \Xi_n T_i$$

and since Ξ_n commutes with T_i for $n > i+1$ we see that

$$\lim_n \epsilon^{(n)} \circ \Xi_n T_i = \lim_n \epsilon^{(n)} T_i \circ \Xi_n = \lim_n \epsilon^{(n)} \circ \Xi_n = \epsilon.$$

□

Corollary 2.4.18. *For all $k \geq 0$ the sequence $(\epsilon_k^{(n)})_{n>k}$ converges to an idempotent operator $\epsilon_k : \mathscr{P}_{as}^+ \to \mathscr{P}(k)^+$ such that for all $i \geq k+1$, $\epsilon_k T_i = \epsilon_k$.*

Proof. This follows immediately from Proposition 2.4.17 after shifting indices and noting that the operators $\epsilon_k^{(n)}$ commute with multiplication by $x_1, \ldots, x_k$. □

Now we will extend our definition of the stable-limit symmetrization operator $\widetilde{\sigma}$ to partial symmetrization operators in the natural way.

DEFINITION 2.4.19. *For $k \geq 0$ let $\widetilde{\sigma}_k : \mathscr{P}_{as}^{+} \to \mathscr{P}(k)^{+}$ be defined on $g \in \mathscr{P}(n)^{+}$ for $n \geq k$ by*

$$\widetilde{\sigma}_k(g) := \partial_{-}^{(k+1)} \cdots \partial_{-}^{(n)}(g). \tag{2.7}$$

REMARK. *The operators $\widetilde{\sigma}_k$ are well defined by Lemma 2.4.7. In particular, if $g \in \mathscr{P}(\ell)^{+}$ for $0 \leq \ell \leq k$ then $\mathscr{P}(\ell)^{+} \subset \mathscr{P}(k)^{+}$ and there is no ambiguity in defining $\widetilde{\sigma}_k(g) = \partial_{-}^{(k+1)} \cdots \partial_{-}^{(n)}(g)$ as above. Note that $\widetilde{\sigma}_0 = \widetilde{\sigma}$. Further, for all $(\mu|\lambda) \in \Sigma$ we see that in this new terminology*

$$\widetilde{E}_{(\mu|\lambda)} = \widetilde{\sigma}_{\ell(\mu)}(\widetilde{E}_{\mu*\lambda}).$$

Further, if $k \leq \ell$ then $\widetilde{\sigma}_k \widetilde{\sigma}_\ell = \widetilde{\sigma}_k$.

We will now show that as operators on $\mathscr{P}_{as}^{+}$, $\epsilon_\ell = \widetilde{\sigma}_\ell$ for all $\ell \geq 0$.

PROPOSITION 2.4.20. *For all $\ell \geq 0$, $\epsilon_\ell = \widetilde{\sigma}_\ell$.*

PROOF. By shifting indices it suffices to just prove that $\epsilon = \widetilde{\sigma}$, i.e., the $\ell = 0$ case. Further, since both maps are T_i-equivariant Λ-module maps (see Corollary 2.4.9) it suffices to show that for all partitions λ, $\epsilon(x^\lambda) = \widetilde{\sigma}(x^\lambda)$. From the proof of Proposition 2.4.17 we saw that $\epsilon(x^\lambda) = Q_\lambda[X;t]$ whereas it follows from the definition of the Jing vertex operators that $\widetilde{\sigma}(x^\lambda) = \mathcal{P}_\lambda[X]$. Therefore, it suffices to argue that $Q_\lambda[X;t] = \mathcal{P}_\lambda[X]$. To this end we will prove that

$$\mathcal{P}_\lambda[X] = \langle z_1^{\lambda_1} \cdots z_r^{\lambda_r} \rangle \operatorname{Exp}[(1-t)(z_1 + \ldots + z_r)X] \operatorname{Exp}[(t-1) \sum_{1 \leq i < j \leq r} \frac{z_j}{z_i}] \tag{2.8}$$

which by 2.15 in Macdonald Chapter 3 [30] is an alternative definition for $Q_\lambda[X;t]$.

Suppose $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a partition. Note first that by definition $\mathcal{P}_\lambda[X] = \mathscr{B}_{\lambda_1} \cdots \mathscr{B}_{\lambda_r}(1)$. We will now induct on the number of operators $\mathscr{B}$ acting on 1 in the expression $\mathscr{B}_{\lambda_1} \cdots \mathscr{B}_{\lambda_r}(1)$. As a base case

$$\mathscr{B}_{\lambda_r}(1) = \langle z_r^{\lambda_r} \rangle 1[X - z_r^{-1}] \operatorname{Exp}[(1-t) z_r X] = \langle z_r^{\lambda_r} \rangle \operatorname{Exp}[(1-t) z_r X].$$

We claim that for all $1 \leq k \leq r$

$$\mathscr{B}_{\lambda_k}\cdots\mathscr{B}_{\lambda_r}(1) = \langle z_k^{\lambda_k}\cdots z_r^{\lambda_r}\rangle\,\mathrm{Exp}[(1-t)(z_k+\ldots+z_r)X]\,\mathrm{Exp}[(t-1)\sum_{k\le i<j\le r}\frac{z_j}{z_i}]. \tag{2.9}$$

Suppose the above is true for some $1 < k \le r$. Then

$$\begin{aligned}
&\mathscr{B}_{\lambda_{k-1}}\mathscr{B}_{\lambda_k}\cdots\mathscr{B}_{\lambda_r}(1)\\
&= \mathscr{B}_{\lambda_{k-1}}\left(\langle z_k^{\lambda_k}\cdots z_r^{\lambda_r}\rangle\,\mathrm{Exp}[(1-t)(z_k+\ldots+z_r)X]\,\mathrm{Exp}[(t-1)\sum_{k\le i<j\le r}\frac{z_j}{z_i}]\right)\\
&= \langle z_{k-1}^{\lambda_{k-1}}\rangle\langle z_k^{\lambda_k}\cdots z_r^{\lambda_r}\rangle\,\mathrm{Exp}[(1-t)(z_k+\ldots+z_r)(X-z_{k-1}^{-1})]\,\mathrm{Exp}[(t-1)\sum_{k\le i<j\le r}\frac{z_j}{z_i}]\\
&\times \mathrm{Exp}[(1-t)z_{k-1}X].
\end{aligned}$$

Now we use the additive property of the plethystic exponential namely,

$$\mathrm{Exp}[A+B] = \mathrm{Exp}[A]\,\mathrm{Exp}[B]$$

, to rearrange terms and get

$$\begin{aligned}
&\langle z_{k-1}^{\lambda_{k-1}}\cdots z_r^{\lambda_r}\rangle\,\mathrm{Exp}[(1-t)(z_k+\ldots+z_r)X]\,\mathrm{Exp}[(1-t)z_{k-1}X]\,\mathrm{Exp}[(t-1)\sum_{k\le i<j\le r}\frac{z_j}{z_i}]\\
&\times \mathrm{Exp}[(t-1)(\frac{z_k}{z_{k-1}}+\ldots+\frac{z_r}{z_{k-1}})]
\end{aligned}$$

which simplifies to

$$\langle z_{k-1}^{\lambda_{k-1}}\cdots z_r^{\lambda_r}\rangle\,\mathrm{Exp}[(1-t)(z_{k-1}+z_k+\ldots+z_r)X]\,\mathrm{Exp}[(t-1)\sum_{k-1\le i<j\le r}\frac{z_j}{z_i}]$$

showing that the formula (2.9) holds for all $1 \le k \le r$. Taking $k = 1$ shows equation (2.8) holds. $\square$

As an immediate consequence of Proposition 2.4.20 we find the following enlightening description for the $\widetilde{E}_{(\mu|\lambda)}$ functions.

Corollary 2.4.21. *For all $(\mu|\lambda) \in \Sigma$,*

$$\widetilde{E}_{(\mu|\lambda)} = \lim_n \epsilon^{(n)}_{\ell(\mu)}(E_{\mu*\lambda*0^{n-(\ell(\mu)+\ell(\lambda))}}). \tag{2.10}$$

In particular, for partitions λ, $\mathcal{A}_\lambda[X] = (1-t)^{\ell(\lambda)} v_\lambda(t) P_\lambda[X; q^{-1}, t]$ where $P_\lambda[X; q^{-1}, t]$ is the symmetric Macdonald function. As a consequence the set $\{\mathcal{A}_\lambda : \lambda \in \mathbb{Y}\}$ is a basis of Λ.

Remark. *The $P_\lambda[X; q, t]$ are the symmetric Macdonald functions as defined by Macdonald in [30] and seen in Cherednik's work [9] not to be confused with the modified symmetric Macdonald functions $\widetilde{H}_\mu$ seen in many places but in particular in the work of Haiman [24]. Further, Corollary 2.4.21 gives an interpretation of the $\widetilde{E}_{(\mu|\lambda)}$ as limits of partially symmetrized non-symmetric Macdonald polynomials. Goodberry in [18] and Lapointe in [28] have investigated similar families of partially symmetric Macdonald polynomials. Up to a change of variables and limiting these different notions are likely directly related.*

In order to prove the first main theorem in this chapter, Theorem 4.2.12, we will require the following straightforward lemma.

Lemma 2.4.22. *For any composition μ there is some nonzero scalar $\gamma_\mu \in \mathbb{Q}(q,t)$ such that*

$$\widetilde{\sigma}(\widetilde{E}_\mu) = \gamma_\mu \mathcal{A}_{\mathrm{sort}(\mu)}$$

where $\gamma_\mu = 1$ when μ is a partition.

Proof. We know that for all partitions λ, $\widetilde{\sigma}(\widetilde{E}_\lambda) = \mathcal{A}_\lambda$ so this lemma holds trivially for partitions. Now we proceed by induction on Bruhat order similarly to the argument in the proof of Theorem 2.4.3. To show the lemma holds it suffices to show that if μ is a composition and k such that $s_k(\mu) > \mu$ in Bruhat order and $\widetilde{\sigma}(\widetilde{E}_\mu) = \gamma_\mu \mathcal{A}_{\mathrm{sort}(\mu)}$ for $\gamma_\mu \neq 0$ then $\widetilde{\sigma}(\widetilde{E}_{s_k(\mu)}) = \gamma_{s_k(\mu)} \mathcal{A}_{\mathrm{sort}(\mu)}$ for $\gamma_{s_k(\mu)} \neq 0$. To this end fix such μ and k. Then by Corollary 2.4.4

$$\widetilde{E}_{s_k(\mu)} = \left(T_k + \frac{(1-t)\widetilde{\alpha}_\mu(k+1)}{\widetilde{\alpha}_\mu(k) - \widetilde{\alpha}_\mu(k+1)} \right) \widetilde{E}_\mu.$$

From Proposition 2.4.20 $\widetilde{\sigma} = \lim_m \epsilon^{(m)}$ so that $\widetilde{\sigma} T_k = \widetilde{\sigma}$. Therefore,

$$\begin{aligned}\widetilde{\sigma}(\widetilde{E}_{s_k(\mu)}) &= \widetilde{\sigma}\left(\left(T_k + \frac{(1-t)\widetilde{\alpha}_\mu(k+1)}{\widetilde{\alpha}_\mu(k) - \widetilde{\alpha}_\mu(k+1)}\right)\widetilde{E}_\mu\right) \\ &= \left(1 + \frac{(1-t)\widetilde{\alpha}_\mu(k+1)}{\widetilde{\alpha}_\mu(k) - \widetilde{\alpha}_\mu(k+1)}\right)\widetilde{\sigma}(\widetilde{E}_\mu) \\ &= \left(\frac{\widetilde{\alpha}_\mu(k) - t\widetilde{\alpha}_\mu(k+1)}{\widetilde{\alpha}_\mu(k) - \widetilde{\alpha}_\mu(k+1)}\right)\gamma_\mu \mathcal{A}_{\mathrm{sort}(\mu)}.\end{aligned}$$

By Lemma 2.4.1 we see that since $s_k(\mu) > \mu$ it follows that $\widetilde{\alpha}_\mu(k) \neq t\widetilde{\alpha}_\mu(k+1)$. Hence, $\gamma_{s_k(\mu)} := \left(\frac{\widetilde{\alpha}_\mu(k)-t\widetilde{\alpha}_\mu(k+1)}{\widetilde{\alpha}_\mu(k)-\widetilde{\alpha}_\mu(k+1)}\right)\gamma_\mu \neq 0$ so the result follows. $\square$

REMARK. *Note that using the recursive formula* $\gamma_{s_k(\mu)} = \left(\frac{\widetilde{\alpha}_\mu(k)-t\widetilde{\alpha}_\mu(k+1)}{\widetilde{\alpha}_\mu(k)-\widetilde{\alpha}_\mu(k+1)}\right)\gamma_\mu$ *in the proof of Lemma 2.4.22, the formula for the eigenvalues* $\widetilde{\alpha}_\mu(k)$ *in Lemma 2.4.1, and the base condition* $\gamma_\mu = 1$ *for* μ *a partition, it is possible to give an explicit expression for* γ_μ *for any composition* μ*. However, all we need for the purposes of this chapter is that* $\gamma_\mu \neq 0$ *so we will not find such an explicit expression for* γ_μ*.*

2.4.3.1. *First Main Theorem and a Full* $\mathscr{Y}$*-Weight Basis of* $\mathscr{P}_{as}^+$*.* Finally, we prove that the stable-limit non-symmetric Macdonald functions are a basis for $\mathscr{P}_{as}^+$. To do this we will use the stable-limit symmetrization operator to help distinguish between stable-limit non-symmetric Macdonald functions with the same $\mathscr{Y}$-weight.

THEOREM 2.4.23. *(First Main Theorem) The* $\widetilde{E}_{(\mu|\lambda)}$ *are a* $\mathscr{Y}$*-weight basis for* $\mathscr{P}_{as}^+$*.*

PROOF. As there are sufficiently many $\widetilde{E}_{(\mu|\lambda)}$ in each graded component of every $\mathscr{P}(k)^+$ it suffices to show that these functions are linearly independent. Certainly, weight vectors in distinct weight spaces are linearly independent. Using Lemmas 2.4.1 and 2.4.12, we deduce that if $\widetilde{E}_{(\mu^{(1)}|\lambda^{(1)})}$ and $\widetilde{E}_{(\mu^{(2)}|\lambda^{(2)})}$ have the same weight then necessarily $\mu^{(1)} = \mu^{(2)}$. Hence, we can restrict to the case where we have a dependence relation

$$c_1\widetilde{E}_{(\mu|\lambda^{(1)})} + \ldots + c_N\widetilde{E}_{(\mu|\lambda^{(N)})} = 0$$

for $\lambda^{(1)}, \ldots, \lambda^{(N)}$ distinct partitions. By applying the stable-limit symmetrization operator we see that

$$\widetilde{\sigma}(c_1 \widetilde{E}_{(\mu|\lambda^{(1)})} + \ldots + c_N \widetilde{E}_{(\mu|\lambda^{(N)})}) = \widetilde{\sigma}(c_1 \widetilde{E}_{\mu*\lambda^{(1)}} + \ldots + c_N \widetilde{E}_{\mu*\lambda^{(N)}}) = 0.$$

Now by Lemma 2.4.22, $\widetilde{\sigma}(\widetilde{E}_{\mu*\lambda^{(i)}}) = \gamma_{\mu*\lambda^{(i)}} \mathcal{A}_{\mathrm{sort}(\mu*\lambda^{(i)})}$ with nonzero scalars $\gamma_{\mu*\lambda^{(i)}}$ yielding

$$0 = c'_1 \mathcal{A}_{\mathrm{sort}(\mu*\lambda^{(1)})} + \ldots + c'_n \mathcal{A}_{\mathrm{sort}(\mu*\lambda^{(N)})}.$$

The partitions $\lambda^{(i)}$ are distinct so we know that the partitions $\mathrm{sort}(\mu * \lambda^{(i)})$ are distinct as well. By Corollary 2.4.21 the symmetric functions $\mathcal{A}_{\mathrm{sort}(\mu*\lambda^{(i)})}$ are linearly independent. Thus $c'_i = 0$ implying $c_i = 0$ for all $1 \leq i \leq N$ as desired. $\square$

2.5. Some Recurrence Relations for the $\widetilde{E}_{(\mu|\lambda)}$

In this section we will discuss a few recurrence relations for the stable-limit non-symmetric Macdonald functions. We start by looking at the action of the Demazure-Lusztig operators T_i and the lowering operators ∂_-.

Proposition 2.5.1. *For $(\mu|\lambda) = (\mu_1, \ldots, \mu_r|\lambda_1, \ldots, \lambda_k) \in \Sigma$ if $\mu_r \geq \lambda_1$ and $\mu_{r-1} \neq 0$ then*

$$\partial_-^{(r)} \left(\widetilde{E}_{(\mu_1,\ldots,\mu_r|\lambda_1,\ldots,\lambda_k)} \right) = \widetilde{E}_{(\mu_1,\ldots,\mu_{r-1}|\mu_r,\lambda_1,\ldots,\lambda_k)}.$$

Proof. This follows immediately from the definitions of $\widetilde{E}_{(\mu|\lambda)}$ and $\partial_-^{(r)}$. $\square$

Proposition 2.5.2. *Take $(\mu|\lambda) \in \Sigma$ and suppose $1 \leq i \leq \ell(\mu) - 1$ such that $s_i(\mu) > \mu$ and $s_i(\mu) \in \mathrm{Comp}^{red}$. Then*

$$\widetilde{E}_{(s_i(\mu)|\lambda)} = \left(T_i + \frac{(1-t)\widetilde{\alpha}_{\mu*\lambda}(i+1)}{\widetilde{\alpha}_{\mu*\lambda}(i) - \widetilde{\alpha}_{\mu*\lambda}(i+1)} \right) \widetilde{E}_{(\mu|\lambda)}.$$

Proof. Since $s_i(\mu) > \mu$ we know that $s_i(\mu * \lambda) > \mu * \lambda$ so by Corollary 2.4.4

$$\widetilde{E}_{s_i(\mu*\lambda)} = \left(T_i + \frac{(1-t)\widetilde{\alpha}_{\mu*\lambda}(i+1)}{\widetilde{\alpha}_{\mu*\lambda}(i) - \widetilde{\alpha}_{\mu*\lambda}(i+1)} \right) \widetilde{E}_{\mu*\lambda}.$$

Now we know T_i commutes with the operators $\partial_-^{(\ell(\mu)+1)}, \ldots, \partial_-^{(\ell(\mu)+\ell(\lambda))}$ and thus we see that

$$\begin{aligned}
\widetilde{E}_{(s_i(\mu)|\lambda)} &= \partial_-^{(\ell(\mu)+1)} \cdots \partial_-^{(\ell(\mu)+\ell(\lambda))}(\widetilde{E}_{s_i(\mu*\lambda)}) \\
&= \partial_-^{(\ell(\mu)+1)} \cdots \partial_-^{(\ell(\mu)+\ell(\lambda))}\left(\left(T_i + \frac{(1-t)\widetilde{\alpha}_{\mu*\lambda}(i+1)}{\widetilde{\alpha}_{\mu*\lambda}(i) - \widetilde{\alpha}_{\mu*\lambda}(i+1)}\right)\widetilde{E}_{\mu*\lambda}\right) \\
&= \left(T_i + \frac{(1-t)\widetilde{\alpha}_{\mu*\lambda}(i+1)}{\widetilde{\alpha}_{\mu*\lambda}(i) - \widetilde{\alpha}_{\mu*\lambda}(i+1)}\right)\partial_-^{(\ell(\mu)+1)} \cdots \partial_-^{(\ell(\mu)+\ell(\lambda))}(\widetilde{E}_{\mu*\lambda}) \\
&= \left(T_i + \frac{(1-t)\widetilde{\alpha}_{\mu*\lambda}(i+1)}{\widetilde{\alpha}_{\mu*\lambda}(i) - \widetilde{\alpha}_{\mu*\lambda}(i+1)}\right)\widetilde{E}_{(\mu|\lambda)}.
\end{aligned}$$

□

PROPOSITION 2.5.3. *For $(\mu|\lambda) = (\mu_1, \ldots, \mu_r|\lambda) \in \Sigma$ we have that*

$$T_r\widetilde{E}_{(\mu|\lambda)} = \frac{\gamma_{\mu*\lambda}}{\gamma_{(\mu_1,\ldots,\mu_{r-1},0,\mu_r)*\lambda}}\widetilde{E}_{(\mu_1,\ldots,\mu_{r-1},0,\mu_r|\lambda)}.$$

PROOF. First note that by Corollary 2.4.12

$$\begin{aligned}
&\phi_r(\widetilde{E}_{(\mu|\lambda)}) \\
&= (T_r(\mathscr{Y}_r - \mathscr{Y}_{r+1}) + (1-t)\mathscr{Y}_{r+1})\widetilde{E}_{(\mu|\lambda)} \\
&= (\widetilde{\alpha}_{\mu*\lambda}(r) - 0)T_r\widetilde{E}_{(\mu|\lambda)} + (1-t)(0)\widetilde{E}_{(\mu|\lambda)} \\
&= \widetilde{\alpha}_{\mu*\lambda}(r)T_r\widetilde{E}_{(\mu|\lambda)}.
\end{aligned}$$

and by Lemma 2.4.1 $\widetilde{\alpha}_{\mu*\lambda}(r) \neq 0$ since $\mu_r \neq 0$. Therefore, $\phi_r(\widetilde{E}_{(\mu|\lambda)})$ is nonzero and therefore must be a $\mathscr{Y}$-weight vector with weight $(\widetilde{\alpha}_{\mu*\lambda}(1), \ldots, \widetilde{\alpha}_{\mu*\lambda}(r-1), 0, \widetilde{\alpha}_{\mu*\lambda}(r), 0, \ldots)$. By using the explicit formula for the eigenvalues $\widetilde{\alpha}_{\mu*\lambda}(i)$ from Lemma 2.4.1 we see that for $1 \leq i \leq r$, $\widetilde{\alpha}_{\mu*\lambda}(i) = 0$ exactly when $\mu_i = 0$ and further, for all $1 \leq i \leq r$ with $\mu_i \neq 0$, $\widetilde{\alpha}_{\mu*\lambda}(i) = q^{\mu_i}t^{b_i}$ for some b_i. Hence by Theorem 4.2.12 and Corollary 2.4.12, $\phi_r(\widetilde{E}_{(\mu|\lambda)})$ is of the form

$$\phi_r(\widetilde{E}_{(\mu|\lambda)}) = \sum_\nu a_\nu \widetilde{E}_{(\mu_1,\ldots,\mu_{r-1},0,\mu_r|\nu)}$$

ν ranges over a finite set of partitions ν and a_ν are some scalars. Note that we have

$$\widetilde{\sigma}(\phi_r(\widetilde{E}_{(\mu|\lambda)})) = \widetilde{\sigma}(\widetilde{\alpha}_{\mu*\lambda}(r)T_r\widetilde{E}_{(\mu|\lambda)})$$

and since $\widetilde{\sigma}T_r = \widetilde{\sigma}$

$$\widetilde{\sigma}(\phi_r(\widetilde{E}_{(\mu|\lambda)})) = \widetilde{\alpha}_{\mu*\lambda}(r)\widetilde{\sigma}(\widetilde{E}_{(\mu|\lambda)}) = \widetilde{\alpha}_{\mu*\lambda}(r)\gamma_{\mu*\lambda}\mathcal{A}_{\mathrm{sort}(\mu*\lambda)}$$

using Lemma 2.4.22. Similarly, we see that

$$\widetilde{\sigma}\left(\sum_\nu a_\nu \widetilde{E}_{(\mu_1,\ldots,\mu_{r-1},0,\mu_r|\nu)}\right) = \sum_\nu a_\nu \gamma_{(\mu_1,\ldots,\mu_{r-1},0,\mu_r)*\nu}\mathcal{A}_{\mathrm{sort}(\mu*\nu)}$$

since $\mathrm{sort}((\mu_1,\ldots,\mu_{r-1},0,\mu_r)*\nu) = \mathrm{sort}(\mu*\nu)$ for all ν.

Thus

$$\mathcal{A}_{\mathrm{sort}(\mu*\lambda)} = \sum_\nu a'_\nu \mathcal{A}_{\mathrm{sort}(\mu*\nu)}$$

where

$$a'_\nu := \frac{a_\nu \gamma_{(\mu_1,\ldots,\mu_{r-1},0,\mu_r)*\nu}}{\widetilde{\alpha}_{\mu*\lambda}(r)\gamma_{\mu*\lambda}}.$$

By Corollary 2.4.21 we know that the $\mathcal{A}_\beta$ are a basis for Λ and so we see that the only possible partition ν that can contribute a nonzero term in the above expansion is $\nu = \lambda$. Further, $a'_\lambda = 1$ and thus $a_\lambda = \frac{\widetilde{\alpha}_{\mu*\lambda}(r)\gamma_{\mu*\lambda}}{\gamma_{(\mu_1,\ldots,\mu_{r-1},0,\mu_r)*\lambda}}$.

Therefore,

$$\phi_r(\widetilde{E}_{(\mu|\lambda)}) = \widetilde{\alpha}_{\mu*\lambda}(r)T_r\widetilde{E}_{(\mu|\lambda)} = \frac{\widetilde{\alpha}_{\mu*\lambda}(r)\gamma_{\mu*\lambda}}{\gamma_{(\mu_1,\ldots,\mu_{r-1},0,\mu_r)*\lambda}}\widetilde{E}_{(\mu_1,\ldots,\mu_{r-1},0,\mu_r|\lambda)}$$

which yields

$$T_r\widetilde{E}_{(\mu|\lambda)} = \frac{\gamma_{\mu*\lambda}}{\gamma_{(\mu_1,\ldots,\mu_{r-1},0,\mu_r)*\lambda}}\widetilde{E}_{(\mu_1,\ldots,\mu_{r-1},0,\mu_r|\lambda)}.$$

□

Definition 2.5.4. *Define* $\widetilde{\pi}_m := X_1T_1^{-1}\cdots T_{m-1}^{-1}$ *considered as an operator on* $\mathscr{P}_m^+$.

Remark. *These operators are the same as the corresponding operators of the same name defined by Ion and Wu up to inversion and some scalars. We have defined the operators as above for convenience. The operators* π_m *and* $\widetilde{\pi}_m$ *are used by Ion and Wu [26] to give operators analogous to the* d_+, d_+^* *operators in* $\mathbb{A}_{t,q}$.

LEMMA 2.5.5. *The sequences of operators $(\widetilde{\pi}_m)_{m\geq 1}$ and $(\pi_m)_{m\geq 1}$ converge to operators $\widetilde{\pi}, \pi : \mathscr{P}_{as}^+ \to \mathscr{P}_{as}^+$ respectively with actions given by*

- $\widetilde{\pi}(x_1^{a_1}\cdots x_k^{a_k}F[X]) = x_1T_1^{-1}\cdots T_k^{-1}x_1^{a_1}\cdots x_k^{a_k}F[X]$
- $\pi(x_1^{a_1}\cdots x_k^{a_k}F[X]) = x_2^{a_1}\cdots x_{k+1}^{a_k}F[X+(q-1)x_1]$.

PROOF. Let $(f_m)_{m\geq 1}$ be a convergent sequence with limit $f \in \mathscr{P}(k)^+$. We start by showing the sequence $(\widetilde{\pi}_m(f_m))_{m\geq 1}$ converges to an element of $\mathscr{P}_{as}^+$. It follows directly by the definition of convergence that there exists some $M > k$ such that for all i and m with $m \geq M$ and $k+1 \leq i \leq m-1$, $T_if_m = f_m$. Therefore, for all $m \geq M$

$$\widetilde{\pi}_m(f_m) = x_1T_1^{-1}\cdots T_k^{-1}f_m$$

which clearly converges to $x_1T_1^{-1}\cdots T_k^{-1}f$. It follows then that the sequence of operators $(\widetilde{\pi}_m)_{m\geq 1}$ converges to an operator which we call $\widetilde{\pi}$. By considering $f = x_1^{a_1}\cdots x_k^{a_k}F[X]$ with $F \in \Lambda$ we get the first formula in the lemma statement above.

Next we will show the sequence $(\pi_m(\Xi_m(f)))_{m\geq 1}$ converges. Expand f as

$$f = \sum_{i=1}^{N} c_i x^{\mu^{(i)}} F_i[X]$$

for $c_i \in \mathbb{Q}(q,t)$, compositions $\mu^{(i)}$, and $F_i \in \Lambda$ where we may assume each composition $\mu^{(i)}$ has length k so that for all $m \geq k$

$$\Xi_m(f) = \sum_{i=1}^{N} c_i x^{\mu^{(i)}} F_i[x_1+\ldots+x_m].$$

Applying π_m to $\Xi_m(f)$ gives for $m \geq k$

$$\pi_m(\Xi_m(f)) = \sum_{i=1}^{N} c_i x_2^{\mu_1^{(i)}}\cdots x_{k+1}^{\mu_k^{(i)}} F[qx_1+x_2+\ldots+x_m]$$

so therefore we get

$$\lim_m \pi_m(\Xi_m(f)) = \sum_{i=1}^{N} c_i x_2^{\mu_1^{(i)}}\cdots x_{k+1}^{\mu_k^{(i)}} F[X+(q-1)x_1].$$

Thus the sequence of operators $(\pi_m)_{m\geq 1}$ converges to an operator which we call π. Lastly, by applying this formula to $f = x_1^{a_1}\cdots x_k^{a_k}F[X]$ with $F \in \Lambda$ to see the second formula given in the lemma statement. $\square$

In line with the above results in this section we will now give a partial generalization of the classical Knop-Sahi relation regarding the action of the π operators on Macdonald polynomials.

Proposition 2.5.6. *For all compositions μ*

$$t^{\#\{j:\mu_j\neq 0\}}\widetilde{\pi}(\widetilde{E}_\mu) = x_1\pi(\widetilde{E}_\mu) = \widetilde{E}_{1*\mu}.$$

Proof. Suppose $\ell(\mu) = n$. Recall that for all $m \geq 1$

$$(Y_{n+m}^{(n+m)})^{-1} = t^{n+m-1}\pi_{n+m}^{-1}T_1^{-1}\cdots T_{n+m-1}^{-1}.$$

Therefore, by recalling the eigenvalue notation in Lemma 2.4.1 we have

$$t^{n+m-1}\pi_{n+m}^{-1}T_1^{-1}\cdots T_{n+m-1}^{-1}E_{\mu*0^m} = (Y_{n+m}^{(n+m)})^{-1}E_{\mu*0^m} = \alpha_\mu^{(m)}(n+m)^{-1}E_{\mu*0^m}$$

so that

$$t^{n+m-1}\alpha_\mu^{(m)}(n+m)x_1T_1^{-1}\cdots T_{n+m-1}^{-1}E_{\mu*0^m} = x_1\pi_{n+m}E_{\mu*0^m}.$$

From Lemma 2.4.1 we see that

$$t^{n+m-1}\alpha_\mu^{(m)}(n+m) = t^{\#\{j:\mu_j\neq 0\}}$$

which gives

$$t^{\#\{j:\mu_j\neq 0\}}x_1T_1^{-1}\cdots T_{n+m-1}^{-1}E_{\mu*0^m} = t^{\#\{j:\mu_j\neq 0\}}\widetilde{\pi}_{n+m}(E_{\mu*0^m}) = x_1\pi_{n+m}E_{\mu*0^m}.$$

From the classical Knop-Sahi relations (see [19]) applied to $E_{\mu*0^m}$ we get

$$x_1\pi_{n+m}E_{\mu*0^m} = E_{1*\mu*0^{m-1}}.$$

Applying Corollary 2.3.2 and Lemma 2.5.5 as $m \to \infty$ now gives

$$t^{\#\{j:\mu_j\neq 0\}}\widetilde{\pi}(\widetilde{E}_\mu) = x_1\pi(\widetilde{E}_\mu) = \widetilde{E}_{1*\mu}.$$

$\square$

2.6. Constructing $\widetilde{E}_{(\mu|\lambda)}$-Diagonal Operators from Symmetric Functions

The main goal of the following section of this chapter is to construct an operator on $\mathscr{P}_{as}^{+}$ which is diagonal in the stable-limit Macdonald function basis, commutes with the limit Cherednik operators $\mathscr{Y}_i$, but does not annihilate Λ. This operator will be constructed from a limit of operators arising from the action of $t^mY_1^{(m)}+\ldots+t^mY_m^{(m)}$ on $\mathscr{P}_m^{+}$. After finding the eigenvalues of this new operator we will show that the addition of this operator to the algebra generated by the limit Cherednik operators has simple spectrum on $\mathscr{P}_{as}^{+}$.

We begin with the following natural definition.

Definition 2.6.1. *For $F \in \Lambda$ define the operator $\Psi_F^{(m)} : \mathscr{P}_m^{+} \to \mathscr{P}_m^{+}$ by*

$$\Psi_F^{(m)} := F[t^mY_1^{(m)} + \ldots + t^mY_m^{(m)}]. \tag{2.11}$$

Further, for a composition μ with $\ell(\mu) = n$ and $m \geq 0$ define the scalar $\kappa_\mu^{(m)}(q,t)$ as

$$\kappa_\mu^{(m)}(q,t) := \sum_{i=1}^{n+m} t^{n+m}\alpha_\mu^{(m)}(i).$$

Recall from Lemma 2.4.1 that $\alpha_\mu^{(m)}(i)$ is given by $Y_i^{(n+m)}E_{\mu*0^m} = \alpha_\mu^{(m)}(i)E_{\mu*0^m}$.

Lemma 2.6.2. *For all compositions μ the sequence $(\kappa_\mu^{(m)}(q,t))_{m\geq 0}$ converges to some $\kappa_\mu(q,t) \in \mathbb{Q}(q,t)$. Further, $\kappa_\mu(q,t) = \kappa_{\mu*0^k}(q,t)$ for all $k \geq 0$ and $\kappa_\mu(q,t) = \kappa_{s_i(\mu)}(q,t)$ for all $1 \leq i \leq \ell(\mu)-1$.*

Proof. Using Lemma 2.4.1 we get the following:

$$\begin{aligned}
\kappa_\mu^{(m)}(q,t) &= \sum_{i=1}^{n+m} t^{n+m}\alpha_\mu^{(m)}(i) \\
&= \sum_{i=1}^{n} t^{n+m}\alpha_\mu^{(m)}(i) + \sum_{i=n+1}^{n+m} t^{\#\{j:\mu_j\neq 0\}}t^{m+1-(i-n)} \\
&= \sum_{i=1}^{n} t^{n}\alpha_\mu^{(0)}(i)t^{m\mathbb{1}(\mu_i=0)} + t^{\#\{j:\mu_j\neq 0\}}\sum_{i=1}^{m} t^{m-i+1} \\
&= \sum_{\mu_i\neq 0} t^{n}\alpha_\mu^{(0)}(i) + t^m\sum_{\mu_i=0} t^{n}\alpha_\mu^{(0)}(i) + t^{\#\{j:\mu_j\neq 0\}}\sum_{i=1}^{m} t^{i}.
\end{aligned}$$

Therefore,

$$\kappa_\mu(q,t) := \lim_m \kappa_\mu^{(m)}(q,t) = \left(\sum_{i:\mu_i\neq 0} t^n \alpha_\mu^{(0)}(i)\right) + \frac{t^{1+\#\{j:\mu_j\neq 0\}}}{1-t} \in \mathbb{Q}(q,t). \tag{2.12}$$

The last statement regarding $\kappa_{\mu*0^k}(q,t)$ and $\kappa_{s_i(\mu)}(q,t)$ follows now directly from Lemma 2.4.1 and classical DAHA intertwiner theory. $\square$

Remark. *Recall from the proof of Lemma 2.4.1 that*

$$t^n\alpha_\mu^{(0)} = q^{\mu_i} t^{n+1-\beta_\mu(i)}.$$

Applying this to the Lemma 2.6.2 gives the combinatorial formula

$$\kappa_\mu(q,t) = \frac{t^{1+\#\{j:\mu_j\neq 0\}}}{1-t} + \sum_{\mu_i\neq 0} q^{\mu_i} t^{n+1-\beta_\mu(i)}.$$

If we consider the partition λ to have an infinite string of 0's attached to its tail then

$$\kappa_\lambda(q,t) = \sum_{i=1}^{\infty} q^{\lambda_i} t^i.$$

Notice that this is exactly equal to

$$\frac{t}{1-t}\left(1-(1-t)(1-q)B_\lambda(q,t)\right)$$

where $B_\lambda(q,t)$ is the diagram generator of λ in [22].

Corollary 2.6.3. *Let λ and ν be partitions. Then $\kappa_\lambda(q,t) = \kappa_\nu(q,t)$ if and only if $\lambda = \nu$.*

Proof. This follows readily from the identity

$$\kappa_\lambda(q,t) = \sum_{i=1}^{\infty} q^{\lambda_i} t^i$$

given in the prior remark. $\square$

In this next result we will show that the sequence of operators $(\Psi_{p_1}^{(m)})_{m\geq 1}$ converges to a well defined map on $\mathscr{P}_{as}^+$. As expected these operators are well-behaved on sequences of the form $\epsilon_{\ell(\mu)}^{(m)}(E_{\mu*\lambda*0^{m-(\ell(\mu)+\ell(\lambda))}})$. In fact it is not hard to show that $(\Psi_{p_1}^{(m)})_{m\geq 1}$ converges on the former sequences. However, this is not a sufficient argument to show the convergence of the $(\Psi_{p_1}^{(m)})_{m\geq 1}$.

In order to obtain a well-defined operator on $\mathscr{P}_{as}^+$ from the sequence of operators $(\Psi_{p_1}^{(m)})_{m\geq 1}$ one needs to show that given an arbitrary convergent sequence $(f_m)_{m\geq 1}$ the corresponding sequence $(\Psi_{p_1}^{(m)}(f_m))_{m\geq 1}$ converges. Therefore, the difficulty in the following proof is to show that the $\Psi_{p_1}^{(m)}$ are well behaved in general.

THEOREM 2.6.4. *The sequence of operators* $(\Psi_{p_1}^{(m)})_{m\geq 1}$ *converges to an operator* $\Psi_{p_1} : \mathscr{P}_{as}^+ \to \mathscr{P}_{as}^+$ *which is diagonal in the* $\widetilde{E}_{(\mu|\lambda)}$ *basis with*

$$\Psi_{p_1}(\widetilde{E}_{(\mu|\lambda)}) = \kappa_{\mu*\lambda}(q,t)\widetilde{E}_{(\mu|\lambda)}.$$

PROOF. Notice that every element of $\mathscr{P}_{as}^+$ is a finite $\mathbb{Q}(q,t)$-linear combination of terms of the form $T_\sigma x^\lambda F[X]$ where σ is a permutation, λ is a partition, and $F \in \Lambda$. Therefore, to show that the sequence of operators $(\Psi_{p_1}^{(m)})_{m\geq 1}$ converges it suffices using Remark 13 to show that sequences of the form

$$(\Psi_{p_1}^{(m)}(T_\sigma x^\lambda F[x_1 + \ldots + x_m]))_{m\geq 1}$$

converge. For m sufficiently large, T_σ commutes with $\Psi_{p_1}^{(m)} = t^m(Y_1^{(m)} + \ldots + Y_m^{(m)})$ so it suffices to consider only sequences of the form

$$(\Psi_{p_1}^{(m)}(x^\lambda F[x_1 + \ldots + x_m]))_{m\geq 1}.$$

Let λ be a partition, $k := \ell(\lambda)$, $F \in \Lambda$, and take $m > k$. Recall that $\widetilde{Y}_1^{(m)} X_1 = t^m Y_1^{(m)} X_1$ from which it follows directly that $\widetilde{Y}_i^{(m)} X_i = t^m Y_i^{(m)} X_i$ for all $1 \leq i \leq m$. Then for all $1 \leq i \leq k$ we have that since $\lambda_i \neq 0$,

$$t^m Y_i^{(m)}(x^\lambda F[x_1 + \ldots + x_m]) = \widetilde{Y}_i^{(m)}(x^\lambda F[x_1 + \ldots + x_m]).$$

Therefore,

$$t^m(Y_1^{(m)} + \ldots + Y_k^{(m)})(x^\lambda F[x_1 + \ldots + x_m]) = (\widetilde{Y}_1^{(m)} + \ldots + \widetilde{Y}_k^{(m)})(x^\lambda F[x_1 + \ldots + x_m]).$$

Now since $x^\lambda F[x_1 + \ldots + x_m]$ is symmetric in the variables $\{k+1, \ldots, m\}$ we see that

$$t^m(Y_{k+1}^{(m)}+\ldots+Y_m^{(m)})(x^\lambda F[x_1+\ldots+x_m])$$

$$= (t^{m-k}T_k\cdots T_1\pi_m T_{m-1}^{-1}\cdots T_{k+1}^{-1}+t^{m-k-1}T_{k+1}\cdots T_1\pi_m T_{m-1}^{-1}\cdots T_{k+2}^{-1}+\ldots+tT_{m-1}\cdots T_1\pi_m)$$

$$\times\,(x^\lambda F[x_1+\ldots+x_m])$$

$$= (t^{m-k}T_k\cdots T_1+t^{m-k-1}T_{k+1}\cdots T_1+\ldots+tT_{m-1}\cdots T_1)\pi_m(x^\lambda F[x_1+\ldots+x_m])$$

$$= (t^{m-k}T_k\cdots T_1+t^{m-k-1}T_{k+1}\cdots T_1+\ldots+tT_{m-1}\cdots T_1)(x_2^{\lambda_1}\cdots x_{k+1}^{\lambda_k}F[qx_1+x_2+\ldots+x_m])$$

$$= (t^{m-k}+t^{m-k-1}T_{k+1}+\ldots+tT_{m-1}\cdots T_{k+1})\left(T_k\cdots T_1x_2^{\lambda_1}\cdots x_{k+1}^{\lambda_k}F[qx_1+x_2+\ldots+x_m]\right).$$

Notice that since $T_k\cdots T_1x_2^{\lambda_1}\cdots x_{k+1}^{\lambda_k}F[qx_1+x_2+\ldots+x_m]$ is symmetric in the variables $\{k+2,\ldots,m\}$

$$\epsilon_{k+1}^{(m)}(T_k\cdots T_1x_2^{\lambda_1}\cdots x_{k+1}^{\lambda_k}F[qx_1+x_2+\ldots+x_m])=T_k\cdots T_1x_2^{\lambda_1}\cdots x_{k+1}^{\lambda_k}F[qx_1+x_2+\ldots+x_m].$$

Therefore,

$$t^m(Y_{k+1}^{(m)}+\ldots+Y_m^{(m)})(x^\lambda F[x_1+\ldots+x_m])$$

$$= (t^{m-k}+\ldots+tT_{m-1}\cdots T_{k+1})\epsilon_{k+1}^{(m)}(T_k\cdots T_1x_2^{\lambda_1}\cdots x_{k+1}^{\lambda_k}F[qx_1+x_2+\ldots+x_m])$$

$$= t(t^{m-k-1}+\ldots+1)\epsilon_k^{(m)}(T_k\cdots T_1x_2^{\lambda_1}\cdots x_{k+1}^{\lambda_k}F[qx_1+x_2+\ldots+x_m])$$

where the last equality follows from

$$\left(\frac{t^{m-k-1}+t^{m-k-2}T_{k+1}+\ldots+T_{m-1}\cdots T_{k+1}}{t^{m-k-1}+t^{m-k-2}+\ldots+1}\right)\epsilon_{k+1}^{(m)}=\epsilon_k^{(m)}.$$

Putting it all together we see that

$$\Psi_{p_1}^{(m)}(x^\lambda F[x_1+\ldots+x_m])$$

$$= t^m(Y_1^{(m)}+\ldots+Y_m^{(m)})(x^\lambda F[x_1+\ldots+x_m])$$

$$= t^m(Y_1^{(m)}+\ldots+Y_k^{(m)})(x^\lambda F[x_1+\ldots+x_m])+t^m(Y_{k+1}^{(m)}+\ldots+Y_m^{(m)})(x^\lambda F[x_1+\ldots+x_m])$$

$$= (\widetilde{Y}_1^{(m)}+\ldots+\widetilde{Y}_k^{(m)})(x^\lambda F[x_1+\ldots+x_m])+t(t^{m-k-1}+\ldots+1)$$

$$\times\,\epsilon_k^{(m)}(T_k\cdots T_1x_2^{\lambda_1}\cdots x_{k+1}^{\lambda_k}F[qx_1+x_2+\ldots+x_m])$$

which by Theorem 2.1.5 and Corollary 2.4.18 converges to

$$(\mathscr{Y}_1 + \ldots + \mathscr{Y}_k)(x^\lambda F[X]) + \frac{t}{1-t}\epsilon_k(T_k \cdots T_1 x_2^{\lambda_1} \cdots x_{k+1}^{\lambda_k} F[X + (q-1)x_1]).$$

Therefore, the limit operator $\Psi_{p_1} := \lim_m \Psi_{p_1}^{(m)}$ is well defined.

We will now show that the $\widetilde{E}_{(\mu|\lambda)}$ are weight vectors of Ψ_{p_1} and compute their corresponding weight values. Let $(\mu|\lambda) \in \Sigma$. By Corollary 2.4.21 we have that

$$\widetilde{E}_{(\mu|\lambda)} = \lim_m \epsilon_{\ell(\mu)}^{(m)}(E_{\mu*\lambda*0^{m-(\ell(\mu)+\ell(\lambda))}}).$$

Therefore, by Proposition 6.21 from [26], Lemma 2.6.2, and the fact that symmetric polynomials in the Y_i variables commute with the T_j elements it follows that

$$\begin{aligned}
&\Psi_{p_1}(\widetilde{E}_{(\mu|\lambda)}) \\
&= \lim_m \Psi_{p_1}^{(m)}(\epsilon_{\ell(\mu)}^{(m)}(E_{\mu*\lambda*0^{m-(\ell(\mu)+\ell(\lambda))}})) \\
&= \lim_m t^m(Y_1^{(m)} + \ldots + Y_m^{(m)})\epsilon_{\ell(\mu)}^{(m)}(E_{\mu*\lambda*0^{m-(\ell(\mu)+\ell(\lambda))}}) \\
&= \lim_m \epsilon_{\ell(\mu)}^{(m)}(t^m(Y_1^{(m)} + \ldots + Y_m^{(m)})E_{\mu*\lambda*0^{m-(\ell(\mu)+\ell(\lambda))}}) \\
&= \lim_m \kappa_{\mu*\lambda}^{(m-(\ell(\mu)+\ell(\lambda)))}(q,t)\epsilon_{\ell(\mu)}^{(m)}(E_{\mu*\lambda*0^{m-(\ell(\mu)+\ell(\lambda))}}) \\
&= \kappa_{\mu*\lambda}(q,t)\widetilde{E}_{(\mu|\lambda)}.
\end{aligned}$$

□

REMARK 14. *From the proof of Theorem 2.6.4 we see that in particular, for partitions λ we have that*

$$\Psi_{p_1}(\mathcal{A}_\lambda[X]) = \frac{t}{1-t}(1 - (1-t)(1-q)B_\lambda(q,t))\mathcal{A}_\lambda[X].$$

We saw that in Corollary 2.4.21 $\mathcal{A}_\lambda[X] = (1-t)^{\ell(\lambda)}v_\lambda(t)P_\lambda[X; q^{-1}, t]$ so, following the argument of Haiman in [22], the operator $t^{-1}(1-t)\Psi_{p_1}$ is up to a change of variables equal to Δ'. Therefore, we can view $t^{-1}(1-t)\Psi_{p_1}$ in a certain sense (after changing variables) as extending the operator Δ' from Λ to $\mathscr{P}_{as}^+$. Further, Theorem 2.6.4 does not follow immediately from the work of Ion and

Wu in [26] and in particular,

$$\Psi_{p_1} \neq \mathscr{Y}_1 + \mathscr{Y}_2 + \dots$$

although the latter operator is certainly well defined in a weak sense as a diagonal operator in the $\widetilde{E}_{(\mu|\lambda)}$ basis. The easiest way to see this is to note that $\mathscr{Y}_1 + \mathscr{Y}_2 + \dots$ will annihilate Λ whereas Ψ_{p_1} acting on the basis $\mathcal{A}_\lambda$ of Λ has nonzero eigenvalues $\kappa_\lambda(q,t) \neq 0$.

THEOREM 2.6.5 (Second Main Theorem). *Let $\widetilde{Y}$ denote the $\mathbb{Q}(q,t)$-subalgebra of $\mathrm{End}_{\mathbb{Q}(q,t)}(\mathscr{P}_{as}^+)$ generated by Ψ_{p_1} and $\mathscr{Y}_i$ for $i \geq 1$. $\mathscr{P}_{as}^+$ has a basis of $\widetilde{Y}$-weight vectors and every $\widetilde{Y}$-weight space of $\mathscr{P}_{as}^+$ is 1-dimensional.*

PROOF. Since Ψ_{p_1} is diagonal in the $\widetilde{E}_{(\mu|\lambda)}$ basis, see Theorem 4.2.12, it commutes with each $\mathscr{Y}_i$. Therefore, $\widetilde{Y}$ is a commutative algebra of operators on $\mathscr{P}_{as}^+$ so it makes sense to ask about its weights in $\mathscr{P}_{as}^+$. To show that the $\widetilde{Y}$-weight spaces of $\mathscr{P}_{as}^+$ are 1-dimensional it suffices to show that if $(\mu^{(1)}|\lambda^{(1)}) \neq (\mu^{(2)}|\lambda^{(2)})$ for $(\mu^{(1)}|\lambda^{(1)}), (\mu^{(2)}|\lambda^{(2)}) \in \Sigma$ with $\widetilde{E}_{(\mu^{(1)}|\lambda^{(1)})}$ and $\widetilde{E}_{(\mu^{(2)}|\lambda^{(2)})}$ having the same $\mathscr{Y}$-weight then the Ψ_{p_1} eigenvalues for $\widetilde{E}_{(\mu^{(1)}|\lambda^{(1)})}$ and $\widetilde{E}_{(\mu^{(2)}|\lambda^{(2)})}$ are distinct. Necessarily, from the proof of Theorem 4.2.12, if $\widetilde{E}_{(\mu^{(1)}|\lambda^{(1)})}$ and $\widetilde{E}_{(\mu^{(2)}|\lambda^{(2)})}$ have the same $\mathscr{Y}$-weight then $\mu^{(1)} = \mu^{(2)} = \mu$. Since $(\mu|\lambda^{(1)}) \neq (\mu|\lambda^{(2)})$ it follows that $\lambda^{(1)} \neq \lambda^{(2)}$ so that $\mathrm{sort}(\mu * \lambda^{(1)}) \neq \mathrm{sort}(\mu * \lambda^{(2)})$. From Lemma 2.6.2 we then know that $\kappa_{\mu*\lambda^{(1)}} \neq \kappa_{\mu*\lambda^{(2)}}$ so lastly by Theorem 2.6.4 we see that the Ψ_{p_1} eigenvalues for $\widetilde{E}_{(\mu|\lambda^{(1)})}$ and $\widetilde{E}_{(\mu|\lambda^{(2)})}$ are distinct. Hence, the $\widetilde{Y}$-weight spaces of $\mathscr{P}_{as}^+$ are 1-dimensional.

$\square$

Theorem 2.6.4 motivates the following definition.

DEFINITION 2.6.6. *For $F \in \Lambda$ let $\Psi_F : \mathscr{P}_{as}^+ \to \mathscr{P}_{as}^+$ be the diagonal operator in $\mathrm{End}_{\mathbb{Q}(q,t)}(\mathscr{P}_{as}^+)$ in the $\{\widetilde{E}_{(\mu|\lambda)} : (\mu|\lambda) \in \Sigma\}$ basis given by*

$$\Psi_F(\widetilde{E}_{(\mu|\lambda)}) := F[\kappa_{\mu*\lambda}(q,t)]\widetilde{E}_{(\mu|\lambda)}.$$

Notice that by construction every operator Ψ_F commutes with each of the operators $\mathscr{Y}_i$ since from Corollary 4.2.12 we know that the $\widetilde{E}_{(\mu|\lambda)}$ are a basis of $\mathscr{P}_{as}^+$.

2.7. Higher Delta Operators

At the end of the author's prior paper [3] it is conjectured that for any symmetric function $F \in \Lambda$ the sequence of operators $(\Psi_F^{(n)})_{n\geq 1}$ converges to an operator on $\mathscr{P}_{as}^{+}$. An affirmation of this conjecture has direct implications related to the conjectural partially symmetric elliptic Hall algebras mentioned by Carlsson-Mellit in [8] and the extended double Dyck path algebra $\mathbb{B}_{q,t}^{ext}$ of González-Gorsky-Simental in [17]. The main purpose of this section is to give a proof of this conjecture. The proof involves a detailed computation which will be done in stages. We will start with some of the required preliminaries.

2.7.1. Preliminaries. There are a few elementary technical results we will need before we are able to prove the main result of this section Theorem 2.7.8.

For the remainder of this section we consider n, k, r with $n > k + r \geq 1$. We begin by expressing certain partially symmetric polynomials in the Cherednik elements Y_i in terms of products of consecutive Cherednik elements.

LEMMA 2.7.1.

$$e_r\left[t^nY_{k+1}^{(n)} + \ldots + t^nY_n^{(n)}\right] = \sum_{\sigma\in\mathfrak{S}_{(1^k,n-k)}/\mathfrak{S}_{(1^k,r,n-k-r)}} t^{-\ell(\sigma)}T_\sigma t^{rn}Y_{k+1}^{(n)}\cdots Y_{k+r}^{(n)}T_{\sigma^{-1}}$$

PROOF. Notice that for $\sigma \in \mathfrak{S}_{(1^k,n-k)}/\mathfrak{S}_{(1^k,r,n-k-r)}$ the values $\sigma(k+1),\ldots,\sigma(k+r)$ are increasing i.e. $k+1 \leq \sigma(k+1) < \ldots < \sigma(k+r) \leq n-k$ and uniquely determine σ. As such there is a natural bijection $\mathfrak{S}_{(1^k,n-k)}/\mathfrak{S}_{(1^k,r,n-k-r)} \to \{(i_1,\ldots,i_r)|k+1 \leq i_1 < \ldots < i_r \leq n-k\}$ given by $\sigma \to (\sigma(k+1),\ldots,\sigma(k+r))$. Hence, it suffices to show that for all

$$Y_{\sigma(k+1)}^{(n)}\cdots Y_{\sigma(k+r)}^{(n)} = t^{-\ell(\sigma)}T_\sigma Y_{k+1}^{(n)}\cdots Y_{k+r}^{(n)}T_{\sigma^{-1}}.$$

We proceed by induction on the Bruhat order on $\mathfrak{S}_{(1^k,n-k)}/\mathfrak{S}_{(1^k,r,n-k-r)}$. Clearly, this formula holds for $\sigma = 1$.

Suppose $k+1 = i_0 \leq i_1 < \ldots < i_r \leq i_{r+1} = n$ with $i_{j+1} - i_j > 1$ for some $0 \leq j \leq r$. Then

$$\begin{aligned}
&Y_{i_1}^{(n)} \cdots Y_{i_{j-1}}^{(n)} Y_{i_j+1}^{(n)} Y_{i_{j+1}}^{(n)} Y_{i_{j+2}}^{(n)} \cdots Y_n^{(n)} \\
&= Y_{i_1}^{(n)} \cdots Y_{i_{j-1}}^{(n)} (t^{-1} T_{i_j} Y_{i_j}^{(n)} T_{i_j}) Y_{i_{j+1}}^{(n)} Y_{i_{j+2}}^{(n)} \cdots Y_n^{(n)} \\
&= t^{-1} T_{i_j} Y_{i_1}^{(n)} \cdots Y_{i_r}^{(n)} T_{i_j}.
\end{aligned}$$

Now if $\sigma, \sigma' \in \mathfrak{S}_{(1^k, n-k)}/\mathfrak{S}_{(1^k, r, n-k-r)}$ are the unique elements with $\sigma(k+\ell) = i_\ell$ and $\sigma' = s_{i_j}\sigma$. Suppose that $Y_{\sigma(k+1)}^{(n)} \cdots Y_{\sigma(k+r)}^{(n)} = t^{-\ell(\sigma)} T_\sigma Y_{k+1}^{(n)} \cdots Y_{k+r}^{(n)} T_{\sigma^{-1}}$. Then from the above we find that

$$\begin{aligned}
&Y_{\sigma'(k+1)}^{(n)} \cdots Y_{\sigma'(k+r)}^{(n)} \\
&= Y_{i_1}^{(n)} \cdots Y_{i_{j-1}}^{(n)} Y_{i_j+1}^{(n)} Y_{i_{j+1}}^{(n)} Y_{i_{j+2}}^{(n)} \cdots Y_n^{(n)} \\
&= t^{-1} T_{i_j} Y_{i_1}^{(n)} \cdots Y_{i_r}^{(n)} T_{i_j} \\
&= t^{-1} T_{i_j} Y_{\sigma(k+1)}^{(n)} \cdots Y_{\sigma(k+r)}^{(n)} T_{i_j} \\
&= t^{-(1+\ell(\sigma))} T_{i_j} T_\sigma Y_{k+1}^{(n)} \cdots Y_{k+r}^{(n)} T_{\sigma^{-1}} T_{i_j} \\
&= t^{-\ell(\sigma')} T_{\sigma'} Y_{k+1}^{(n)} \cdots Y_{k+r}^{(n)} T_{(\sigma')^{-1}}.
\end{aligned}$$

□

Now we may write a product of consecutive Cherednik elements in terms of π_n.

Lemma 2.7.2.

$$\begin{aligned}
&t^{rn} Y_{k+1}^{(n)} \cdots Y_{k+r}^{(n)} \\
&= t^{(n-k)+\ldots+(n-k-r+1)} (T_k \cdots T_1)(T_{k+1} \cdots T_2) \cdots (T_{k+r-1} \cdots T_r) \pi_n^r (T_{n-r}^{-1} \cdots T_{k+1}^{-1}) \cdots (T_{n-1}^{-1} \cdots T_{k+r}^{-1}).
\end{aligned}$$

Proof. We will show this result by induction. For $r = 1$ we see that

$$t^n Y_{k+1}^{(n)} = t^{n-k} T_k \cdots T_1 \pi_n T_{n-1}^{-1} \cdots T_{k+1}^{-1}.$$

Now we find

$$
\begin{aligned}
&t^{(r+1)n}Y_{k+1}^{(n)}\cdots Y_{k+r+1}^{(n)}\\
&=t^{rn}Y_{k+1}^{(n)}\cdots Y_{k+r}^{(n)}t^nY_{k+r+1}^{(n)}\\
&=t^{(n-k)+\ldots+(n-k-r+1)}(T_k\cdots T_1)\cdots(T_{k+r-1}\cdots T_r)\pi_n^r(T_{n-r}^{-1}\cdots T_{k+1}^{-1})\cdots(T_{n-1}^{-1}\cdots T_{k+r}^{-1})t^nY_{k+r+1}^{(n)}\\
&=t^{(n-k)+\ldots+(n-k-r+1)}(T_k\cdots T_1)\cdots(T_{k+r-1}\cdots T_r)\pi_n^r(T_{n-r}^{-1}\cdots T_{k+1}^{-1})\cdots(T_{n-1}^{-1}\cdots T_{k+r}^{-1})\\
&\times\left(t^{n-k-r}T_{k+r}\cdots T_1\pi_nT_{n-1}^{-1}\cdots T_{k+r+1}^{-1}\right)\\
&=t^{(n-k)+\ldots+(n-k-r)}(T_k\cdots T_1)\cdots(T_{k+r-1}\cdots T_r)\pi_n^r(T_{n-r}^{-1}\cdots T_{k+1}^{-1})\cdots(T_{n-1}^{-1}\cdots T_{k+r}^{-1})\\
&\times T_{k+r}\cdots T_1\pi_nT_{n-1}^{-1}\cdots T_{k+r+1}^{-1}.
\end{aligned}
$$

Looking closer we see

$$
\begin{aligned}
&(T_{n-r}^{-1}\cdots T_{k+1}^{-1})\cdots(T_{n-1}^{-1}\cdots T_{k+r}^{-1})T_{k+r}\cdots T_1\\
&=(T_{n-r}^{-1}\cdots T_{k+1}^{-1})\cdots(T_{n-1}^{-1}\cdots T_{k+r+1}^{-1})T_{k+r-1}\cdots T_1\\
&=(T_{n-r}^{-1}\cdots T_{k+1}^{-1})\cdots(T_{n-2}^{-1}\cdots T_{k+r-1}^{-1})T_{k+r-1}\cdots T_1(T_{n-1}^{-1}\cdots T_{k+r+1}^{-1})\\
&=\ldots\\
&=(T_k\cdots T_1)(T_{n-r}^{-1}\cdots T_{k+2}^{-1})\cdots(T_{n-1}^{-1}\cdots T_{k+r+1}^{-1}).
\end{aligned}
$$

Therefore,

$$(T_k \cdots T_1) \cdots (T_{k+r-1} \cdots T_r)\pi_n^r(T_{n-r}^{-1} \cdots T_{k+1}^{-1}) \cdots (T_{n-1}^{-1} \cdots T_{k+r}^{-1})T_{k+r} \cdots T_1\pi_n T_{n-1}^{-1} \cdots T_{k+r+1}^{-1}$$
$$= (T_k \cdots T_1) \cdots (T_{k+r-1} \cdots T_r)\pi_n^r(T_k \cdots T_1)(T_{n-r}^{-1} \cdots T_{k+2}^{-1}) \cdots (T_{n-1}^{-1} \cdots T_{k+r+1}^{-1})\pi_n T_{n-1}^{-1} \cdots T_{k+r+1}^{-1}$$
$$= (T_k \cdots T_1) \cdots (T_{k+r-1} \cdots T_r)(T_{k+r} \cdots T_{r+1})\pi_n^r(T_{n-r}^{-1} \cdots T_{k+2}^{-1}) \cdots (T_{n-1}^{-1} \cdots T_{k+r+1}^{-1})$$
$$\times \pi_n T_{n-1}^{-1} \cdots T_{k+r+1}^{-1}$$
$$= (T_k \cdots T_1) \cdots (T_{k+r-1} \cdots T_r)(T_{k+r} \cdots T_{r+1})\pi_n^{r+1}(T_{n-r-1}^{-1} \cdots T_{k+1}^{-1}) \cdots (T_{n-2}^{-1} \cdots T_{k+r}^{-1})$$
$$\times (T_{n-1}^{-1} \cdots T_{k+r+1}^{-1})$$

so that

$$t^{(r+1)n} Y_{k+1}^{(n)} \cdots Y_{k+r+1}^{(n)}$$
$$= t^{(n-k)+\ldots+(n-k-r)}(T_k \cdots T_1) \cdots (T_{k+r} \cdots T_{r+1})\pi_n^{r+1}(T_{n-r-1}^{-1} \cdots T_{k+1}^{-1}) \cdots (T_{n-1}^{-1} \cdots T_{k+r+1}^{-1}).$$

□

We need the following standard result.

Lemma 2.7.3.

$$\sum_{\sigma \in \mathfrak{S}_{(1^k, n-k)} / \mathfrak{S}_{(1^k, r, n-k-r)}} t^{\binom{n-k}{2} - \binom{n-k-r}{2} - \binom{r}{2} - \ell(\sigma)} = \frac{[n-k]_t!}{[n-k-r]_t! [r]_t!}$$

Proof. We see the following:

$$
\begin{aligned}
&[n-k]_t! \\
&= \sum_{\sigma\in\mathfrak{S}_{(1^k,n-k)}} t^{\binom{n-k}{2}-\ell(\sigma)} \\
&= \sum_{\sigma\in\mathfrak{S}_{(1^k,n-k)}/\mathfrak{S}_{(1^k,r,n-k-r)}} \sum_{\gamma\in\mathfrak{S}_{(1^k,r,n-k-r)}} t^{\binom{n-k}{2}-\ell(\sigma\gamma)} \\
&= \sum_{\sigma\in\mathfrak{S}_{(1^k,n-k)}/\mathfrak{S}_{(1^k,r,n-k-r)}} \sum_{\gamma\in\mathfrak{S}_{(1^k,r,n-k-r)}} t^{\binom{n-k}{2}-\ell(\sigma)-\ell(\gamma)} \\
&= \sum_{\sigma\in\mathfrak{S}_{(1^k,n-k)}/\mathfrak{S}_{(1^k,r,n-k-r)}} t^{\binom{n-k}{2}-\binom{n-k-r}{2}-\binom{r}{2}-\ell(\sigma)} \sum_{\gamma\in\mathfrak{S}_{(1^k,r,n-k-r)}} t^{\binom{n-k-r}{2}+\binom{r}{2}-\ell(\gamma)} \\
&= [n-k-r]_t![r]_t! \sum_{\sigma\in\mathfrak{S}_{(1^k,n-k)}/\mathfrak{S}_{(1^k,r,n-k-r)}} t^{\binom{n-k}{2}-\binom{n-k-r}{2}-\binom{r}{2}-\ell(\sigma)}.
\end{aligned}
$$

The result follows. □

Using the prior lemmas in this section now shows the following:

Lemma 2.7.4.

$$
\begin{aligned}
&e_r\left[t^nY_{k+1}^{(n)}+\ldots+t^nY_n^{(n)}\right]\epsilon_k^{(n)} \\
&= t^{\binom{r+1}{2}}\left(\frac{1-t^{n-k-r+1}}{1-t}\right)\cdots\left(\frac{1-t^{n-k}}{1-t^r}\right)\epsilon_k^{(n)}(T_k\cdots T_1)(T_{k+1}\cdots T_2)\cdots(T_{k+r-1}\cdots T_r)\pi_n^r\epsilon_k^{(n)}.
\end{aligned}
$$

Proof. Putting together Lemmas 2.7.1, 2.7.2, and 2.7.3 we get the following computation:

$$
\begin{aligned}
& e_r\left[t^nY^{(n)}_{k+1}+\ldots+t^nY^{(n)}_{n}\right]\epsilon^{(n)}_k \\
&= e_r\left[t^nY^{(n)}_{k+1}+\ldots+t^nY^{(n)}_{n}\right](\epsilon^{(n)}_k)^2 \\
&= \epsilon^{(n)}_k e_r\left[t^nY^{(n)}_{k+1}+\ldots+t^nY^{(n)}_{n}\right]\epsilon^{(n)}_k \\
&= \sum_{\sigma\in\mathfrak{S}_{(1^k,n-k)}/\mathfrak{S}_{(1^k,r,n-k-r)}} t^{-\ell(\sigma)}\epsilon^{(n)}_k T_\sigma t^{rn}Y^{(n)}_{k+1}\cdots Y^{(n)}_{k+r}T_{\sigma^{-1}}\epsilon^{(n)}_k \\
&= \sum_{\sigma\in\mathfrak{S}_{(1^k,n-k)}/\mathfrak{S}_{(1^k,r,n-k-r)}} t^{-\ell(\sigma)}\epsilon^{(n)}_k t^{rn}Y^{(n)}_{k+1}\cdots Y^{(n)}_{k+r}\epsilon^{(n)}_k \\
&= \left(\sum_{\sigma\in\mathfrak{S}_{(1^k,n-k)}/\mathfrak{S}_{(1^k,r,n-k-r)}} t^{-\ell(\sigma)}\right)\epsilon^{(n)}_k t^{(n-k)+\ldots+(n-k-r+1)}(T_k\cdots T_1)(T_{k+1}\cdots T_2)\cdots(T_{k+r-1}\cdots T_r) \\
&\times \pi^r_n(T^{-1}_{n-r}\cdots T^{-1}_{k+1})\cdots(T^{-1}_{n-1}\cdots T^{-1}_{k+r})\epsilon^{(n)}_k \\
&= \left(\sum_{\sigma\in\mathfrak{S}_{(1^k,n-k)}/\mathfrak{S}_{(1^k,r,n-k-r)}} t^{-\ell(\sigma)}\right)\epsilon^{(n)}_k t^{(n-k)+\ldots+(n-k-r+1)}(T_k\cdots T_1)(T_{k+1}\cdots T_2)\cdots(T_{k+r-1}\cdots T_r) \\
&\times \pi^r_n\epsilon^{(n)}_k \\
&= t^{\binom{r}{2}+r}\left(\sum_{\sigma\in\mathfrak{S}_{(1^k,n-k)}/\mathfrak{S}_{(1^k,r,n-k-r)}} t^{\binom{n-k}{2}-\binom{n-k-r}{2}-\binom{r}{2}-\ell(\sigma)}\right) \\
&\times \epsilon^{(n)}_k(T_k\cdots T_1)(T_{k+1}\cdots T_2)\cdots(T_{k+r-1}\cdots T_r)\pi^r_n\epsilon^{(n)}_k \\
&= t^{\binom{r+1}{2}}\frac{[n-k]_t!}{[n-k-r]_t![r]_t!}\epsilon^{(n)}_k(T_k\cdots T_1)(T_{k+1}\cdots T_2)\cdots(T_{k+r-1}\cdots T_r)\pi^r_n\epsilon^{(n)}_k \\
&= t^{\binom{r+1}{2}}\left(\frac{1-t^{n-k-r+1}}{1-t}\right)\cdots\left(\frac{1-t^{n-k}}{1-t^r}\right)\epsilon^{(n)}_k(T_k\cdots T_1)(T_{k+1}\cdots T_2)\cdots(T_{k+r-1}\cdots T_r)\pi^r_n\epsilon^{(n)}_k.
\end{aligned}
$$

□

The next result will be important for the proof of Theorem 2.7.7 where we will need to argue that the operator $e_r\left[t^nY^{(n)}_{k+1}+\ldots+t^nY^{(n)}_{n}\right]$ preserves the space $x_1\ldots x_k\mathbb{Q}(q,t)[x_1,\ldots,x_n]^{\mathfrak{S}_{(1^k,n-k)}}$ in the polynomial representation.

Lemma 2.7.5.

$$
\begin{aligned}
& e_r \left[t^n Y_{k+1}^{(n)} + \ldots + t^n Y_n^{(n)} \right] \epsilon_k^{(n)} X_1 \cdots X_k \\
& = X_1 \cdots X_k t^{rk + \binom{r+1}{2}} \left(\frac{1 - t^{n-k-r+1}}{1-t} \right) \cdots \left(\frac{1 - t^{n-k}}{1-t^r} \right) \\
& \times \epsilon_k^{(n)} (T_k^{-1} \cdots T_1^{-1})(T_{k+1}^{-1} \cdots T_2^{-1}) \cdots (T_{k+r-1}^{-1} \cdots T_r^{-1}) \pi_n^r \epsilon_k^{(n)}
\end{aligned}
$$

Proof.

$$
\begin{aligned}
& e_r \left[t^n Y_{k+1}^{(n)} + \ldots + t^n Y_n^{(n)} \right] \epsilon_k^{(n)} X_1 \cdots X_k \\
& = t^{\binom{r+1}{2}} \left(\frac{1 - t^{n-k-r+1}}{1-t} \right) \cdots \left(\frac{1 - t^{n-k}}{1-t^r} \right) \epsilon_k^{(n)} (T_k \cdots T_1)(T_{k+1} \cdots T_2) \cdots (T_{k+r-1} \cdots T_r) \\
& \times \pi_n^r \epsilon_k^{(n)} X_1 \cdots X_k \\
& = t^{\binom{r+1}{2}} \left(\frac{1 - t^{n-k-r+1}}{1-t} \right) \cdots \left(\frac{1 - t^{n-k}}{1-t^r} \right) \epsilon_k^{(n)} (T_k \cdots T_1)(T_{k+1} \cdots T_2) \cdots (T_{k+r-1} \cdots T_r) \\
& \times \pi_n^r X_1 \cdots X_k \epsilon_k^{(n)} \\
& = t^{\binom{r+1}{2}} \left(\frac{1 - t^{n-k-r+1}}{1-t} \right) \cdots \left(\frac{1 - t^{n-k}}{1-t^r} \right) \\
& \times \epsilon_k^{(n)} (T_k \cdots T_1)(T_{k+1} \cdots T_2) \cdots (T_{k+r-1} \cdots T_r) X_{r+1} \cdots X_{k+r} \pi_n^r \epsilon_k^{(n)}.
\end{aligned}
$$

Further,

$$
\begin{aligned}
& T_{k+r-1} \cdots T_r X_{r+1} \cdots X_{r+k} \\
& = T_{k+r-1} \cdots T_{r+1} (T_r X_{r+1}) X_{r+2} \cdots X_{r+k} \\
& = T_{k+r-1} \cdots T_{r+1} (t X_r T_r^{-1}) X_{r+2} \cdots X_{r+k} \\
& = t T_{k+r-1} \cdots T_{r+1} X_r X_{r+2} \cdots X_{r+k} T_r^{-1} \\
& = t X_r T_{k+r-1} \cdots T_{r+1} X_{r+2} \cdots X_{r+k} T_r^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&= tX_rT_{k+r-1}\cdots T_{r+2}tX_{r+1}T_{r+1}^{-1}X_{r+3}\cdots X_{r+k}T_r^{-1}\\
&= t^2X_rX_{r+1}T_{k+r-1}\cdots T_{r+2}X_{r+3}\cdots X_{r+k}T_{r+1}^{-1}T_r^{-1}\\
&= \dots\\
&= t^kX_r\cdots X_{r+k-1}T_{k+r-1}^{-1}\cdots T_r^{-1}.
\end{aligned}
$$

By applying this argument repeatedly we find that

$$
\begin{aligned}
&(T_k\cdots T_1)\cdots(T_{k+r-2}\cdots T_{r-1})(T_{k+r-1}\cdots T_r)X_{r+1}\cdots X_{k+r}\\
&= t^k(T_k\cdots T_1)\cdots(T_{k+r-2}\cdots T_{r-1})X_r\cdots X_{k+r-1}(T_{k+r-1}^{-1}\cdots T_r^{-1})\\
&= \dots\\
&= t^{rk}X_1\cdots X_k(T_k^{-1}\cdots T_1^{-1})\cdots(T_{k+r-1}^{-1}\cdots T_r^{-1})
\end{aligned}
$$

so therefore,

$$
\begin{aligned}
&t^{\binom{r+1}{2}}\left(\frac{1-t^{n-k-r+1}}{1-t}\right)\cdots\left(\frac{1-t^{n-k}}{1-t^r}\right)\epsilon_k^{(n)}(T_k\cdots T_1)(T_{k+1}\cdots T_2)\cdots(T_{k+r-1}\cdots T_r)\\
&\times X_{r+1}\cdots X_{k+r}\pi_n^r\epsilon_k^{(n)}\\
&= t^{\binom{r+1}{2}}\left(\frac{1-t^{n-k-r+1}}{1-t}\right)\cdots\left(\frac{1-t^{n-k}}{1-t^r}\right)\epsilon_k^{(n)}t^{rk}X_1\cdots X_k(T_k^{-1}\cdots T_1^{-1})\cdots(T_{k+r-1}^{-1}\cdots T_r^{-1})\pi_n^r\epsilon_k^{(n)}\\
&= X_1\cdots X_kt^{rk+\binom{r+1}{2}}\left(\frac{1-t^{n-k-r+1}}{1-t}\right)\cdots\left(\frac{1-t^{n-k}}{1-t^r}\right)\epsilon_k^{(n)}(T_k^{-1}\cdots T_1^{-1})\cdots(T_{k+r-1}^{-1}\cdots T_r^{-1})\pi_n^r\epsilon_k^{(n)}.
\end{aligned}
$$

□

Lastly, we have the standard coproduct formula for the elementary symmetric functions.

Lemma 2.7.6. $e_r[X+Y]=\sum_{s=0}^r e_s[X]e_{r-s}[Y]$.

PROOF. Using the definition of e_r we see that if $Z = z_1 + z_2 + \ldots$ then

$$e_r[Z] = \sum_{i_1 < \ldots < i_r} z_{i_1} \cdots z_{i_r}$$

so if we set $Z = X + Y$ we have

$$e_r[X+Y] = \sum_{s=0}^{r} \sum_{\substack{i_1<\ldots<i_\ell \\ j_1<\ldots<j_{r-s}}} X_{i_1} \cdots X_{i_s} Y_{j_1} \cdots Y_{j_{r-s}} = \sum_{s=0}^{r} e_\ell[X] e_{r-s}[Y].$$

□

2.7.2. Proof of Convergence. We will now use all of the lemmas proven above to show the following result:

PROPOSITION 2.7.7. *For $r \geq 1$ the sequence of operators $(\Psi_{e_r}^{(n)})_{n\geq 1}$ converges to an operator $e_r[\Delta]$ on $\mathscr{P}_{as}^{+}$. The operators $e_r[\Delta]$ satisfy the following properties:*

- $e_r[\Delta](\widetilde{E}_{(\mu|\lambda)}) = e_r[\kappa_{\mathrm{sort}(\mu*\lambda)}(q,t)]\widetilde{E}_{(\mu|\lambda)}$
- $[e_r[\Delta], \mathscr{Y}_i] = 0$
- $[e_r[\Delta], T_i] = 0$
- $[e_r[\Delta], e_s[\Delta]] = 0$
-

$$e_r[\Delta]|_{x_1 \cdots x_k \, \mathscr{P}(k)^+} = \sum_{s=0}^{r} \prod_{i=1}^{s} \left(\frac{t^i}{1-t^i} \right) e_{r-s}(\mathscr{Y}_1, \ldots, \mathscr{Y}_k) \epsilon_k (T_k \cdots T_1)(T_{k+1} \cdots T_2) \cdots (T_{k+s-1} \cdots T_s) \pi^s.$$

PROOF. The structure of the following argument is similar to the proof of Theorem 59 in [3].

Notice that every element of $\mathscr{P}_{as}^{+}$ is a finite $\mathbb{Q}(q,t)$-linear combination of terms of the form $T_\sigma x^\lambda F[X]$ where σ is a permutation, λ is a partition, and $F \in \Lambda$. Therefore, to show that the sequence of operators $(\Psi_{e_r}^{(n)})_{n\geq 1}$ converges it suffices to show that sequences of the form

$$(\Psi_{e_r}^{(n)}(T_\sigma x^\lambda F[x_1 + \ldots + x_n]))_{n\geq 1}$$

converge. For n sufficiently large, T_σ commutes with $\Psi_{e_r}^{(n)} = e_r[t^n Y_1^{(n)} + \ldots + t^n Y_n^{(n)}]$ so it suffices to consider only sequences of the form

$$(\Psi_{e_r}^{(n)}(x^\lambda F[x_1 + \ldots + x_n]))_{n\geq 1}.$$

Let λ be a partition, $k := \ell(\lambda)$, $F \in \Lambda$, and take $n > k + r$. Set $\lambda' := (\lambda_1 - 1, \ldots, \lambda_k - 1)$. Recall that $\widetilde{Y}_1^{(n)} X_1 = t^n Y_1^{(n)} X_1$ from which it follows directly that $\widetilde{Y}_i^{(n)} X_i = t^n Y_i^{(n)} X_i$ for all $1 \leq i \leq n$. Then for all $1 \leq i \leq k$ we have that

$$t^n Y_i^{(n)} X_1 \cdots X_k = \widetilde{Y}_i^{(n)} X_1 \cdots X_k.$$

This means that as operators on $x_1 \ldots x_k \mathscr{P}_k^+$, $t^n Y_i^{(n)} = \widetilde{Y}_i^{(n)}$. Note that these operators preserve the subspace $x_1 \ldots x_k \mathscr{P}_k^+$. Further, we may naturally extend this argument to show that as operators on $x_1 \cdots x_k \mathscr{P}_k^+$ for any $a_1, \ldots, a_k \geq 0$, and any permutation $\gamma \in \mathfrak{S}_k$,

$$(t^n Y_{\gamma(1)}^{(n)})^{a_1} \cdots (t^n Y_{\gamma(k)}^{(n)})^{a_k}|_{x_1 \cdots x_k \mathscr{P}_k^+} = (\widetilde{Y}_{\gamma(1)}^{(n)})^{a_1} \cdots (\widetilde{Y}_{\gamma(k)}^{(n)})^{a_k}|_{x_1 \cdots x_k \mathscr{P}_k^+}.$$

This is notable because, unlike the Cherednik operators $Y_i^{(n)}$, the deformed Cherednik operators $\widetilde{Y}_i^{(n)}$ do not mutually commute. Therefore again as operators on $\mathscr{P}_k^+$, for all $0 \leq s \leq r$

$$e_{r-s}[t^n Y_1^{(n)} + \ldots + t^n Y_k^{(n)}] X_1 \cdots X_k = e_{r-s}(\widetilde{Y}_1^{(n)}, \ldots, \widetilde{Y}_k^{(n)}) X_1 \cdots X_k.$$

Using Lemma 2.7.6 we now find the following:

$$\begin{aligned}
&e_r[t^n Y_1^{(n)} + \ldots + t^n Y_n^{(n)}](x^\lambda F[x_1 + \ldots + x_n]) \\
&= e_r[(t^n Y_1^{(n)} + \ldots + t^n Y_k^{(n)}) + (t^n Y_{k+1}^{(n)} + \ldots + t^n Y_n^{(n)})](x^\lambda F[x_1 + \ldots + x_n]) \\
&= \sum_{s=0}^{r} e_{r-s}[t^n Y_1^{(n)} + \ldots + t^n Y_k^{(n)}] e_s[t^n Y_{k+1}^{(n)} + \ldots + t^n Y_n^{(n)}](x^\lambda F[x_1 + \ldots + x_n]).
\end{aligned}$$

Importantly, since $x^\lambda F[x_1 + \ldots + x_n]$ is symmetric in $k+1, \ldots, n$,

$$\epsilon_k^{(n)}(x^\lambda F[x_1 + \ldots + x_n]) = x^\lambda F[x_1 + \ldots + x_n]$$

so that by using Lemmas 2.7.4 and 2.7.5 we find

$$
\begin{aligned}
&\sum_{s=0}^{r} e_{r-s}[t^nY_1^{(n)}+\ldots+t^nY_k^{(n)}]e_s[t^nY_{k+1}^{(n)}+\ldots+t^nY_n^{(n)}](x^\lambda F[x_1+\ldots+x_n])\\
&=\sum_{s=0}^{r} e_{r-s}[t^nY_1^{(n)}+\ldots+t^nY_k^{(n)}]e_s[t^nY_{k+1}^{(n)}+\ldots+t^nY_n^{(n)}]\epsilon_k^{(n)}X_1\cdots X_k(x^{\lambda'}F[x_1+\ldots+x_n])\\
&=\sum_{s=0}^{r} e_{r-s}[t^nY_1^{(n)}+\ldots+t^nY_k^{(n)}]X_1\cdots X_k t^{sk+\binom{s+1}{2}}\left(\frac{1-t^{n-k-s+1}}{1-t}\right)\cdots\left(\frac{1-t^{n-k}}{1-t^s}\right)\epsilon_k^{(n)}\\
&\times(T_k^{-1}\cdots T_1^{-1})(T_{k+1}^{-1}\cdots T_2^{-1})\cdots(T_{k+s-1}^{-1}\cdots T_s^{-1})\pi_n^r\epsilon_k^{(n)}(x^{\lambda'}F[x_1+\ldots+x_n])\\
&=\sum_{s=0}^{r} e_{r-s}(t^n\widetilde{Y}_1^{(n)},\ldots,t^n\widetilde{Y}_k^{(n)})X_1\cdots X_k t^{sk+\binom{s+1}{2}}\left(\frac{1-t^{n-k-s+1}}{1-t}\right)\cdots\left(\frac{1-t^{n-k}}{1-t^s}\right)\epsilon_k^{(n)}\\
&\times(T_k^{-1}\cdots T_1^{-1})(T_{k+1}^{-1}\cdots T_2^{-1})\cdots(T_{k+s-1}^{-1}\cdots T_s^{-1})\pi_n^r\epsilon_k^{(n)}(x^{\lambda'}F[x_1+\ldots+x_n])\\
&=\sum_{s=0}^{r} e_{r-s}(t^n\widetilde{Y}_1^{(n)},\ldots,t^n\widetilde{Y}_k^{(n)})t^{\binom{s+1}{2}}\left(\frac{1-t^{n-k-s+1}}{1-t}\right)\cdots\left(\frac{1-t^{n-k}}{1-t^s}\right)\epsilon_k^{(n)}\\
&\times(T_k\cdots T_1)(T_{k+1}\cdots T_2)\cdots(T_{k+s-1}\cdots T_s)\pi_n^s(x^\lambda F[x_1+\ldots+x_n]).
\end{aligned}
$$

From here it is clear that

$$
\begin{aligned}
&\lim_n \Psi_{e_r}^{(n)}(x^\lambda F[x_1+\ldots+x_n])\\
&=\sum_{s=0}^{r}\prod_{i=1}^{s}\left(\frac{t^i}{1-t^i}\right)e_{r-s}(\mathscr{Y}_1,\ldots,\mathscr{Y}_k)\epsilon_k(T_k\cdots T_1)(T_{k+1}\cdots T_2)\cdots(T_{k+s-1}\cdots T_s)\pi^s(x^\lambda F[X])
\end{aligned}
$$

which is evidently an element of $\mathscr{P}(k)^+ \subset \mathscr{P}_{as}^+$. Therefore, the sequence of operators $(\Psi_{e_r}^{(n)})_{n\geq 1}$ converges to an operator on $\mathscr{P}_{as}^+$ which we will call $e_r[\Delta]$.

We will now prove various properties of $e_r[\Delta]$. For all $1\leq i\leq k-1$ and $0\leq s\leq r$

$$
\begin{aligned}
&\epsilon_k(T_k\cdots T_1)(T_{k+1}\cdots T_2)\cdots(T_{k+s-1}\cdots T_s)\pi^s T_i\\
&=\epsilon_k(T_k\cdots T_1)(T_{k+1}\cdots T_2)\cdots(T_{k+s-1}\cdots T_s)T_{i+s}\pi^s\\
&=\epsilon_k(T_k\cdots T_1)(T_{k+1}\cdots T_2)\cdots(T_{k+s-2}\cdots T_{s-2})T_{i+s-1}(T_{k+s-1}\cdots T_s)\pi^s\\
&=\dots\\
&=\epsilon_k T_i(T_k\cdots T_1)(T_{k+1}\cdots T_2)\cdots(T_{k+s-1}\cdots T_s)\pi^s\\
&=T_i\epsilon_k(T_k\cdots T_1)(T_{k+1}\cdots T_2)\cdots(T_{k+s-1}\cdots T_s)\pi^s.
\end{aligned}
$$

Therefore, for any $f\in x_1\cdots x_k\,\mathscr{P}(k)^+$ by expanding f into a sum of terms of the form $T_\sigma x^\lambda F[X]$ where $\sigma\in\mathfrak{S}_k$ and λ is a partition with $\ell(\lambda)=k$ we find that

$$
e_r[\Delta](f)=\sum_{s=0}^{r}\prod_{i=1}^{s}\left(\frac{t^i}{1-t^i}\right)e_{r-s}(\mathscr{Y}_1,\dots,\mathscr{Y}_k)\epsilon_k(T_k\cdots T_1)(T_{k+1}\cdots T_2)\cdots(T_{k+s-1}\cdots T_s)\pi^s(f).
$$

Now let $(\mu|\lambda)\in\Phi$. Using Corollary 47 in [3] (see Definition 2.4.10) and Proposition 1.7.5 we have that

$$
\begin{aligned}
&e_r[\Delta](\widetilde{E}_{(\mu|\lambda)})\\
&=\lim_n e_r[t^nY_1^{(n)}+\cdots+t^nY_n^{(n)}](\epsilon^{(n)}_{\ell(\mu)}(E_{\mu*\lambda*0^{n-(\ell(\mu)+\ell(\lambda))}}))\\
&=\lim_n \epsilon^{(n)}_{\ell(\mu)}e_r[t^nY_1^{(n)}+\cdots+t^nY_n^{(n)}](E_{\mu*\lambda*0^{n-(\ell(\mu)+\ell(\lambda))}})\\
&=\lim_n \epsilon^{(n)}_{\ell(\mu)}\left(e_r\left[\sum_{i=1}^{n}q^{\mathrm{sort}(\mu*\lambda)_i}t^i\right]E_{\mu*\lambda*0^{n-(\ell(\mu)+\ell(\lambda))}}\right)\\
&=\lim_n e_r\left[\sum_{i=1}^{n}q^{\mathrm{sort}(\mu*\lambda)_i}t^i\right]\epsilon^{(n)}_{\ell(\mu)}(E_{\mu*\lambda*0^{n-(\ell(\mu)+\ell(\lambda))}})\\
&=e_r[\kappa_{\mathrm{sort}(\mu*\lambda)}(q,t)]\widetilde{E}_{(\mu|\lambda)}.
\end{aligned}
$$

To see that $[e_r[\Delta], T_i] = 0$ we may check directly:

$$\begin{aligned}
&e_r[\Delta]T_i \\
&= \lim_n e_r[t^n Y_1^{(n)} + \ldots + t^n Y_n^{(n)}]T_i \\
&= \lim_n T_i e_r[t^n Y_1^{(n)} + \ldots + t^n Y_n^{(n)}] \\
&= T_i e_r[\Delta].
\end{aligned}$$

Lastly, since the $\widetilde{E}_{(\mu|\lambda)}$ are a basis of $\mathscr{P}_{as}^+$ (Theorem 4.2.12) it follows that for all $i, r, s \geq 1$,

- $[e_r[\Delta], \mathscr{Y}_i] = 0$
- $[e_r[\Delta], e_s[\Delta]] = 0$.

$\square$

As an immediate consequence we have the following result confirming the conjecture posed in [3].

THEOREM 2.7.8. *For any symmetric function $F \in \Lambda$ the sequence of operators $(\Psi_F^{(n)})_{n\geq 1}$ converges to an operator on $\mathscr{P}_{as}^+$ which we may call $F[\Delta]$. These operators satisfy the following properties:*

- $F[\Delta](\widetilde{E}_{(\mu|\lambda)}) = F[\kappa_{\mathrm{sort}(\mu*\lambda)}(q,t)]\widetilde{E}_{(\mu|\lambda)}$
- $[F[\Delta], \mathscr{Y}_i] = 0$
- $[F[\Delta], T_i] = 0$
- $[F[\Delta], G[\Delta]] = 0$.

PROOF. Recall that the ring of symmetric functions Λ is generated algebraically by the elementary symmetric polynomials $e_1, e_2, \ldots$. For any $F \in \Lambda$ we may write $F = f(e_1, e_2, \ldots, e_r)$ so that for all $n \geq 1$

$$\Psi_F^{(n)} = f(\Psi_{e_1}^{(n)}, \ldots, \Psi_{e_r}^{(n)}).$$

By applying Propositions 1.7.6 and 2.7.7 we find that $(\Psi_F^{(n)})_{n\geq 1}$ converges and that

$$F[\Delta] := \lim_n \Psi_F^{(n)} = f(\lim_n \Psi_{e_1}^{(n)}, \ldots, \lim_n \Psi_{e_r}^{(n)}) = f(e_1[\Delta], \ldots, e_r[\Delta]).$$

For $(\mu|\lambda) \in \Phi$ we see that

$$
\begin{aligned}
& F[\Delta](\widetilde{E}_{(\mu|\lambda)}) \\
& = f(e_1[\Delta], \ldots, e_r[\Delta])(\widetilde{E}_{(\mu|\lambda)}) \\
& = f(e_1[\kappa_{\mathrm{sort}(\mu*\lambda)}(q,t)], \ldots, e_r[\kappa_{\mathrm{sort}(\mu*\lambda)}(q,t)])(\widetilde{E}_{(\mu|\lambda)}) \\
& = F[\kappa_{\mathrm{sort}(\mu*\lambda)}(q,t)]\widetilde{E}_{(\mu|\lambda)}.
\end{aligned}
$$

The other properties follow directly from Theorem 4.2.12 and Proposition 2.7.7.

$\square$

EXAMPLE. *The operator $e_2[\Delta]$ acts on $x_1x_2x_3\,\mathscr{P}(3)^+$ as*

$$
(\mathscr{Y}_1\mathscr{Y}_2 + \mathscr{Y}_1\mathscr{Y}_3 + \mathscr{Y}_2\mathscr{Y}_3) + \frac{t}{1-t}(\mathscr{Y}_1 + \mathscr{Y}_2 + \mathscr{Y}_3)\epsilon_3 T_3T_2T_1\pi + \frac{t^3}{(1-t)(1-t^2)}\epsilon_3 T_3T_2T_1T_4T_3T_2\pi^2.
$$

If we instead consider $e_4[\Delta]$ acting on $\mathscr{P}(0)^+ = \Lambda[X]$ then we get

$$
\frac{t^{10}}{(1-t)(1-t^2)(1-t^3)(1-t^4)}\epsilon\pi^4.
$$

As an example computation we have that

$$
\begin{aligned}
& p_2[\Delta](\widetilde{E}_{(4,1,2|5,4,2,2,1)}) = \\
& \left(q^{10}t^2 + q^8t^4 + q^8t^6 + q^4t^8 + q^4t^{10} + q^4t^{12} + q^2t^{14} + q^2t^{16} + t^{18} + \ldots\right)\widetilde{E}_{(4,1,2|5,4,2,2,1)}.
\end{aligned}
$$

In the next section we will explore a few interesting commutation relations satisfied by the Δ-operators on $\mathscr{P}_{as}^{+}$.

2.7.3. Interesting Relations. In this section we compute some of the commutation relations between the Δ-operators $F[\Delta]$ and the operator $\widetilde{\pi}$ on $\mathscr{P}_{as}^{+}$. These relations are conceptually important because in the case of the finite rank DAHA in type GL_n the analogous commutation relations allow for one to develop a theory of intertwiners and the Knop-Sahi relations for non-symmetric Macdonald polynomials.

We start with the following result which will follow easily using the properties of Ion-Wu limits and particularly Proposition 1.7.6.

PROPOSITION 2.7.9. *For $F \in \Lambda$*

$$\widetilde{\pi}F[\Delta] = F[\Delta + (q^{-1}-1)\mathscr{Y}_1]\widetilde{\pi}.$$

PROOF. Let $F \in \Lambda$ and $G_i, H_i \in \Lambda$ such that

$$F[X+Y] = \sum_i G_i[X]H_i[Y].$$

We may compute directly:

$$\begin{aligned}
\widetilde{\pi}F[\Delta] &= \lim_n \widetilde{\pi}_n F[t^nY_1^{(n)} + \ldots + t^nY_n^{(n)}] \\
&= \lim_n F[t^nY_2^{(n)} + \ldots + t^nY_n^{(n)} + q^{-1}t^nY_1^{(n)}]\widetilde{\pi}_n \\
&= \lim_n F[t^nY_1^{(n)} + \ldots + t^nY_n^{(n)} + (q^{-1}-1)t^nY_1^{(n)}]\widetilde{\pi}_n \\
&= \lim_n \sum_i G_i[t^nY_1^{(n)} + \ldots + t^nY_n^{(n)}]H_i[(q^{-1}-1)t^nY_1^{(n)}]\widetilde{\pi}_n \\
&= \lim_n \sum_i G_i[t^nY_1^{(n)} + \ldots + t^nY_n^{(n)}]H_i[(q^{-1}-1)t^nY_1^{(n)}]X_1T_1^{-1}\cdots T_{n-1}^{-1} \\
&= \lim_n \sum_i G_i[t^nY_1^{(n)} + \ldots + t^nY_n^{(n)}]H_i[(q^{-1}-1)\widetilde{Y}_1^{(n)}]X_1T_1^{-1}\cdots T_{n-1}^{-1} \\
&= \lim_n \sum_i G_i[t^nY_1^{(n)} + \ldots + t^nY_n^{(n)}]H_i[(q^{-1}-1)\widetilde{Y}_1^{(n)}]\widetilde{\pi}_n \\
&= \sum_i \left(\lim_n G_i[t^nY_1^{(n)} + \ldots + t^nY_n^{(n)}]\right)\left(\lim_n H_i[(q^{-1}-1)\widetilde{Y}_1^{(n)}]\right)\left(\lim_n \widetilde{\pi}_n\right) \\
&= \sum_i G_i[\Delta]H_i[(q^{-1}-1)\mathscr{Y}_1]\widetilde{\pi} \\
&= F[\Delta + (q^{-1}-1)\mathscr{Y}_1]\widetilde{\pi}.
\end{aligned}$$

$\square$

By applying Proposition 2.7.9 to $F = p_r$ we see the following:

Corollary 2.7.10. *For every* $r \geq 1$

$$[\widetilde{\pi}, p_r[\Delta]] = (q^{-r} - 1)\mathscr{Y}_1^r \widetilde{\pi}.$$

Proof. Using Proposition 2.7.9 applied to $F[X] = p_r[X]$ gives

$$\widetilde{\pi} p_r[\Delta] = p_r[\Delta + (q^{-1} - 1)\mathscr{Y}_1]\widetilde{\pi} = (p_r[\Delta] + (q^{-r} - 1)\mathscr{Y}_1^r)\widetilde{\pi}.$$

□

Lastly, we compute the full commutation relations between the limit Cherednik operators $\mathscr{Y}_i$ and $\widetilde{\pi}$. Interestingly, most of these relations mimic the standard finite rank DAHA situation except for $\mathscr{Y}_1 \widetilde{\pi}$ which now involves Δ.

Proposition 2.7.11.

$$\mathscr{Y}_i \widetilde{\pi} = \begin{cases} \widetilde{\pi} \mathscr{Y}_{i-1} & i > 1 \\ [\widetilde{\pi}, \Delta]/(q^{-1} - 1) & i = 1. \end{cases}$$

Proof. For $i = 1$ we have that from Proposition 2.7.9

$$\widetilde{\pi}\Delta - \Delta\widetilde{\pi} = (q^{-1} - 1)\mathscr{Y}_1 \widetilde{\pi}$$

and hence

$$\mathscr{Y}_1 \widetilde{\pi} = [\widetilde{\pi}, \Delta]/(q^{-1} - 1).$$

Let $i > 1$. We see that

$$\begin{aligned}
&\mathscr{Y}_i\widetilde{\pi}\\
&= \lim_n \widetilde{Y}_i^{(n)}\widetilde{\pi}_n\\
&= \lim_n \left(t^{n-i+1}T_{i-1}\cdots T_1\rho\pi_n T_{n-1}^{-1}\cdots T_i^{-1}\right)\left(X_1T_1^{-1}\cdots T_{n-1}^{-1}\right)\\
&= \lim_n t^{n-i+1}T_{i-1}\cdots T_1\rho\pi_n X_1 T_{n-1}^{-1}\cdots T_i^{-1}T_1^{-1}\cdots T_{n-1}^{-1}\\
&= \lim_n t^{n-i+1}T_{i-1}\cdots T_1\rho X_2\pi_n T_{n-1}^{-1}\cdots T_i^{-1}T_1^{-1}\cdots T_{n-1}^{-1}\\
&= \lim_n t^{n-i+1}T_{i-1}\cdots T_1 X_2\rho\pi_n T_{n-1}^{-1}\cdots T_i^{-1}T_1^{-1}\cdots T_{n-1}^{-1}\\
&= \lim_n t^{n-i+1}T_{i-1}\cdots T_2(tX_1T_1^{-1})\rho\pi_n T_{n-1}^{-1}\cdots T_i^{-1}T_1^{-1}\cdots T_{n-1}^{-1}\\
&= \lim_n t^{n-i+2}X_1T_{i-1}\cdots T_2T_1^{-1}\rho\pi_n T_1^{-1}\cdots T_{i-2}^{-1}T_{n-1}^{-1}\cdots T_i^{-1}T_{i-1}^{-1}\cdots T_{n-1}^{-1}\\
&= \lim_n t^{n-i+2}X_1T_{i-1}\cdots T_2T_1^{-1}\rho T_2^{-1}\cdots T_{i-1}^{-1}\pi_n T_{n-1}^{-1}\cdots T_i^{-1}T_{i-1}^{-1}\cdots T_{n-1}^{-1}\\
&= \lim_n t^{n-i+2}X_1T_{i-1}\cdots T_2T_1^{-1}T_2^{-1}\cdots T_{i-1}^{-1}\rho\pi_n T_{n-1}^{-1}\cdots T_i^{-1}T_{i-1}^{-1}\cdots T_{n-1}^{-1}\\
&= \lim_n t^{n-i+2}X_1T_1^{-1}\cdots T_{i-1}^{-1}T_{i-2}\cdots T_1\rho\pi_n T_{i-1}^{-1}\cdots T_{n-2}^{-1}T_{n-1}^{-1}\cdots T_{i-1}^{-1}\\
&= \lim_n t^{n-i+2}X_1T_1^{-1}\cdots T_{i-1}^{-1}T_{i-2}\cdots T_1\rho T_i^{-1}\cdots T_{n-1}^{-1}\pi_n T_{n-1}^{-1}\cdots T_{i-1}^{-1}\\
&= \lim_n t^{n-i+2}X_1T_1^{-1}\cdots T_{i-1}^{-1}T_{i-2}\cdots T_1T_i^{-1}\cdots T_{n-1}^{-1}\rho\pi_n T_{n-1}^{-1}\cdots T_{i-1}^{-1}\\
&= \lim_n t^{n-i+2}X_1T_1^{-1}\cdots T_{i-1}^{-1}T_i^{-1}\cdots T_{n-1}^{-1}T_{i-2}\cdots T_1\rho\pi_n T_{n-1}^{-1}\cdots T_{i-1}^{-1}\\
&= \lim_n \left(X_1T_1^{-1}\cdots T_{n-1}^{-1}\right)\left(t^{n-i+2}T_{i-2}\cdots T_1\rho\pi_n T_{n-1}^{-1}\cdots T_{i-1}^{-1}\right)\\
&= \lim_n \widetilde{\pi}_n\widetilde{Y}_{i-1}^{(n)}\\
&= \widetilde{\pi}\mathscr{Y}_{i-1}.
\end{aligned}$$

□

2.8. Specialization at $t = 0, q = \infty$

The goal of this section is to determine the specialization of the stable-limit non-symmetric Macdonald functions $\widetilde{E}_{(\mu|\lambda)}$ at $q = \infty$ and $t = 0$. After adjusting for the (q,t)-conventions in this thesis, we will see that this specialization generalizes the well known specialization result of Ion [25] about the non-symmetric Macdonald polynomials. We will show that the specializations of the $\widetilde{E}_{(\mu|\lambda)}$ give an almost symmetric generalization of the Schur functions $s_{(\mu|\lambda)}$ which satisfy some positivity properties. Further, we will give an interpretation for the these almost symmetric Schur functions in terms of Demazure characters.

In order to state and prove the main results of this section we will first need to review some relevant information about Weyl symmetrization, isobaric divided difference operators, and key polynomials.

2.8.1. Weyl Symmetrization and Isobaric Divided Difference Operators. We now recall the definition of the Weyl symmetrization map and its partial symmetrization analogues. Informally, these maps are the $t = 0$ specialization of the $\epsilon_k^{(n)}$ maps defined previously.

DEFINITION 2.8.1. *Let $0 \leq k \leq n$. We define the **partial Weyl symmetrizer**, $W_k^{(n)}$, to be the map*

$$W_k^{(n)} : \mathbb{Q}(q,t)[x_1, \ldots, x_n] \to \mathbb{Q}(q,t)[x_1, \ldots, x_n]^{\mathfrak{S}_{(1^k, n-k)}}$$

given by

$$W_k^{(n)}(f(x_1, \ldots, x_n)) := \sum_{\sigma \in \mathfrak{S}_{(1^k, n-k)}} \sigma \left(f(x) \prod_{k+1 \leq i < j \leq n} \left(\frac{1}{1 - x_j/x_i} \right) \right).$$

REMARK 15. *Notice that these maps are defined over $\mathbb{Q}$ (over $\mathbb{Z}$ in fact) and hence in fact define maps*

$$W_k^{(n)} : \mathbb{Q}[x_1, \ldots, x_n] \to \mathbb{Q}[x_1, \ldots, x_n]^{\mathfrak{S}_{(1^k, n-k)}}.$$

It is not immediately obvious that $W_k^{(n)}(f(x_1, \ldots, x_n))$ is a well defined polynomial due to presence of the nontrivial rational function

$$\prod_{k+1 \leq i < j \leq n} \left(\frac{1}{1 - x_j/x_i} \right).$$

However, we may rewrite the given definition of $W_k^{(n)}$ as follows. Let $\delta_k^{(n)} := 0^k * (n-k-1, \ldots, 1, 0)$.

$$
\begin{aligned}
&W_k^{(n)}(f(x_1,\dots,x_n))\\
&= \sum_{\sigma\in\mathfrak{S}_{(1^k,n-k)}} \sigma\left(f(x)\prod_{k+1\le i<j\le n}\left(\frac{1}{1-x_j/x_i}\right)\right)\\
&= \sum_{\sigma\in\mathfrak{S}_{(1^k,n-k)}} \sigma\left(f(x)\prod_{k+1\le i<j\le n}\left(\frac{x_i}{x_i-x_j}\right)\right)\\
&= \sum_{\sigma\in\mathfrak{S}_{(1^k,n-k)}} \sigma\left(x^{\delta_k^{(n)}}f(x)\prod_{k+1\le i<j\le n}\left(\frac{1}{x_i-x_j}\right)\right)\\
&= \frac{\sum_{\sigma\in\mathfrak{S}_{(1^k,n-k)}}(-1)^{\ell(\sigma)}\sigma\left(x^{\delta_k^{(n)}}f(x)\right)}{\prod_{k+1\le i<j\le n}(x_i-x_j)}.
\end{aligned}
$$

Since the numerator of the above fraction is an alternating polynomial, i.e. $s_i(g)=-g$ *for* $k+1\le i\le n-1$, *it must be divisible by the Vandermonde determinant* $\prod_{k+1\le i<j\le n}(x_i-x_j)$. *Thus* $W_k^{(n)}(f(x_1,\dots,x_n))$ *must be a polynomial.*

LEMMA 2.8.2. *As elements of* $\mathrm{End}_{\mathbb{Q}(q,t)}(\mathscr{P}_n)$ *the operators* $W_k^{(n)}$ *satisfy the following:*

- $(W_k^{(n)})^2 = W_k^{(n)}$
- $\sigma W_k^{(n)} = W_k^{(n)}\sigma$ *for* $\sigma\in\mathfrak{S}_{(k,n-k)}$
- $\sigma W_k^{(n)} = W_k^{(n)}$ *for* $\sigma\in\mathfrak{S}_{(1^k,n-k)}$
- $W_k^{(n)}W_j^{(n)} = W_k^{(n)}$ *for* $k<j$.

PROOF. These properties are straightforward to verify and we leave their verification to the reader. $\square$

LEMMA 2.8.3. *For* $0\le k\le n$

$$W_k^{(n)}\Xi^{(n)} = \Xi^{(n)}W_k^{(n+1)}.$$

PROOF. We begin with the following computation:

$$
\begin{aligned}
&W_k^{(n+1)}(f)\\
&= \frac{\sum_{\sigma\in\mathfrak{S}_{(1^k,n+1-k)}}(-1)^{\ell(\sigma)}\sigma\left(x^{\delta_k^{(n+1)}}f(x)\right)}{\prod_{k+1\le i<j\le n+1}(x_i-x_j)}\\
&= \frac{1}{\prod_{k+1\le i<j\le n+1}(x_i-x_j)}\sum_{\sigma\in\mathfrak{S}_{(1^k,n-k,1)}}\sum_{\gamma\in\mathfrak{S}_{(1^k,n-k,1)}\backslash\mathfrak{S}_{(1^k,n+1-k)}}(-1)^{\ell(\sigma\gamma)}\sigma\gamma\left(x^{\delta_k^{(n+1)}}f(x)\right)\\
&= \frac{1}{\prod_{k+1\le i<j\le n+1}(x_i-x_j)}\sum_{\sigma\in\mathfrak{S}_{(1^k,n-k,1)}}(-1)^{\ell(\sigma)}\sigma\sum_{\gamma\in\mathfrak{S}_{(1^k,n-k,1)}\backslash\mathfrak{S}_{(1^k,n+1-k)}}(-1)^{\ell(\gamma)}\gamma\left(x^{\delta_k^{(n+1)}}f(x)\right)\\
&= \frac{1}{\prod_{k+1\le i<j\le n+1}(x_i-x_j)}\\
&\times\sum_{\sigma\in\mathfrak{S}_{(1^k,n-k,1)}}(-1)^{\ell(\sigma)}\sigma\left(1-s_n\pm\ldots+(-1)^{n-k+1}s_n\cdots s_k\right)\left(x^{\delta_k^{(n+1)}}f(x)\right).
\end{aligned}
$$

Now notice that for all $k\le i\le n$, $s_n\cdots s_iX^{\delta_k^{(n+1)}}=X_{n+1}s_n\cdots s_i(X^{\delta_k^{(n+1)}-e_i})$. Further, if $x_{n+1}|g$ then $x_{n+1}|\sigma(g)$ for any $\sigma\in\mathfrak{S}_{(1^k,n-k,1)}$. Thus we may write

$$W_k^{(n+1)}(f)=\frac{1}{\prod_{k+1\le i<j\le n+1}(x_i-x_j)}\left(\sum_{\sigma\in\mathfrak{S}_{(1^k,n-k,1)}}(-1)^{\ell(\sigma)}\sigma\left(x^{\delta_k^{(n+1)}}f(x)\right)+x_{n+1}g\right)$$

for some polynomial g.

Therefore,

$$
\begin{aligned}
&\Xi^{(n)}W_k^{(n+1)}(f)\\
&=\Xi^{(n)}\frac{1}{\prod_{k+1\le i<j\le n+1}(x_i-x_j)}\left(\sum_{\sigma\in\mathfrak{S}_{(1^k,n-k,1)}}(-1)^{\ell(\sigma)}\sigma\left(x^{\delta_k^{(n+1)}}f(x)\right)+x_{n+1}g\right)\\
&=\frac{x_{k+1}^{-1}\cdots x_n^{-1}}{\prod_{k+1\le i<j\le n}(x_i-x_j)}\left(\sum_{\sigma\in\mathfrak{S}_{(1^k,n-k,1)}}(-1)^{\ell(\sigma)}\sigma\left(\Xi^{(n)}(x^{\delta_k^{(n+1)}}f(x))\right)+0\right)\\
&=\frac{x_{k+1}^{-1}\cdots x_n^{-1}}{\prod_{k+1\le i<j\le n}(x_i-x_j)}\left(\sum_{\sigma\in\mathfrak{S}_{(1^k,n-k,1)}}(-1)^{\ell(\sigma)}\sigma\left(x^{\delta_k^{(n+1)}}\Xi^{(n)}(f(x))\right)\right).
\end{aligned}
$$

Since we have that $x_{k+1}^{-1}\cdots x_n^{-1}x^{\delta_k^{(n+1)}} = x^{\delta_k^{(n)}}$ we see

$$\begin{aligned}
&\Xi^{(n)}W_k^{(n+1)}(f)\\
&= \frac{1}{\prod_{k+1\leq i<j\leq n}(x_i-x_j)}\left(\sum_{\sigma\in\mathfrak{S}_{(1^k,n-k,1)}}(-1)^{\ell(\sigma)}\sigma\left(x^{\delta_k^{(n)}}\Xi^{(n)}(f(x))\right)\right)\\
&= W_k^{(n)}(\Xi^{(n)}(f(x))).
\end{aligned}$$

□

The above lemma allows for the following definition.

Definition 2.8.4. *Let $k\geq 0$ define the operator W_k on $\mathscr{P}_{as}^+$ as*

$$W_k := \lim_n W_k^{(n)}.$$

As we will prove later, the operators W_k are the $t=0$ specializations of the partial Hecke symmetrizers ϵ_k.

Definition 2.8.5. *Define the* ***isobaric divided difference operators****, $\xi_1,\ldots\xi_{n-1}$, on $\mathscr{P}_n$ by*

$$\xi_i(f) = \frac{x_if - x_{i+1}s_i(f)}{x_i - x_{i+1}}.$$

Lemma 2.8.6. *We have the following relations:*

- $\xi_i^2 = \xi_i$
- $\xi_i\xi_{i+1}\xi_i = \xi_{i+1}\xi_i\xi_{i+1}$
- $\xi_i\xi_j = \xi_j\xi_i$ *for* $|i-j|>1$.

Proof. This will follow from Lemma 2.8.15 proven independently later in this section. □

The above are the generating relations for the ***0-Hecke algebra***. For any $\sigma\in\mathfrak{S}_n$ with a reduced expression $\sigma = s_{i_1}\cdots s_{i_r}$ we define

$$\xi_\sigma = \xi_{i_1}\cdots\xi_{i_r}.$$

The following lemma relates the Weyl symmetrizers $W_k^{(n)}$ to be the isobaric divided difference operators ξ_i.

Lemma 2.8.7. *We have the recursion relation:*

$$W_k^{(n)} = \xi_{n-1}\cdots\xi_{k+1}W_{k+1}^{(n)}.$$

Proof. We leave this as an exercise to the reader. $\square$

One of the main utilities for defining the maps $W_k^{(n)}$ is that they generate the Schur polynomials in the following way.

Proposition 2.8.8 (Weyl Character Formula for GL_n). *For $\lambda \in \mathbb{Y}$ and $n \geq \ell(\lambda)$*

$$W_0^{(n)}(x^\lambda) = s_\lambda(x_1,\ldots,x_n).$$

2.8.2. Key Polynomials. Here we review some relevant information about the key polynomials.

Definition 2.8.9. *Let $n \geq 1$. Define the* ***key polynomials*** *to be the unique collection of polynomials $\{\mathcal{K}_\alpha(x_1,\ldots,x_n)\}_{\alpha\in\mathbb{Z}_{\geq 0}^n}$ determined by the following properties:*

- *If $\alpha_1 \geq \ldots \geq \alpha_n$ then*
$$\mathcal{K}_\alpha(x_1,\ldots,x_n) := x^\alpha.$$
- *Whenever $\alpha_i > \alpha_{i+1}$*
$$\mathcal{K}_{s_i(\alpha)}(x_1,\ldots,x_n) = \xi_i(\mathcal{K}_\alpha(x_1,\ldots,x_n)).$$

By a simple induction argument we see that for $\alpha \in \mathbb{Z}_{\geq 0}^n$

$$\mathcal{K}_{(\alpha_1,\ldots,\alpha_n,0)}(x_1,\ldots,x_n,x_{n+1}) = \mathcal{K}_{(\alpha_1,\ldots,\alpha_n)}(x_1,\ldots,x_n).$$

As such we will refer to $\mathcal{K}_\mu(x)$ for $\mu \in \mathrm{Comp}^{red}$ unambiguously as an element of $\mathbb{Z}_{\geq 0}[x_1,x_2,x_3,\ldots] \subset \mathscr{P}_{as}^+$.

Remark 16. *It is known that the key polynomials $\{\mathcal{K}_\alpha | \alpha \in \mathbb{Z}_{\geq 0}^k\}$ for a basis for $\mathscr{P}_k^+$.*
For $\lambda \in \mathbb{Y}$ and $n \geq \ell(\lambda)$

$$\mathcal{K}_{0^{n-\ell(\lambda)}*\mathrm{rev}(\lambda)}(x_1,\ldots,x_n) = s_\lambda(x_1,\ldots,x_n).$$

Further, if $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ *and there exists some* $1 \leq i < j \leq n$ *with* $\alpha_i < \ldots < \alpha_j$ *then* $\mathcal{K}_\alpha(x_1, \ldots, x_n)$ *is symmetric in the variables* $x_i, \ldots, x_j$.

2.8.3. Specialization at $t = 0$, $q = \infty$.

Definition 2.8.10. *Define* $\mathcal{O} \subset \mathscr{P}^+_{as}$ *to be the set of* $f(x) \in \mathscr{P}^+_{as}$ *such that*

$$f(x) = f(x_1, x_2, \ldots; q^{-1}, t) = \sum_i c^{(i)} x^{\mu^{(i)}} m_{\lambda^{(i)}}[X]$$

for some scalars $c^{(i)} = c^{(i)}(q^{-1}, t) \in \mathbb{Q}[q^{-1}][[t]] \cap \mathbb{Q}(q,t)$, $(\mu^{(i)}|\lambda^{(i)}) \in \Sigma$. *Let* $\mathscr{P}^+_{as,\mathbb{Q}}$ *denote the set of* $f(x) \in \mathscr{P}^+_{as}$ *such that*

$$f(x) = \sum_i c^{(i)} x^{\mu^{(i)}} m_{\lambda^{(i)}}[X]$$

for some scalars $c^{(i)} \in \mathbb{Q}$, $(\mu^{(i)}|\lambda^{(i)}) \in \Sigma$. *Define the* $\mathbb{Q}$*-algebra homomorphism* $\Upsilon : \mathcal{O} \to \mathscr{P}^+_{as,\mathbb{Q}}$ *by*

$$\Upsilon(f(x_1, x_2, \ldots; q^{-1}, t)) := f(x_1, x_2, \ldots; 0, 0).$$

Equivalently,

$$\Upsilon(f) := \lim_{q \to \infty} \lim_{t \to 0} f.$$

We will need the following lemma.

Lemma 2.8.11. *Let* $f_n \in \mathscr{P}^+_n \cap \mathcal{O}$ *with* $\lim_n f_n = f \in \mathscr{P}^+_{as}$. *Then* $f \in \mathcal{O}$ *and*

$$\Upsilon(f) = \lim_n \Upsilon(f_n).$$

Proof. By the definition of convergence (see Definition 1.7.3) we know that we have for all $n \geq 1$

$$f_n = \sum_{i=1}^N c_n^{(i)} x^{\mu^{(i)}} m_{\lambda^{(i)}}[x_1 + \ldots + x_n]$$

where $c_n^{(i)} \in \mathbb{Q}(q,t)$, $(\mu^{(i)}|\lambda^{(i)}) \in \Sigma$ with $\lim_n c_n^{(i)} = c^{(i)} \in \mathbb{Q}(q,t)$ convergent t-adically. Since $f_n \in \mathscr{P}^+_n \cap \mathcal{O}$ we know that $c_n^{(i)} = c_n^{(i)}(q^{-1}, t) \in \mathbb{Q}[q^{-1}][[t]] \cap \mathbb{Q}(q,t)$. Since $\mathbb{Q}[q^{-1}][[t]]$ is complete t-adically we must have $c^{(i)} \in \mathbb{Q}[q^{-1}][[t]] \cap \mathbb{Q}(q,t)$. Then it is clear that

$$f = \sum_{i=1}^N c^{(i)} x^{\mu^{(i)}} m_{\lambda^{(i)}}[X] \in \mathcal{O}.$$

A simple topological argument shows that

$$\lim_{q\to\infty}\lim_{t\to 0} c^{(i)}(q^{-1},t) = \lim_{n}\lim_{q\to\infty}\lim_{t\to 0} c_n^{(i)}(q^{-1},t).$$

Then we find

$$\begin{aligned}
&\lim_{n} \Upsilon(f_n)\\
&= \lim_{n} \Upsilon\left(\sum_{i=1}^{N} c_n^{(i)} x^{\mu^{(i)}} m_{\lambda^{(i)}}[x_1+\ldots+x_n]\right)\\
&= \lim_{n}\lim_{q\to\infty}\lim_{t\to 0}\sum_{i=1}^{N} c_n^{(i)} x^{\mu^{(i)}} m_{\lambda^{(i)}}[x_1+\ldots+x_n]\\
&= \lim_{n}\sum_{i=1}^{N}\left(\lim_{q\to\infty}\lim_{t\to 0} c_n^{(i)}\right) x^{\mu^{(i)}} m_{\lambda^{(i)}}[x_1+\ldots+x_n]\\
&= \sum_{i=1}^{N}\left(\lim_{q\to\infty}\lim_{t\to 0} c^{(i)}\right) x^{\mu^{(i)}} m_{\lambda^{(i)}}[X]\\
&= \Upsilon(f).
\end{aligned}$$

□

Adjusting to the (q,t)-conventions in this thesis we may restate a result of Ion [25] relating the non-symmetric Macdonald polynomials to the key polynomials.

Theorem 2.8.12. *[25] For $\alpha \in \mathbb{Z}^n_{\geq 0}$*

$$\Upsilon(E_\alpha) = \mathcal{K}_\alpha.$$

From Ion's result we find a known combinatorial formula for the key polynomials using the HHL combinatorial formula (see 2.2.2) for the non-symmetric Macdonald polynomials. For $\alpha \in \mathbb{Z}^n_{\geq 0}$ denote by $\mathcal{L}(\alpha)$ the set of non-attacking labellings $\sigma : \alpha \to [n]$ such that $\mathrm{maj}(\widehat{\sigma}) = \mathrm{coinv}(\widehat{\sigma}) = 0$.

Proposition 2.8.13. *For $\alpha \in \mathbb{Z}^n_{\geq 0}$,*

$$\mathcal{K}_\alpha = \sum_{\sigma\in\mathcal{L}(\alpha)} x^\sigma.$$

PROOF. From the combinatorial formula for E_α (Theorem 2.2.2) we see that

$$E_\alpha = \sum_{\substack{\sigma:\alpha\to[n]\\ \text{non-attacking}}} x^\sigma q^{-\operatorname{maj}(\widehat{\sigma})} t^{\operatorname{coinv}(\widehat{\sigma})} \prod_{\substack{u\in dg'(\alpha)\\ \widehat{\sigma}(u)\neq\widehat{\sigma}(d(u))}} \left(\frac{1-t}{1-q^{-(\lg(u)+1)}t^{(a(u)+1)}}\right).$$

Note that the values $\operatorname{leg}(u)$ and $\operatorname{arm}(u)$ are both non-negative so that $E_\alpha \in \mathcal{O}$. Therefore, when we specialize $q\to\infty$ and $t\to 0$ we find that

$$\lim_{q\to\infty}\lim_{t\to 0} q^{-\operatorname{maj}(\widehat{\sigma})} t^{\operatorname{coinv}(\widehat{\sigma})} \prod_{\substack{u\in dg'(\alpha)\\ \widehat{\sigma}(u)\neq\widehat{\sigma}(d(u))}} \left(\frac{1-t}{1-q^{-(\lg(u)+1)}t^{(a(u)+1)}}\right) = \mathbb{1}\left(\operatorname{maj}(\widehat{\sigma}) = \operatorname{coinv}(\widehat{\sigma}) = 0\right).$$

Hence, from Theorem 2.8.12

$$\mathcal{K}_\alpha = \Upsilon(E_\alpha) = \sum_{\substack{\sigma:\alpha\to[n]\\ \text{non-attacking}\\ \operatorname{maj}(\widehat{\sigma})=0\\ \operatorname{coinv}(\widehat{\sigma})=0}} x^\sigma = \sum_{\sigma\in\mathcal{L}(\alpha)} x^\sigma.$$

□

REMARK 17. *Note that* $\operatorname{maj}(\widehat{\sigma}) = 0$ *is equivalent to* $\operatorname{Des}(\widehat{\sigma}) = \emptyset$ *which in turn is equivalent to* $\widehat{\sigma}(u) \leq \widehat{\sigma}(d(u))$ *i.e.* $\widehat{\sigma}$ *is weakly decreasing upwards along columns. The requirement that* $\operatorname{coinv}(\widehat{\sigma}) = 0$ *is equivalent to the statement that* $\widehat{\sigma}$ *has no* ***co-inversion triples*** *(see Definition 2.2.3). Importantly, for a non-attacking filling* $\sigma : \alpha \to [n]$, $\operatorname{coinv}(\widehat{\sigma})$ *is equal to the number of co-inversion triples of* $\widehat{\sigma}$*. Thus a non-attacking filling* σ *is in* $\mathcal{L}(\alpha)$ *if* $\widehat{\sigma}$ *is weakly decreasing upwards along columns and has no co-inversion triples.*

As an easy application of Ion's result we may compute the specializations of all $\widetilde{E}_{(\mu|\emptyset)}$.

PROPOSITION 2.8.14. *For all* $\mu \in \operatorname{Comp}^{red}$, $\widetilde{E}_{(\mu|\emptyset)} \in \mathcal{O}$ *and*

$$\Upsilon(\widetilde{E}_{(\mu|\emptyset)}) = \mathcal{K}_\mu.$$

PROOF. Let $\mu \in \operatorname{Comp}^{red}$. From the combinatorial formula Corollary 2.3.2 we may observe directly that $\widetilde{E}_{(\mu|\emptyset)} \in \mathcal{O}$. To see this note that each of the scalar coefficients of the expansion of $\widetilde{E}_{(\mu|\emptyset)}$ has the form

$$q^{-a}t^b \prod_i \left(\frac{1-t}{1-q^{-c_i}t^{d_i}}\right)$$

for some $a, b, c_i, d_i \geq 0$. By expanding the denominators

$$\frac{1}{1 - q^{-c_i} t^{d_i}} = \sum_{m \geq 0} q^{-mc_i} t^{md_i}$$

we see that

$$q^{-a} t^b \prod_i \left(\frac{1-t}{1 - q^{-c_i} t^{d_i}} \right) \in \mathbb{Q}[q^{-1}][[t]]$$

as required.

As $\Upsilon(\widetilde{E}_\mu)$ is now well defined, we may compute directly using Lemma 2.8.11 to find

$$\begin{aligned}
&\Upsilon(\widetilde{E}_\mu) \\
&= \lim_n \Upsilon(E_{\mu * 0^n}) \\
&= \lim_n \mathcal{K}_{\mu * 0^n} \\
&= \lim_n \mathcal{K}_\mu \\
&= \mathcal{K}_\mu .
\end{aligned}$$

$\square$

In the next lemma we will formalize the notion that the operators ξ_i, W_k are the $q = \infty$ and $t = 0$ specializations of T_i, ϵ_k respectively. This result is standard but we will include its proof for the sake of completeness.

Lemma 2.8.15. *For all $k \geq 0$ and $i \geq 1$, $\Upsilon \circ T_i|_{\mathcal{O}} = \xi_i \circ \Upsilon|_{\mathcal{O}}$ and $\Upsilon \circ \epsilon_k|_{\mathcal{O}} = W_k \circ \Upsilon|_{\mathcal{O}}$.*

Proof. Let $f = f(x; q^{-1}, t) \in \mathscr{P}_{as}^+ \cap \mathcal{O}$. Let $i \geq 1$ and $k \geq 0$.

First, we have

$$
\begin{aligned}
&\Upsilon \circ T_i(f) \\
&= \Upsilon\left(s_i(f) + (1-t)x_i \frac{f - s_i(f)}{x_i - x_{i+1}}\right) \\
&= s_i\Upsilon(f) + (1-0)x_i \frac{\Upsilon(f) - s_i\Upsilon(f)}{x_i - x_{i+1}} \\
&= \left(s_i + x_i \frac{1 - s_i}{x_i - x_{i+1}}\right) f(x;0,0) \\
&= \left(\frac{(x_i - x_{i+1})s_i + x_i(1 - s_i)}{x_i - x_{i+1}}\right) f(x;0,0) \\
&= \left(\frac{x_i - x_{i+1}s_i}{x_i - x_{i+1}}\right) f(x;0,0) \\
&= \xi_i f(x;0,0) \\
&= \xi_i \circ \Upsilon(f).
\end{aligned}
$$

If $f \in \mathscr{P}(k)^+$ then

$$\Upsilon \circ \epsilon_k(f) = \Upsilon(f)$$

and

$$W_k \circ \Upsilon(f) = \Upsilon(f).$$

Thus we may assume that $f \in \mathscr{P}(k+r)^+$ for some $r \geq 1$ in which case using Lemma 2.8.11 we see

$$
\begin{aligned}
&\Upsilon \circ \epsilon_k(f) \\
&= \Upsilon\left(\lim_n \epsilon_k^{(n)}(\Xi^{(n)}(f))\right) \\
&= \Upsilon\left(\lim_n \frac{1}{[n-k]_t!} \sum_{\sigma \in \mathfrak{S}_{(1^k, n-k)}} t^{\binom{n-k}{2} - \ell(\sigma)} T_\sigma \Xi^{(n)}(f)\right) \\
&= \lim_n \Upsilon\left(\frac{1}{[n-k]_t!} \sum_{\sigma \in \mathfrak{S}_{(1^k, n-k)}} t^{\binom{n-k}{2} - \ell(\sigma)} T_\sigma \Xi^{(n)}(f)\right)
\end{aligned}
$$

$$= \lim_n \sum_{\sigma \in \mathfrak{S}_{(1^k, n-k)}} \mathbb{1}\left(\binom{n-k}{2} = \ell(\sigma)\right) \Upsilon\left(T_\sigma \Xi^{(n)}(f)\right)$$

$$= \lim_n (T_{n-1} \cdots T_{k+1}) \cdots (T_{n-1} \cdots T_{k+r}) \Upsilon(\Xi^{(n)}(f))$$

$$= \lim_n (\xi_{n-1} \cdots \xi_{k+1}) \cdots (\xi_{n-1} \cdots \xi_{k+r}) f(x_1, \ldots, x_n, 0, \ldots; 0, 0)$$

$$= \lim_n W_k^{(n)} f(x_1, \ldots, x_n, 0, \ldots; 0, 0)$$

$$= W_k \circ \Upsilon(f)$$

□

2.8.4. Almost Symmetric Schur Functions. The stable-limit non-symmetric Macdonald functions $\widetilde{E}_{(\mu|\lambda)}$ were defined in Definition 2.4.10 by applying successive partial-symmetrization operators to the functions $\widetilde{E}_{(\mu*\lambda|\emptyset)}$. Given that the operator T_i, ϵ_k specialize to the ξ_i, W_k respectively we may define a set of almost symmetric functions $s_{(\mu|\lambda)}$ analogously.

DEFINITION 2.8.16. *Define the* ***almost symmetric Schur functions****,* $s_{(\mu|\lambda)} = s_{(\mu|\lambda)}(x_1, x_2, \ldots)$*, for* $(\mu|\lambda) \in \Sigma$*. by the following recursive formula:*

- $s_{(\mu|\emptyset)} = \mathcal{K}_\mu$
- *If* $\mu_r \geq \lambda_1$ *then*

$$s_{(\mu_1, \ldots, \mu_{r-1}|\mu_r, \lambda_1, \ldots, \lambda_\ell)} = W_{r-1}(s_{(\mu_1, \ldots, \mu_{r-1}, \mu_r|\lambda_1, \ldots, \lambda_\ell)}).$$

REMARK 18. *We note that from the above recursion it follows that for any* $\lambda \in \mathbb{Y}$ $s_{(\emptyset|\lambda)} = s_\lambda$*. Thus the almost symmetric Schur functions interpolate between the key polynomials and the Schur functions in infinitely many variables* $x_1, x_2, \ldots$*. Lapointe in their paper [28] defines the* ***m-symmetric Schur functions*** $s_{(a;\lambda)}(x;t)$*. These functions have the property that* $s_{(a;\emptyset)}(x;1) = \mathcal{K}_a(x)$ *and* $s_{(\emptyset;\lambda)}(x;1) = s_\lambda(x)$ *similarly to the functions* $s_{(\mu|\lambda)}(x)$ *defined above. Further, there give a basis for* $\mathscr{P}(m)^+$*. However, it is not clear to this author how Lapointe's m-symmetric Schur functions are related to the almost symmetric Schur functions. Lapointe defines* $s_{(a;\lambda)}(x;t)$ *by first defining the* ***dual*** *m-symmetric Schur functions* $s^*_{(a;\lambda)}(x;t)$ *as an explicit linear combination of the non-symmetric Hall-Littlewood-Schur basis* $H_\alpha(x;t)s_\nu[X]$ *(note that* $X = x_1 + \ldots$ *here instead of*

$\mathfrak{X}_m = x_{m+1} + \ldots$) with explicit combinatorial coefficients (involving certain weighted skew tableaux) along with a non-degenerate pairing on $\mathscr{P}(m)^+$. The $s_{(a;\lambda)}(x;t)$ are then defined as the dual basis to the $s^*_{(a;\lambda)}(x;t)$. In this thesis we have used a purely algebraic/recursive approach in defining our generalized Schur functions. Any proof that relates these two types of functions would likely be nontrivial and combinatorial in nature. However, it seems fruitful to understand how these notions are related as this will likely provide additional insight into the properties of the almost symmetric functions.

EXAMPLE. *Here we calculate $s_{(2|3,1)}$ directly using the operators ξ_i and W_k:*

$$\begin{aligned}
&s_{(2|3,1)} \\
&= W_1W_2(s_{(2,3,1|\emptyset)}) \\
&= W_1W_2\xi_1(s_{(3,2,1|\emptyset)}) \\
&= W_1W_2\xi_1(x_1^3x_2^2x_3) \\
&= W_1W_2(x_1^3x_2^2x_3 + x_1^2x_2^3x_3) \\
&= W_1(x_1^3x_2^2s_1[\mathfrak{X}_2] + x_1^2x_2^3s_1[\mathfrak{X}_2]) \\
&= x_1^3s_{(2,1)}[\mathfrak{X}_1] + x_1^2s_{(3,1)}[\mathfrak{X}_1]
\end{aligned}$$

EXAMPLE. *Here we give a list of some examples of almost symmetric Schur functions that are neither symmetric Schur functions nor key polynomials.*

- $s_{(0,1|2)} = x_1^2x_2 + x_1^2s_1[\mathfrak{X}_2] + x_2^2x_1 + x_2^2s_1[\mathfrak{X}_2] + x_1s_2[\mathfrak{X}_2] + x_2s_2[\mathfrak{X}_2] + 2x_1x_2s_1[\mathfrak{X}_2]$
- $s_{(2|3,1)} = x_1^3s_{(2,1)}[\mathfrak{X}_1] + x_1^2s_{(3,1)}[\mathfrak{X}_1]$
- $s_{(2,1|1)} = x_1^2x_2s_1[\mathfrak{X}_2]$
- $s_{(1,2|1)} = x_1^2x_2s_1[\mathfrak{X}_2] + x_1x_2^2s_1[\mathfrak{X}_2]$
- $s_{(1|2,1)} = x_1^2s_{(1,1)}[\mathfrak{X}_1] + x_1s_{(2,1)}[\mathfrak{X}_1]$.

We are now ready to compute the specializations of the stable-limit non-symmetric Macdonald functions $\widetilde{E}_{(\mu|\lambda)}$ at $q = \infty$ and $t = 0$.

Theorem 2.8.17. *For* $(\mu|\lambda) \in \Sigma$, $\widetilde{E}_{(\mu|\lambda)} \in \mathcal{O}$ *and*

$$\Upsilon(\widetilde{E}_{(\mu|\lambda)}) = s_{(\mu|\lambda)}(x).$$

Proof. Let $(\mu|\lambda) \in \Sigma$. In order to show that $\widetilde{E}_{(\mu|\lambda)} \in \mathcal{O}$ it suffices by induction to verify that each $\epsilon_k(f) \in \mathcal{O}$ for every $f \in \mathcal{O}$. However, this is easy to see using the explicit formula for the action of ϵ_k using the Jing vertex operators $\mathscr{B}_r$ (see 2.4.5). We now proceed by direct computation using Lemma 2.8.15 and Proposition 2.8.14.

$$\begin{aligned}
&\Upsilon(\widetilde{E}_{(\mu|\lambda)})\\
&= \Upsilon(\epsilon_{\ell(\mu)}(\widetilde{E}_{(\mu*\lambda|\emptyset)}))\\
&= W_{\ell(\mu)}(\Upsilon(\widetilde{E}_{(\mu*\lambda|\emptyset)}))\\
&= W_{\ell(\mu)}(\mathcal{K}_{\mu*\lambda})\\
&= W_{\ell(\mu)}(s_{(\mu*\lambda|\emptyset)})\\
&= s_{(\mu|\lambda)}.
\end{aligned}$$

□

2.8.5. Combinatorial Formula for Almost Symmetric Schur Functions. In this section we will compute an explicit combinatorial formula for the monomial expansion of the almost symmetric Schur functions. Further, we will use this expansion to show that a generalization of the classical Kostka coefficients for Schur functions are non-negative integers.

Proposition 2.8.18. *For* $(\mu|\lambda) \in \Sigma$,

$$s_{(\mu|\lambda)} = \lim_n \mathcal{K}_{\mu*0^n*\mathrm{rev}(\lambda)}.$$

Proof. We proceed by direct calculation:

$$\begin{aligned}
& s_{(\mu|\lambda)} \\
& = W_{\ell(\mu)} \cdots W_{\ell(\mu)+\ell(\lambda)} s_{(\mu*\lambda|\emptyset)} \\
& = W_{\ell(\mu)} s_{(\mu*\lambda|\emptyset)} \\
& = W_{\ell(\mu)} \mathcal{K}_{\mu*\lambda} \\
& = \lim_n W_{\ell(\mu)}^{(\ell(\mu)+\ell(\lambda)+n)} (\mathcal{K}_{\mu*\lambda*0^n}) \\
& = \lim_n (\xi_{\ell(\mu)+\ell(\lambda)+n-1} \cdots \xi_{\ell(\mu)+1}) \cdots (\xi_{\ell(\mu)+\ell(\lambda)+n-1} \cdots \xi_{\ell(\mu)+\ell(\lambda)}) (\mathcal{K}_{\mu*\lambda*0^n}) \\
& = \lim_n \mathcal{K}_{\mu*0^n*\mathrm{rev}(\lambda)}.
\end{aligned}$$

□

As an immediate consequence we get the following:

Corollary 2.8.19. *The set* $\{s_{(\mu|\lambda)}(x) | (\mu|\lambda) \in \Sigma\}$ *is a homogeneous* $\mathbb{Q}$*-basis for* $\mathscr{P}^+_{as,\mathbb{Q}}$.

Proof. Since the key polynomials are homogeneous and the operators W_k are clearly homogeneous, we see that the $s_{(\mu|\lambda)}$ are homogeneous as well. Following similarly to the proof of Theorem 4.2.12, we see that as there are sufficiently many $s_{(\mu|\lambda)}$ in each homogeneous component of $\mathscr{P}(k)^+$, it suffices to show that the $s_{(\mu|\lambda)}$ are linearly independent (over $\mathbb{Q}$). Let $(\mu^{(1)}|\lambda^{(1)}), \ldots, (\mu^{(m)}|\lambda^{(m)}) \in \Sigma$ be distinct. Set $r^{(i)} := \ell(\mu^{(i)}) + \ell(\lambda^{(i)})$. Suppose that for some $a^{(i)} \in \mathbb{Q}$, $\sum_{i=1}^m a^{(i)} s_{(\mu^{(i)}|\lambda^{(i)})} = 0$. Then

$$\begin{aligned}
0 &= \sum_{i=1}^m a^{(i)} s_{(\mu^{(i)}|\lambda^{(i)})} \\
&= \sum_{i=1}^m a^{(i)} \lim_n \mathcal{K}_{\mu^{(i)}*0^{n-r^{(i)}}*\mathrm{rev}(\lambda^{(i)})} \\
&= \lim_n \sum_{i=1}^m a^{(i)} \mathcal{K}_{\mu^{(i)}*0^{n-r^{(i)}}*\mathrm{rev}(\lambda^{(i)})}.
\end{aligned}$$

Now we see that for all sufficiently large n,

$$\sum_{i=1}^{m} a^{(i)} \mathcal{K}_{\mu^{(i)} * 0^{n-r^{(i)}} * \mathrm{rev}(\lambda^{(i)})} = 0$$

but, since the pairs $(\mu^{(i)}|\lambda^{(i)})$ are distinct, we know that the key polynomials $\mathcal{K}_{\mu^{(i)} * 0^{n-r^{(i)}} * \mathrm{rev}(\lambda^{(i)})}$ are linearly independent. Therefore, $a^{(i)} = 0$ as desired. $\square$

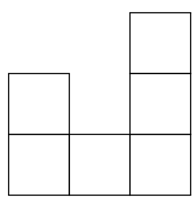
We will need to consider the following combinatorial construction.

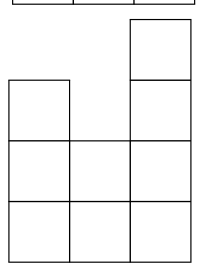
Definition 2.8.20. *Let $(\mu|\lambda) \in \Sigma$. Let ω denote the first infinite ordinal i.e. $n < \omega$ for all $n \in \{1, 2, \ldots\}$. For a labelling $\sigma : dg'(\mu * \mathrm{rev}(\lambda)) \to \{1, 2, \ldots\}$ denote by $\sigma^\star$ the labelling of $\widehat{dg}(\mu * \mathrm{rev}(\lambda))$ given by*

- $\sigma^\star(u) = \sigma(u)$ *if* $u \in dg'(\mu * \mathrm{rev}(\lambda))$
- $\sigma^\star(j, 0) = j$ *for* $1 \leq j \leq \ell(\mu)$
- $\sigma^\star(j, 0) = \omega + j - \ell(\mu) - 1$ *for* $\ell(\mu) + 1 \leq j \leq \ell(\mu) + \ell(\lambda)$.

We naturally extend the definitions in Definition 2.2.1 of non-attacking, coinv*, and* Des *to labellings of the form $\sigma^\star$ which take values in $\{1, 2, \ldots\} \cup \{\omega + 1, \omega + 2, \ldots\}$. Define $\mathcal{L}(\mu|\lambda)$ to be the set of labellings $\sigma : dg'(\mu * \mathrm{rev}(\lambda)) \to \{1, 2, \ldots\}$ such that $\sigma^\star$ is non-attacking,* $\mathrm{coinv}(\sigma^\star) = 0$*, and* $\mathrm{Des}(\sigma^\star) = \emptyset$.

Example. *We will consider in this example two labellings of the type defined above for the pair $(2|3, 1)$. Our diagrams in this case are given as follows:*

$dg'(2, 1, 3) =$ 

$\widehat{dg}(2, 1, 3) =$ 

Consider the labellings $\sigma_1, \sigma_2 : dg'(2, 1, 3) \to \{1, 2, 3, 4\}$ and there corresponding labellings $\sigma_1^\star, \sigma_2^\star : \widehat{dg}(2, 1, 3) \to \{1, 2, 3, 4\}$ given by

$$\sigma_1 = \begin{array}{ccc} & & \boxed{1} \\ \boxed{1} & & \boxed{2} \\ \boxed{1} & \boxed{3} & \boxed{4} \end{array} \to \sigma_1^\star = \begin{array}{ccc} & & \boxed{1} \\ \boxed{1} & & \boxed{2} \\ \boxed{1} & \boxed{3} & \boxed{4} \\ \boxed{1} & \boxed{\omega} & \boxed{\omega+1} \end{array}$$

$$\sigma_2 = \begin{array}{ccc} & & \boxed{1} \\ \boxed{1} & & \boxed{3} \\ \boxed{1} & \boxed{2} & \boxed{4} \end{array} \to \sigma_2^\star = \begin{array}{ccc} & & \boxed{1} \\ \boxed{1} & & \boxed{3} \\ \boxed{1} & \boxed{2} & \boxed{4} \\ \boxed{1} & \boxed{\omega} & \boxed{\omega+1} \end{array}$$

Both σ_1, σ_2 *are non-attacking with* $\mathrm{maj}(\sigma_1^\star) = \mathrm{maj}(\sigma_2^\star) = 0$. *However,* $\mathrm{coinv}(\sigma_1^\star) = 0$ *whereas* $\mathrm{coinv}(\sigma_2^\star) \neq 0$. *To see this note that in the labelling* σ_2*, the boxes*

$$\begin{array}{ccc} & & \boxed{1} \\ \boxed{} & & \boxed{3} \\ \boxed{} & \boxed{2} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{array}$$

form a coinversion-triple in the sense of [19].

The almost symmetric Schur functions have the following monomial expansion.

Theorem 2.8.21. *For* $(\mu|\lambda) \in \Sigma$

$$s_{(\mu|\lambda)} = \sum_{\sigma \in \mathcal{L}(\mu|\lambda)} x^\sigma.$$

Proof. We start by noticing that from Proposition 2.8.18 we have

$$\begin{aligned} &s_{(\mu|\lambda)} \\ &= \lim_n \mathcal{K}_{\mu * 0^n * \mathrm{rev}(\lambda)} \\ &= \lim_n \sum_{\sigma \in \mathcal{L}(\mu * 0^n * \mathrm{rev}(\lambda))} x^\sigma. \end{aligned}$$

For all $n \geq 0$ there is an injection $\mathcal{L}(\mu * 0^n * \mathrm{rev}(\lambda)) \to \mathcal{L}(\mu * 0^{n+1} * \mathrm{rev}(\lambda))$ obtained as follows. Let $\sigma \in \mathcal{L}(\mu * 0^n * \mathrm{rev}(\lambda))$. Consider $\sigma' : dg'(\mu * 0^{n+1} * \mathrm{rev}(\lambda))$ given by

- $\sigma'(u) = \sigma(u)$ if $u \in dg'(\mu)$
- $\sigma'(i,j) = \sigma(i, j-1)$ if (i,j) lies in the $\mathrm{rev}(\lambda)$ component of $dg'(\mu * 0^{n+1} * \mathrm{rev}(\lambda))$.

In other words, we are simply aligning the $\mathrm{rev}(\lambda)$ parts of each of the diagrams $dg'(\mu * 0^{n+1} * \mathrm{rev}(\lambda))$ and $dg'(\mu * 0^n * \mathrm{rev}(\lambda))$ and copying the corresponding values of σ. It is easy to see that $\sigma' \in \mathcal{L}(\mu * 0^{n+1} * \mathrm{rev}(\lambda))$ and that the map $\sigma \to \sigma'$ is injective. Now we may consider the directed union

$$L := \bigcup_{n \geq 0} \mathcal{L}(\mu * 0^n * \mathrm{rev}(\lambda))$$

where we identify the image of $\mathcal{L}(\mu * 0^n * \mathrm{rev}(\lambda))$ in $\mathcal{L}(\mu * 0^{n+1} * \mathrm{rev}(\lambda))$ for all $n \geq 0$. Hence, we have

$$s_{(\mu|\lambda)} = \sum_{\sigma \in L} x^{\sigma}.$$

Lastly, we show that there exists a simple bijection $L \to \mathcal{L}(\mu|\lambda)$ such that $x^{\sigma} = x^{f(\sigma)}$ for all $\sigma \in L$. For $\sigma \in L$ say, $\sigma \in \mathcal{L}(\mu * 0^n * \mathrm{rev}(\lambda))$, we may define $\sigma'' : dg'(\mu * \mathrm{rev}(\lambda)) \to \{1, 2, \ldots\}$ by

- $\sigma''(u) = \sigma(u)$ if $u \in dg'(\mu)$
- $\sigma''(i,j) = \sigma(i+n, j)$ for (i,j) in the $\mathrm{rev}(\lambda)$ component of $dg'(\mu * \mathrm{rev}(\lambda))$.

Then $\sigma'' \in \mathcal{L}(\mu|\lambda)$ and the map $\sigma \to \sigma''$ is injective. We now show this map is also surjective. Let $\gamma \in \mathcal{L}(\mu|\lambda)$ and $N := \max\{\max_{u \in dg'(\mu * \mathrm{rev}(\lambda))} \sigma(u), \ell(\mu) + \ell(\lambda)\}$. Define $\sigma : \mu * 0^{N - \ell(\mu) - \ell(\lambda)} * \mathrm{rev}(\lambda) \to [N]$ similarly to before by copying the values of σ for both the μ and $\mathrm{rev}(\lambda)$ components of $dg'(\mu * \mathrm{rev}(\lambda))$ onto the corresponding components of $dg'(\mu * 0^{N-\ell(\mu)-\ell(\lambda)} * \mathrm{rev}(\lambda))$. Since N was chosen sufficiently large, $\sigma \in \mathcal{L}(\mu * 0^{N-\ell(\mu)-\ell(\lambda)} * \mathrm{rev}(\lambda))$. Now $\sigma'' = \gamma$ and $x^{\sigma''} = x^{\gamma}$. Therefore,

$$s_{(\mu|\lambda)} = \sum_{\sigma \in L} x^{\sigma} = \sum_{\sigma \in \mathcal{L}(\mu|\lambda)} x^{\sigma}.$$

$\square$

Since $s_{(\mu|\lambda)} \in \mathscr{P}(\ell(\mu))^+$ and the set $\{x^{\alpha} m_{\nu}[\mathfrak{X}_{\ell(\mu)}] \mid (\alpha|\nu), \ell(\alpha) \leq \ell(\mu)\}$ is a basis for $\mathscr{P}(k)^+$ we may consider the following definition.

Definition 2.8.22. *Define the* ***almost symmetric Kostka coefficients*** $K^{(\mu|\lambda)}_{(\alpha|\nu)}$ *to be the coefficients of the almost symmetric Schur functions into the monomial basis of* $\mathscr{P}(\ell(\mu))^+$, *i.e.*

$$s_{(\mu|\lambda)} = \sum_{\substack{(\alpha|\nu) \\ \ell(\alpha)\leq \ell(\mu)}} K^{(\mu|\lambda)}_{(\alpha|\nu)} x^{\alpha} m_{\nu}[\mathfrak{X}_{\ell(\mu)}].$$

If $\ell(\alpha) > \ell(\mu)$ *we simply set* $K^{(\mu|\lambda)}_{(\alpha|\nu)} = 0$.

Remark 19. *It is straightforward to check that for*

$$K^{(\emptyset|\lambda)}_{(\mu|\nu)} = \delta_{\mu,\emptyset} K_{\lambda,\nu}$$

meaning that the $K^{(\mu|\lambda)}_{(\alpha|\nu)}$ *generalize the classical Kostka coefficients* $K_{\lambda,\nu}$. *On the other extreme, we find that*

$$K^{(\mu|\emptyset)}_{(\alpha|\lambda)} = 0$$

unless $\lambda = \emptyset$ *in which case* $K^{(\mu|\emptyset)}_{(\alpha|\emptyset)}$ *is the multiplicity of the weight* α *in the Demazure character corresponding to* μ. *In either case, we see that the Kostka coefficients are non-negative.*

Theorem 2.8.23 (Positivity for almost symmetric Kostka coefficients).

$$K^{(\mu|\lambda)}_{(\alpha|\nu)} \in \mathbb{Z}_{\geq 0}$$

Proof. Let $(\mu|\lambda) \in \Sigma$. Using the explicit combinatorial formula in Theorem 2.8.21 we see that

$$s_{(\mu|\lambda)} = \sum_{\sigma\in\mathcal{L}(\mu|\lambda)} x^{\sigma}.$$

However, we know $s_{(\mu|\lambda)}$ is symmetric in the variables $x_{\ell(\mu)+1}, x_{\ell(\mu)+2}, \ldots$ so we may group terms by symmetry to find

$$\sum_{\sigma\in\mathcal{L}(\mu|\lambda)} x^{\sigma} = \sum_{\nu\in\mathbb{Y}} m_{\nu}[\mathfrak{X}_{\ell(\mu)}] \sum_{\sigma\in L_{\nu}(\mu|\lambda)} x_1^{|\sigma^{-1}(1)|}\cdots x_{\ell(\mu)}^{|\sigma^{-1}(\ell(\mu))|}$$

where $L_{\nu}(\mu|\lambda)$ is the set of labellings $\sigma : \mu * \mathrm{rev}(\lambda) \to [\mu + \ell(\nu)]$ such that $\sigma \in \mathcal{L}(\mu|\lambda)$ and for all $1 \leq i \leq \ell(\nu)$, $|\sigma^{-1}(\ell(\mu)+i)| = \nu_i$. Notice that $|L_{\nu}(\mu|\lambda)| < \infty$ for all ν.

We may further subdivide the sets $L_{\nu}(\mu|\lambda)$ now to account for the value of $x_1^{|\sigma^{-1}(1)|}\cdots x_{\ell(\mu)}^{|\sigma^{-1}(\ell(\mu))|}$. For $\ell(\alpha) \leq \ell(\mu)$ let $L_{(\alpha|\nu)}(\mu|\lambda)$ denote the set of all $\sigma \in L_{\nu}(\mu|\lambda)$ such that $|\sigma^{-1}(i)| = (\alpha * 0^{\ell(\mu)-\ell(\alpha)})_i$.

Then

$$s_{(\mu|\lambda)} = \sum_{(\alpha|\nu)} |L_{(\alpha|\nu)}(\mu|\lambda)| x^{\alpha} m_{\nu}[\mathfrak{X}_{\ell(\mu)}].$$

Thus

$$K^{(\mu|\lambda)}_{(\alpha|\nu)} = |L_{(\alpha|\nu)}(\mu|\lambda)| \in \mathbb{Z}_{\geq 0}.$$

$\square$

Remark 20. *Note that* $K^{(\mu|\lambda)}_{(\alpha|\nu)} = |L_{(\alpha|\nu)}(\mu|\lambda)|$ *gives a combinatorial formula for the almost symmetric Kostka coefficients. This formula generalizes the well known formula* $K_{\lambda,\mu} = |\operatorname{SSYT}(\lambda,\mu)|$ *where* $\operatorname{SSYT}(\lambda,\mu)$ *is the set of semistandard Young tableaux with shape* λ *and content* μ*.*

Example. *We saw before that*

$$s_{(2|3,1)} = x_1^3 s_{(2,1)}[\mathfrak{X}_1] + x_1^2 s_{(3,1)}[\mathfrak{X}_1]$$

which we can expand as

$$\begin{aligned} &s_{(2|3,1)} \\ &= x_1^3 m_{(2,1)}[\mathfrak{X}_1] + 2x_1^3 m_{(1,1,1)}[\mathfrak{X}_1] + x_1^2 m_{(3,1)}[\mathfrak{X}_1] + x_1^2 m_{(2,2)}[\mathfrak{X}_1] + 2x_1^2 m_{(2,1,1)}[\mathfrak{X}_1] + 3x_1^2 m_{(1,1,1,1)}[\mathfrak{X}_1]\end{aligned}$$

This gives that, for example, $K^{(2|3,1)}_{(2|1,1,1,1)} = 3$ *which corresponds to the 3 diagrams:*

$$\begin{array}{ccc} & & 2 \\ 1 & & 3 \\ 1 & 4 & 5 \\ 1 & \omega & \omega+1 \end{array} \qquad \begin{array}{ccc} & & 2 \\ 1 & & 4 \\ 1 & 3 & 5 \\ 1 & \omega & \omega+1 \end{array} \qquad \begin{array}{ccc} & & 3 \\ 1 & & 4 \\ 1 & 2 & 5 \\ 1 & \omega & \omega+1 \end{array}$$

Note that the above fillings in the $\operatorname{rev}(\lambda) = (1,3)$ *component are exactly, up to shifting indices, the semistandard Young tableuax of shape* $(3,1)$ *with content* $(1,1,1,1)$ *(and hence standard). This reflects that in the monomial-Schur expansion of* $s_{(2|3,1)}$ *there is one copy of* $x_1^2 s_{(3,1)}[\mathfrak{X}_1]$ *and* $K_{(3,1),(1,1,1,1)} = 3$.

We also have that $K^{(2|3,1)}_{(3|1,1,1,1)} = 2$ *which may be seen by computing the labellings in* $L_{(3|1,1,1,1)}(2|3,1)$ *directly:*

$$\begin{array}{|c|c|c|}\cline{3-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 2 \\ \cline{1-1}\cline{3-3} 1 & \multicolumn{1}{c|}{} & 3 \\ \hline 1 & 1 & 4 \\ \hline 1 & \omega & \omega+1 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\cline{3-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 \\ \cline{1-1}\cline{3-3} 1 & \multicolumn{1}{c|}{} & 2 \\ \hline 1 & 3 & 4 \\ \hline 1 & \omega & \omega+1 \\ \hline \end{array}$$

From the computation of $K^{(2|3,1)}_{(2|1,1,1,1)}$ *one might be tempted to guess that there is always a way to compute the almost symmetric Kostka numbers by classical Kostka numbers in some obvious manner. However, the example of* $K^{(2|3,1)}_{(3|1,1,1,1)}$ *shows that it is not always so simple. It particular, the filling*

$$\begin{array}{|c|c|c|}\cline{3-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 \\ \cline{1-1}\cline{3-3} 1 & \multicolumn{1}{c|}{} & 3 \\ \hline 1 & 2 & 4 \\ \hline 1 & \omega & \omega+1 \\ \hline \end{array}$$

has a reverse standard filling of $\mathrm{rev}(\lambda)$ *but is not in* $L_{(3|1,1,1,1)}(2|3,1)$ *since* $\mathrm{coinv} \neq 0$.

2.8.6. Representation-Theoretic Interpretation. In this section we are going to show that the monomial-Schur expansions of the $s_{(\mu|\lambda)}$ have non-negative coefficients using the Demazure character formula by relating $s_{(\mu|\lambda)}$ to the representation theory of parabolic subgroups of the GL_n .

Definition 2.8.24. *Define the scalars* $M^{(\mu|\lambda)}_{(\alpha|\nu)}$ *to be the coefficients of the expansion of the almost symmetric Schur functions into the monomial-Schur basis of* $\mathscr{P}(\ell(\mu))^+$*, i.e.*

$$s_{(\mu|\lambda)} = \sum_{\substack{(\alpha|\nu) \\ \ell(\alpha)\leq \ell(\mu)}} M^{(\mu|\lambda)}_{(\alpha|\nu)} x^{\alpha} s_{\nu}[\mathfrak{X}_{\ell(\mu)}].$$

If $\ell(\alpha) > \ell(\mu)$ *we simply set* $M^{(\mu|\lambda)}_{(\alpha|\nu)} = 0$.

We wish to show that $M^{(\mu|\lambda)}_{(\alpha|\nu)} \in \mathbb{Z}_{\geq 0}$ but in order to do so we must first review some representation theory in type GL .

Definition 2.8.25. *For $n \geq 1$ define GL_n to be the group of invertible $n \times n$ matrices over $\mathbb{C}$. Let B_n denote the Borel subgroup of upper-triangular matrices in GL_n and let H_n denote the group of diagonal matrices in GL_n. For $0 \leq k \leq n$ denote by $\mathrm{P}_n(k)$ the group of $M \in \mathrm{GL}_n$ such that $M_{ij} = 0$ if either $1 \leq j < i \leq k$ or $j \leq k \leq i-1$. Lastly, let $\mathrm{L}_n(k) = \mathrm{H}_k \times \mathrm{GL}_{n-k} \subset \mathrm{GL}_n$ under the block diagonal embedding $\mathrm{GL}_k \times \mathrm{GL}_{n-k} \to \mathrm{GL}_n$. Let $\mathfrak{b}_n$ denote the Lie algebra of B_n i.e. the set of upper triangular $n \times n$ matrices over $\mathbb{C}$ with the usual commutator product. Let $\mathcal{U}(\mathfrak{b}_n)$ denote the **universal enveloping algebra** of $\mathfrak{b}_n$.*

Remark 21. *Note that*

$$\mathrm{H}_n \subset \mathrm{B}_n \subset \mathrm{GL}_n$$

and for all $0 \leq k \leq n$

$$\mathrm{H}_n \subset \mathrm{B}_n \subset \mathrm{P}_n(k).$$

*Following terminology standard to Lie theory, B_n and H_n are respectively **Borel** and **Cartan** subgroups of GL_n. Further, the group $\mathrm{P}_n(k)$ is a **parabolic** subgroup of GL_n with **Levi** subgroup $\mathrm{L}_n(k)$. Parabolic subgroups of GL_n correspond to pairs of choices of a Borel subgroup and a subset of the set of simple positive roots $\{e_i - e_{i+1} | 1 \leq i \leq n-1\}$ in type A_{n-1}. The group $\mathrm{P}_n(k)$ corresponds to the Borel subgroup B_n and the subset $\{e_i - e_{i+1} | k+1 \leq i \leq n-1\}$. From standard results in the the theory of algebraic groups we know that GL_n is **reductive** meaning that any finite dimensional polynomial (rational) representation of GL_n decomposes as direct sum of irreducible polynomial (rational) representations. Importantly, if V is a polynomial representation of GL_n then H_n acts semi-simply with simultaneous eigenvectors $v \in V$ having eigenvalues indexed by $a \in \mathbb{Z}^n_{\geq 0}$ i.e.*

$$\begin{pmatrix} z_1 & 0 & \dots & 0 \\ 0 & z_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & z_n \end{pmatrix} v = z_1^{a_1} \cdots z_n^{a_n} v.$$

*Thus the H_n-**weights** of polynomials representations of GL_n are indexed by $\mathbb{Z}^n_{\geq 0}$.*

Definition 2.8.26. *Given a finite dimensional polynomial representation V of H_n we will denote by $\mathrm{char}(V) \in \mathbb{Z}[x_1, \ldots, x_n]$ the* ***formal character*** *of V as*

$$\mathrm{char}(V) = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} \dim \mathrm{Hom}_{\mathrm{H}_n}(\alpha, V) x^\alpha.$$

Remark 22. *If V is any polynomial representation of GL_n then $\mathrm{char}(V)$ is a symmetric polynomial since $\mathrm{char}(V)$ must be invariant under the action of the* ***Weyl group*** *of GL_n i.e. $\mathfrak{S}_n$. If W is another polynomial representation then we have that*

- $\mathrm{char}(V \oplus W) = \mathrm{char}(V) + \mathrm{char}(W)$
- $\mathrm{char}(V \otimes W) = \mathrm{char}(V)\,\mathrm{char}(W)$.

Thus we may interpret the map $V \to \mathrm{char}(V)$ as giving a ring homomorphism from the virtual polynomial representation ring of GL_n to the symmetric polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]^{\mathfrak{S}_n}$. This map is an isomorphism.

It follows from standard representation theory of reductive algebraic groups over $\mathbb{C}$ that we have the following description of the irreducible representations of GL_n .

Theorem 2.8.27. *The irreducible polynomial representations of GL_n are indexed by* ***dominant integral weights*** *$\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$ i.e. $\lambda_1 \geq \ldots \geq \lambda_n$. These representations $\mathcal{V}^\lambda$ have the following properties:*

- $\mathrm{char}(\mathcal{V}^\lambda) = s_\lambda(x_1, \ldots, x_n)$*; where we truncate λ when necessary to obtain a partition*
- *There exists a unique* ***highest weight*** *in $\mathcal{V}^\lambda$. Namely, there exists a unique vector $v \in \mathcal{V}^\lambda$ (up to scaling) such that v is a H_n-weight vector with weight λ and $\mathcal{U}(\mathfrak{b}_n)v = 0$.*
- *For all $\sigma \in \mathfrak{S}_n$*

$$\dim \mathrm{Hom}_{\mathrm{H}_n}(\sigma(\lambda), \mathcal{V}^\lambda) = 1.$$

Definition 2.8.28. *Given a dominant integral weight $\lambda \in \mathbb{Z}^n_{\geq 0}$ and $\sigma \in \mathfrak{S}_n$ define the* ***Demazure module*** *$\mathcal{V}^\lambda_{\sigma(\lambda)}$ to be the B_n-module*

$$\mathcal{V}^\lambda_\sigma := \mathcal{U}(\mathfrak{b}_n)v$$

where $v \in \mathcal{V}^\lambda_\sigma$ is any weight vector with weight $\sigma(\lambda)$.

REMARK 23. *Notice that the Demazure module* $\mathcal{V}_\sigma^\lambda$ *is only well defined up to the vector* $\sigma(\lambda)$*. Therefore, we may instead index these modules as*

$$\mathcal{V}_{\sigma(\lambda)}^\lambda := \mathcal{V}_\sigma^\lambda.$$

THEOREM 2.8.29. *(Demazure Character Formula) [1] Given a dominant integral weight* λ *and* $\sigma \in \mathfrak{S}_n$

$$\mathrm{char}(\mathcal{V}_{\sigma(\lambda)}^\lambda) = \mathcal{K}_{\sigma(\lambda)}.$$

REMARK 24. *The Demazure character formula in full generality gives a similar formula to the above for all semisimple Lie types. The first complete proof of the Demazure character formula was given by Andersen [1] by realizing the Demazure modules as spaces of sections of vector bundles of Schubert varieties and showing that the singularities of Schubert varieties are rational.*

DEFINITION 2.8.30. *Let* $(\mu|\lambda) \in \Sigma$*. For all* $n \geq \ell(\mu) + \ell(\lambda)$ *define*

$$\mathcal{V}^{(n)}(\mu|\lambda) := \mathcal{V}_{\mu * 0^{n-\ell(\mu)-\ell(\lambda)} * \mathrm{rev}(\lambda)}^{\mathrm{sort}(\mu*\lambda)*0^{n-\ell(\mathrm{sort}(\mu*\lambda))}}.$$

If $\alpha \in \mathrm{Comp}^{red}$ *and* $\ell(\alpha) \leq k$ *we will write* $\chi^{(n)}(\alpha|\lambda)$ *for the irreducible* $\mathrm{L}_n(k) = \mathrm{H}_k \times \mathrm{GL}_{n-k}$*-module given by*

$$\chi^{(n)}(\alpha|\lambda) := (\alpha * 0^{k-\ell(\alpha)}) \otimes \mathcal{V}^{\lambda * 0^{n-k-\ell(\lambda)}}$$

where we are using the shorthand $\alpha * 0^{k-\ell(\alpha)}$ *to represent the corresponding* 1*-dimensional representation of* H_k*.*

We may relate the almost symmetric Schur functions $s_{(\mu|\lambda)}$ to Demazure characters via key polynomials directly from the following simple lemma.

LEMMA 2.8.31. *Let* $(\mu|\lambda) \in \Sigma$*. Then*

$$s_{(\mu|\lambda)} = \lim_n \mathrm{char}\, \mathcal{V}^{(n)}(\mu|\lambda).$$

PROOF. In Proposition 2.8.18 we saw that

$$s_{(\mu|\lambda)} = \lim_n \mathcal{K}_{\mu*0^n*\mathrm{rev}(\lambda)} = \lim_n \mathcal{K}_{\mu*0^{n-\ell(\mu)-\ell(\lambda)}*\mathrm{rev}(\lambda)}.$$

Using the Demazure character formula we see that

$$\mathcal{K}_{\mu*0^{n-\ell(\mu)-\ell(\lambda)}*\mathrm{rev}(\lambda)} = \mathrm{char}\left(\mathcal{V}^{\mathrm{sort}(\mu*\lambda)*0^{n-\ell(\mathrm{sort}(\mu*\lambda))}}_{\mu*0^{n-\ell(\mu)-\ell(\lambda)}*\mathrm{rev}(\lambda)}\right)$$

so the result follows. $\square$

We require the following simple lemma.

Lemma 2.8.32. *Suppose λ is an integral dominant weight of GL_n and $\alpha * \beta = \sigma(\lambda)$ for some $\sigma \in \mathfrak{S}_n$ with β weakly decreasing. Then $\mathcal{V}^\lambda_{\alpha*\beta}$ is a $\mathrm{P}_n(\ell(\alpha))$ submodule of $\mathcal{V}^\lambda$.*

Proof. Let $k = \ell(\alpha)$. Since $\mathrm{P}_n(k)$ is the semidirect product of B_n and $\mathrm{L}_n(k)$ we only need to show that $\mathcal{V}^\lambda_{\alpha*\beta}$ is preserved under action by both B_n and $\mathrm{L}_n(k)$. Since $\mathcal{V}^\lambda_{\alpha*\beta}$ is by definition a B_n-module it suffices to show that $\mathcal{V}^\lambda_{\alpha*\beta}$ is preserved under the action of $Id_k \times \mathrm{GL}_{n-k}$.

We will proceed by induction using raising and lowering operators. To start fix $v_0 \in \mathcal{V}^\lambda_{\alpha*\beta}$ to be a nonzero vector with weight $\alpha * \beta$. Then for all $k+1 \le i < j \le n$, since β is weakly decreasing, $E_{ji}v = 0 \in \mathcal{V}^\lambda_{\alpha*\beta}$. Suppose now that $v_0, v_1, \ldots, v_{m+1}$ is a sequence of weight vectors in $\mathcal{V}^\lambda_{\alpha*\beta}$ with $v_{r+1} = E_{i_r j_r} v_r$ for all $0 \le r \le m$ for some $1 \le i_r < j_r \le n$ and that

$$E_{ji}v_r \in \mathcal{V}^\lambda_{\alpha*\beta}$$

for all $k+1 \le i < j \le n$ and $0 \le r \le m$. Note that any weight vector in $\mathcal{V}^\lambda_{\alpha*\beta}$ may be obtained using such a chain. Now fix some $k+1 \le i < j \le n$. We see that

$$\begin{aligned}
&E_{ji}v_{m+1}\\
&= E_{ji}E_{i_m j_m} v_m\\
&= \left(E_{i_m j_m}E_{ji} + [E_{ji}, E_{i_m j_m}]\right) v_m\\
&= E_{i_m j_m}\left(E_{ji}v_m\right) + [E_{ji}, E_{i_m j_m}]v_m.
\end{aligned}$$

By assumption $E_{ji}v_m \in \mathcal{V}^\lambda_{\alpha*\beta}$ so that, since $i_m < j_m$, $E_{i_m j_m}\left(E_{ji}v_m\right) \in \mathcal{V}^\lambda_{\alpha*\beta}$. Therefore, it suffices to show that $[E_{ji}, E_{i_m j_m}]v_m \in \mathcal{V}^\lambda_{\alpha*\beta}$.

There are a few cases we must consider. First, assume $i = i_m$. Then

$$[E_{ji}, E_{i_m j_m}]v_m = (E_{jj_m} - \delta_{j,j_m} E_{ii})\, v_m = E_{jj_m} v_m - cv_m$$

for some scalar c. If $j \leq j_m$ then $E_{jj_m} v_m \in \mathcal{V}^{\lambda}_{\alpha*\beta}$ automatically. If instead $j > j_m$, then $k+1 \leq i = i_m < j_m$ so $E_{jj_m} v_m \in \mathcal{V}^{\lambda}_{\alpha*\beta}$ by the inductive hypothesis. Either way $[E_{ji}, E_{i_m j_m}]v_m \in \mathcal{V}^{\lambda}_{\alpha*\beta}$.

Now assume $j = j_m$. Then

$$[E_{ji}, E_{i_m j_m}]v_m = (\delta_{i,i_m} E_{jj} - E_{i_m i})\, v_m = cv_m - E_{i_m i} v_m$$

for some scalar c. If $i_m \leq i$ then $E_{i_m i} v_m \in \mathcal{V}^{\lambda}_{\alpha*\beta}$ automatically. If $i_m > i$ then, since $k+1 \leq i$, $E_{i_m i} v_m \in \mathcal{V}^{\lambda}_{\alpha*\beta}$ by the inductive hypothesis. In either case, $[E_{ji}, E_{i_m j_m}]v_m \in \mathcal{V}^{\lambda}_{\alpha*\beta}$. Lastly, if $i \neq i_m$ and $j \neq j_m$ then $[E_{ji}, E_{i_m j_m}] = 0$ so $[E_{ji}, E_{i_m j_m}]v_m = 0 \in \mathcal{V}^{\lambda}_{\alpha*\beta}$ trivially. $\square$

Since the group $\mathrm{L}_n(k)$ is reductive we obtain the following representation theoretic interpretation for the coefficients $M^{(\mu|\lambda)}_{(\alpha|\gamma)}$.

THEOREM 2.8.33. *Let $(\mu|\lambda), (\alpha|\gamma) \in \Sigma$. For all sufficiently large n*

$$M^{(\mu|\lambda)}_{(\alpha|\gamma)} = \dim \mathrm{Hom}_{\mathrm{L}_n(\ell(\mu))} \left(\chi^{(n)}(\alpha|\nu), \mathcal{V}^{(n)}(\mu|\lambda) \right) \in \mathbb{Z}_{\geq 0}.$$

PROOF. From Lemma 2.8.31 and the definition of the coefficients $M^{(\mu|\lambda)}_{(\alpha|\gamma)}$ we see that for n sufficiently large

$$\sum_{\substack{(\alpha|\nu) \\ \ell(\alpha) \leq \ell(\mu)}} M^{(\mu|\lambda)}_{(\alpha|\nu)} x^{\alpha} s_{\nu}[x_{\ell(\mu)+1} + \ldots + x_n] = \mathrm{char}\, \mathcal{V}^{(n)}(\mu|\lambda).$$

From Lemma 2.8.32 we may decompose $\mathcal{V}^{(n)}(\mu|\lambda)$ into irreducible $\mathrm{L}_n(\ell(\mu))$ submodules as

$$\mathcal{V}^{(n)}(\mu|\lambda) = \bigoplus_{\substack{(\alpha|\nu) \\ \ell(\alpha) \leq \ell(\mu)}} \chi^{(n)}(\alpha|\nu)^{\oplus d^{(n)}_{(\alpha|\nu)}}$$

where $d^{(n)}_{(\alpha|\nu)} = \dim \mathrm{Hom}_{\mathrm{L}_n(\ell(\mu))} \left(\chi^{(n)}(\alpha|\nu), \mathcal{V}^{(n)}(\mu|\lambda) \right)$. Notice that

$$\mathrm{char}\, \chi^{(n)}(\alpha|\nu) = x^{\alpha} s_{\nu}[x_{\ell(\mu)+1} + \ldots + x_n].$$

Putting this together we find that for all n sufficiently large

$$\begin{aligned}
&\sum_{\substack{(\alpha|\nu)\\ \ell(\alpha)\le\ell(\mu)}} M^{(\mu|\lambda)}_{(\alpha|\nu)} x^\alpha s_\nu[x_{\ell(\mu)+1}+\ldots+x_n]\\
&= \operatorname{char}\mathcal{V}^{(n)}(\mu|\lambda)\\
&= \operatorname{char} \bigoplus_{\substack{(\alpha|\nu)\\ \ell(\alpha)\le\ell(\mu)}} \chi^{(n)}(\alpha|\nu)^{\oplus d^{(n)}_{(\alpha|\nu)}}\\
&= \sum_{\substack{(\alpha|\nu)\\ \ell(\alpha)\le\ell(\mu)}} \operatorname{char}\chi^{(n)}(\alpha|\nu)^{\oplus d^{(n)}_{(\alpha|\nu)}}\\
&= \sum_{\substack{(\alpha|\nu)\\ \ell(\alpha)\le\ell(\mu)}} d^{(n)}_{(\alpha|\nu)} \operatorname{char}\chi^{(n)}(\alpha|\nu)\\
&= \sum_{\substack{(\alpha|\nu)\\ \ell(\alpha)\le\ell(\mu)}} \dim\operatorname{Hom}_{\mathrm{L}_n(\ell(\mu))}\left(\chi^{(n)}(\alpha|\nu),\mathcal{V}^{(n)}(\mu|\lambda)\right) x^\alpha s_\nu[x_{\ell(\mu)+1}+\ldots+x_n].
\end{aligned}$$

Lastly, as the terms $x^\alpha s_\nu[x_{\ell(\mu)+1}+\ldots+x_n]$ for $\ell(\alpha)\le\ell(\mu)$ are linearly independent we may compare coefficients to obtain the result. □

As a consequence of the above theorem we obtain a second proof of Theorem 2.8.23.

Corollary 2.8.34. *Let $(\mu|\lambda),(\alpha|\gamma)\in\Sigma$. For all sufficiently large n*

$$|L_{(\alpha|\nu)}(\mu|\lambda)| = \sum_{\gamma\in\mathbb{Y}} |\operatorname{SSYT}(\gamma,\nu)| \times \dim\operatorname{Hom}_{\mathrm{L}_n(\ell(\mu))}\left(\chi^{(n)}(\alpha|\gamma),\mathcal{V}^{(n)}(\mu|\lambda)\right).$$

Proof. First, we expand the Schur functions $s_\gamma[\mathfrak{X}_{\ell(\mu)}]$ into the monomial symmetric function basis:

$$\begin{aligned}
s_{(\mu|\lambda)} &= \sum_{\substack{(\alpha|\gamma)\\ \ell(\alpha)\leq\ell(\mu)}} M_{(\alpha|\gamma)}^{(\mu|\lambda)} x^\alpha s_\gamma[\mathfrak{X}_{\ell(\mu)}] \\
&= \sum_{\substack{(\alpha|\gamma)\\ \ell(\alpha)\leq\ell(\mu)}} M_{(\alpha|\gamma)}^{(\mu|\lambda)} x^\alpha \sum_{\nu\in\mathbb{Y}} K_{\gamma,\nu} m_\nu[\mathfrak{X}_{\ell(\mu)}] \\
&= \sum_{\substack{(\alpha|\nu)\\ \ell(\alpha)\leq\ell(\mu)}} \left(\sum_{\gamma\in\mathbb{Y}} K_{\gamma,\nu} M_{(\alpha|\gamma)}^{(\mu|\lambda)}\right) x^\alpha m_\nu[\mathfrak{X}_{\ell(\mu)}].
\end{aligned}$$

From here we find

$$K_{(\alpha|\nu)}^{(\mu|\lambda)} = \sum_{\gamma\in\mathbb{Y}} K_{\gamma,\nu} M_{(\alpha|\gamma)}^{(\mu|\lambda)}.$$

Lastly, by combining the formula $K_{\gamma,\nu} = |\operatorname{SSYT}(\gamma,\nu)|$, the expression for $M_{(\alpha|\gamma)}^{(\mu|\lambda)}$ in Theorem 2.8.33, and the equation $K_{(\alpha|\nu)}^{(\mu|\lambda)} = |L_{(\alpha|\nu)}(\mu|\lambda)|$ from the proof of Theorem 2.8.23 we conclude the desired result. $\square$

Remark 25. *The* ***inverse Kostka coefficients*** $K_{\gamma,\lambda}^{(-1)}$ *are given by*

$$m_\gamma = \sum_\lambda K_{\gamma,\lambda}^{(-1)} s_\lambda.$$

Notice that

$$\delta_{\gamma,\lambda} = \sum_\mu K_{\gamma,\mu}^{(-1)} K_{\mu,\lambda}.$$

The inverse Kostka coefficients are known from the work of Eǧecioľǧu-Remmel [13] to have an explicit combinatorial formula involving ***signed rim hook tabloids*** *which we will not detail here. In the same way we obtained Corollary 2.8.34 we may instead expand each m_λ into the Schur basis to obtain for all sufficiently large n*

$$\dim \operatorname{Hom}_{\mathrm{L}_n(\ell(\mu))}\left(\chi^{(n)}(\alpha|\nu), \mathcal{V}^{(n)}(\mu|\lambda)\right) = \sum_{\gamma\in\mathbb{Y}} K_{\gamma,\nu}^{(-1)} \times |L_{(\alpha|\gamma)}(\mu|\lambda)|.$$

*Using the combinatorial formula for the $K^{(-1)}_{\gamma,\lambda}$ we see that this gives a purely combinatorial formula. However, this is **not** a non-negative combinatorial formula as the inverse Kostka coefficients are often negative.*

By carefully taking direct limits of groups and their corresponding modules in the right way it is possible to simplify the expression in Theorem 2.8.33:

$$M^{(\mu|\lambda)}_{(\alpha|\gamma)} = \dim \mathrm{Hom}_{\mathrm{L}_\infty(\ell(\mu))}\left(\chi^{(\infty)}(\alpha|\nu), \mathcal{V}^{(\infty)}(\mu|\lambda)\right).$$

CHAPTER 3

Murnaghan-Type Representations for the Elliptic Hall Algebra

3.1. Introduction

The space of symmetric functions, Λ, is a central object in algebraic combinatorics deeply connecting the fields of representation theory, geometry, and combinatorics. In his influential paper [29], Macdonald introduced a special basis $P_\lambda[X;q,t]$ for Λ over $\mathbb{Q}(q,t)$ simultaneously generalizing many other important and well-studied symmetric function bases like the Schur functions $s_\lambda[X]$. These symmetric functions $P_\lambda[X;q,t]$, called the symmetric Macdonald functions, exhibit many striking combinatorial properties and can be defined as the eigenvectors of a certain operator $\Delta : \Lambda \to \Lambda$ called the Macdonald operator constructed using polynomial difference operators. It was discovered through the works of Bergeron, Garsia, Haiman, Tesler, and many others [23] [4] [5] that variants of the symmetric Macdonald functions called the modified Macdonald functions $\widetilde{H}_\lambda[X;q,t]$ have deep ties to the geometry of the Hilbert schemes $\mathrm{Hilb}^n(\mathbb{C}^2)$. On the side of representation theory, it was shown first in full generality by Cherednik [9] that one can recover the symmetric Macdonald functions by considering the representation theory of certain algebras called the spherical double affine Hecke algebras (DAHAs) in type GL_n.

The positive elliptic Hall algebra (EHA), $\mathscr{E}^+$, was introduced by Burban and Schiffmann [6] as the positive subalgebra of the Hall algebra of the category of coherent sheaves on an elliptic curve over a finite field. This algebra has connections to many areas of mathematics including, most importantly for the present situation, to Macdonald theory. In [34], Schiffmann and Vasserot realize $\mathscr{E}^+$ as a stable limit of the positive spherical DAHAs in type GL_n. They show further that there is a natural action of $\mathscr{E}^+$ on Λ aligning with the spherical DAHA representations originally considered by Cherednik. In particular, the action of $P_{0,1} \in \mathscr{E}^+$ gives the Macdonald operator Δ. The action of $\mathscr{E}^+$ on Λ can be realized as the action of certain generalized convolution operators on the torus equivariant K-theory of the schemes $\mathrm{Hilb}^n(\mathbb{C}^2)$.

Dunkl and Luque in [12] introduced symmetric and non-symmetric vector-valued (vv.) Macdonald polynomials. The term vector-valued here refers to polynomial-like objects of the form $\sum_{\alpha} c_{\alpha} X^{\alpha} \otimes v_{\alpha}$ for some scalars c_{α}, monomials X^{α}, and vectors v_{α} lying in some $\mathbb{Q}(q,t)$-vector space. The non-symmetric vv. Macdonald polynomials are distinguished bases for certain DAHA representations built from the irreducible representations of the finite Hecke algebras in type A. These DAHA representations are indexed by Young diagrams and exhibit interesting combinatorial properties relating to periodic Young tableaux. The symmetric vv. Macdonald polynomials are distinguished bases for the spherical (i.e. Hecke-invariant) subspaces of these DAHA representations. Naturally, the spherical DAHA acts on this spherical subspace with the special element $Y_1 + \ldots + Y_n$ of spherical DAHA acting diagonally on the symmetric vv. Macdonald polynomials.

Dunkl and Luque in [12] (and in later work of Colmenarejo, Dunkl, and Luque [10] and Dunkl [11]) only consider the finite rank non-symmetric and symmetric vv. Macdonald polynomials. It is natural to ask if there is an infinite-rank stable-limit construction using the symmetric vv. Macdonald polynomials to give generalized symmetric Macdonald functions and an associated representation of the positive elliptic Hall algebra $\mathscr{E}^{+}$. In this chapter, we will describe such a construction (Thm. 4.2.12). We will obtain a new family of graded $\mathscr{E}^{+}$-representations $\widetilde{W}_{\lambda}$ indexed by Young diagrams λ and a natural generalization of the symmetric Macdonald functions $\mathfrak{P}_{T}$ indexed by certain labellings of infinite Young diagrams built as limits of the symmetric vv. Macdonald polynomials. For combinatorial reasons there is essentially a unique natural way to obtain this construction. For any λ we will consider the increasing chains of Young diagrams $\lambda^{(n)} = (n - |\lambda|, \lambda)$ for $n \geq |\lambda| + \lambda_1$ to build the representations $\widetilde{W}_{\lambda}$. These special sequences of Young diagrams are central to Murnaghan's theorem [32] regarding the reduced Kronecker coefficients. As such we refer to the $\mathscr{E}^{+}$-representations $\widetilde{W}_{\lambda}$ as Murnaghan-type. For $\lambda = \emptyset$ we recover the $\mathscr{E}^{+}$ action on Λ and the symmetric Macdonald functions $P_{\mu}[X;q,t]$. We will obtain a Pieri rule for the action of the multiplication operators $e_r^{\bullet}$ on the generalized symmetric Macdonald function basis $\mathfrak{P}_{T}$. After studying the particular case of the e_1-Pieri coefficients we will show that the modules $\widetilde{W}_{\lambda}$ are cyclic generated by their unique elements of minimal degree $\mathfrak{P}_{T_{\lambda}^{min}}$. Lastly, we will show that these Murnaghan-type representations $\widetilde{W}_{\lambda}$ are mutually non-isomorphic.

The existence of these representations of the elliptic Hall algebra raises many questions about possible new relations between Macdonald theory and geometry. Other authors have constructed families of $\mathscr{E}^+$-representations [14] [15]. Although there should exist a relationship between the Murnaghan-type representations $\widetilde{W}_\lambda$ and those of other authors, the construction in this thesis appears to be distinct from prior $\mathscr{E}^+$-module constructions.

For technical reasons (regarding the misalignment of the spectrum of the Cherednik operators Y_i) we will need to reprove many of the results of Dunkl and Luque in [12] using a re-oriented version of the Cherednik operators θ_i. Since the elements θ_i are not uniformly conjugate to the Y_i on the vector-valued polynomial spaces V_λ, we are not immediately able to use the results of Dunkl and Luque. However, many of these results follow from very similar proofs in this chapter. This alternative choice of conventions greatly assists during the construction of the generalized Macdonald functions $\mathfrak{P}_T$. The θ_i satisfy additional stability properties which the Y_i fail to satisfy. The combinatorics underpinning the non-symmetric vv. Macdonald polynomials originally defined by Dunkl and Luque is also nearly identical but with reversed orientation to the conventions appearing in this chapter.

3.1.1. Overview. Here we will give a brief overview of this chapter. First, in Section 3.2 we will introduce and review relevant combinatorial definitions and notations. In Section 3.3 we will reprove many of the results of Dunkl-Luque but for the re-oriented Cherednik operators including describing the non-symmetric v.v. Macdonald polynomials F_τ and their associated Knop-Sahi relations (Prop. 3.3.5). We define (Def. 3.3.12) the DAHA modules $V_{\lambda^{(n)}}$ and connecting maps $\Phi_\lambda^{(n)} : V_{\lambda^{(n+1)}} \to V_{\lambda^{(n)}}$ which will be used in the stable-limit process. Next in Section 3.4, we describe the spherical subspaces $W_\lambda^{(n)}$ of Hecke invariants of $V_\lambda^{(n)}$ and the symmetric v.v. Macdonald polynomials P_T including an explicit expansion of the P_T into the F_τ (Cor. 3.4.5). We will use the connecting maps to define the stable-limit spaces $\widetilde{W}_\lambda$ and show in Thm. 3.4.13 that they possess a graded action of $\mathscr{E}^+$ having a distinguished basis of generalized symmetric Macdonald functions $\mathfrak{P}_T$. In Section 3.5 we will obtain a Pieri formula (Cor. 3.5.9) for the action of $e_r^\bullet$ on the generalized Macdonald functions $\mathfrak{P}$. Lastly in Section 3.6, we will look at an interesting family of (q,t) product-series identities (Thm. 3.6.12) which follow naturally from the algebra/combinatorics in the prior sections of the chapter.

3.2. Diagrams and Labellings

We start with a description of many of the combinatorial objects which we will need for the remainder of this chapter.

Definition 3.2.1. *Denote by $\mathbb{Y}$ the set of all partitions. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ we set $\ell(\lambda) := r$ and $|\lambda| := \lambda_1 + \ldots + \lambda_r$. For $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Y}$ and $n \geq n_\lambda := |\lambda| + \lambda_1$ we set $\lambda^{(n)} := (n - |\lambda|, \lambda_1, \ldots, \lambda_r)$. We will identify partitions as defined above with **Young diagrams** of the corresponding shape in English notation i.e. justified up and to the left.*

Fix a partition λ with $|\lambda| = n$. We will require each of the following combinatorial constructions for types of labelling of the Young diagram λ. If a diagram λ appears as the domain of a labelling function then we are referring to the set of boxes of λ as the domain.

- *A non-negative **reverse Young tableau** $\mathrm{RYT}_{\geq 0}(\lambda)$ is a labelling $T : \lambda \to \mathbb{Z}_{\geq 0}$ which is weakly decreasing along rows and columns.*
- *A non-negative **reverse semi-standard Young tableau** $\mathrm{RSSYT}_{\geq 0}(\lambda)$ is a labelling $T : \lambda \to \mathbb{Z}_{\geq 0}$ which is weakly decreasing along rows and strictly decreasing along columns.*
- *A **standard Young tableau** $\mathrm{SYT}(\lambda)$ is a labelling $\tau : \lambda \to \{1, \ldots, n\}$ which is strictly increasing along rows and columns.*
- *A non-negative **periodic standard Young tableau** $\mathrm{PSYT}_{\geq 0}(\lambda)$ is a labelling $\tau : \lambda \to \{jq^b : 1 \leq j \leq n, b \geq 0\}$ in which each $1 \leq j \leq n$ occurs in exactly one box of λ and where the labelling is strictly increasing along rows and columns. Here we order the formal products jq^m by $jq^m < kq^\ell$ if $m > \ell$ or in the case that $m = \ell$ we have $j < k$. Note that $SYT(\lambda) \subset \mathrm{PSYT}_{\geq 0}(\lambda)$.*

Define $\tau_\lambda^{rs}, \tau_\lambda^{cs} \in \mathrm{SYT}(\lambda)$ to be the row-standard and column-standard labellings of λ respectively.

Example.
$$\begin{array}{|c|c|c|c|c|c|}
\hline
17q^7 & 15q^5 & 16q^5 & 11q^3 & 7q^1 & 2q^0 \\
\hline
14q^6 & 12q^4 & 13q^4 & 9q^2 & 8q^0 \\
\cline{1-5}
10q^2 & 4q^1 & 5q^1 & 6q^1 \\
\cline{1-4}
3q^1 & 1q^0 \\
\cline{1-2}
\end{array} \in \mathrm{PSYT}_{\geq 0}(6,5,4,2)$$

Definition 3.2.2. *Given a box, $\Box$, in a Young diagram λ we define the* ***content*** *of $\Box$ as $c(\Box) := a - b$ where $\Box = (a,b)$ as drawn in the $\mathbb{N} \times \mathbb{N}$ grid (English notation). Let $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda)$ and $1 \leq i \leq n$. Whenever $\tau(\Box) = iq^b$ for some box $\Box \in \lambda$ we will write*

- $c_\tau(i) := c(\Box)$
- $w_\tau(i) := b$.

Set $w_\tau := (w_\tau(1), \ldots, w_\tau(n)) \in \mathbb{Z}^n_{\geq 0}$. Let $1 \leq j \leq n-1$ and suppose that for some boxes $\Box_1, \Box_2 \in \lambda$ that $\tau(\Box_1) = jq^m$ and $\tau(\Box_2) = (j+1)q^\ell$. Let τ' be the labelling defined by $\tau'(\Box_1) = (j+1)q^m$, $\tau'(\Box_2) = jq^\ell$, and $\tau'(\Box) = \tau(\Box)$ for $\Box \in \lambda \backslash \{\Box_1, \Box_2\}$. If $\tau' \in \mathrm{PSYT}_{\geq 0}(\lambda)$ then we write $s_j(\tau) := \tau'$. Let $\Psi(\tau) \in \mathrm{PSYT}_{\geq 0}(\lambda)$ be the labelling defined by whenever $\tau(\Box) = kq^a$ then either $\Psi(\tau)(\Box) = (k-1)q^a$ when $k \geq 2$ or $\Psi(\tau)(\Box) = nq^{a+1}$ when $k = 1$.

We give the set $\mathrm{PSYT}_{\geq 0}(\lambda)$ a partial order $\geq$ defined by the following cover relations.

- *For all $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda)$, $\Psi(\tau) > \tau$.*
- *If $w_\tau(i) < w_\tau(i+1)$ then $s_i(\tau) > \tau$.*
- *If $w_\tau(i) = w_\tau(i+1)$ and $c_\tau(i) - c_\tau(i+1) > 1$ then $s_i(\tau) > \tau$.*

Define the map $\mathfrak{p}_\lambda : \mathrm{PSYT}_{\geq 0}(\lambda) \to \mathrm{RYT}_{\geq 0}(\lambda)$ by $\mathfrak{p}_\lambda(\tau)(\Box) = b$ whenever $\tau(\Box) = iq^b$. We will write $\mathrm{PSYT}_{\geq 0}(\lambda; T)$ for the set of all $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda)$ with $\mathfrak{p}_\lambda(\tau) = T \in \mathrm{RYT}_{\geq 0}(\lambda)$.

Example.
$$\Psi\left(\begin{array}{cccccc} 1q^7 & 3q^5 & 5q^5 & 8q^2 & 12q^1 & 17q^0 \\ 2q^6 & 4q^5 & 6q^5 & 14q^0 & 16q^0 & \\ 7q^2 & 10q^1 & 11q^1 & 15q^0 & & \\ 9q^1 & 13q^0 & & & & \end{array}\right) = \begin{array}{cccccc} 17q^8 & 2q^5 & 4q^5 & 7q^2 & 11q^1 & 16q^0 \\ 1q^6 & 3q^5 & 5q^5 & 13q^0 & 15q^0 & \\ 6q^2 & 9q^1 & 10q^1 & 14q^0 & & \\ 8q^1 & 12q^0 & & & & \end{array}$$

We will frequently require the basic lemma regarding the ordering $\leq$ on $\mathrm{PSYT}_{\geq 0}(\lambda)$.

Lemma 3.2.3. *Let $\lambda \in \mathbb{Y}$ and $T \in \mathrm{RYT}_{\geq 0}(\lambda)$. There are unique $\min(T), \mathrm{top}(T) \in \mathrm{PSYT}_{\geq 0}(\lambda; T)$ such that for all $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda)$ with $\mathfrak{p}_\lambda(\tau) = T$, $\min(T) \leq \tau \leq \mathrm{top}(T)$.*

Proof. We can explicitly construct the elements $\mathrm{top}(T), \min(T)$ directly. Define $\mathrm{top}(T)$ by first filling in the boxes $\Box \in \lambda$ of λ with the values $q^{T(\Box)}$. Now we label these boxes with the values $\{1, \cdots, n\}$ by first decomposing λ into skew diagrams where T is constant on each sub-diagram. This gives us an increasing chain of Young diagrams $\lambda^{(1)} \subset \ldots \subset \lambda^{(r)} = \lambda$. Next we fill each diagram $\lambda^{(i)}$ with the values $\{|\lambda^{(1)}|+\ldots+|\lambda^{(i-1)}|+1,\ldots,|\lambda^{(1)}|+\ldots+|\lambda^{(i)}|\}$ in column-standard order. This gives a value iq^a in each box of λ.

For $\min(T)$, we proceed similarly by first first filling in the boxes $\Box\in\lambda$ of λ with the values $q^{T(\Box)}$. Then we decompose λ into the same skew diagrams as before. Now we fill each diagram $\lambda^{(i)}$ with the values $\{n-(|\lambda^{(1)}|+\ldots+|\lambda^{(i-1)}|),\ldots,n-(|\lambda^{(1)}|+\ldots+|\lambda^{(i)}|)\}$ in row-standard order. This gives a value iq^a in each box of λ.

$\Box$

Example. *Given* $T = \begin{array}{|c|c|c|c|c|c|} \hline 7 & 5 & 5 & 2 & 1 & 0 \\ \hline 6 & 5 & 5 & 0 & 0 \\ \cline{1-5} 2 & 1 & 1 & 0 \\ \cline{1-4} 1 & 0 \\ \cline{1-2} \end{array} \in \mathrm{RYT}_{\geq 0}(6,5,4,2)$ *we have that*

$$\min(T) = \begin{array}{|c|c|c|c|c|c|} \hline 17q^7 & 12q^5 & 13q^5 & 10q^2 & 6q^1 & 1q^0 \\ \hline 16q^6 & 14q^5 & 15q^5 & 2q^0 & 3q^0 \\ \cline{1-5} 11q^2 & 7q^1 & 8q^1 & 4q^0 \\ \cline{1-4} 9q^1 & 5q^0 \\ \cline{1-2} \end{array} \quad \text{and } \mathrm{top}(T) = \begin{array}{|c|c|c|c|c|c|} \hline 1q^7 & 3q^5 & 5q^5 & 8q^2 & 12q^1 & 17q^0 \\ \hline 2q^6 & 4q^5 & 6q^5 & 14q^0 & 16q^0 \\ \cline{1-5} 7q^2 & 10q^1 & 11q^1 & 15q^0 \\ \cline{1-4} 9q^1 & 13q^0 \\ \cline{1-2} \end{array}.$$

Definition 3.2.4. *Let $\lambda\in\mathbb{Y}$ with $|\lambda|=n$ and $T\in\mathrm{RYT}_{\geq 0}(\lambda)$. Define $\nu(T)\in\mathbb{Z}^n_{\geq 0}$ to be the vector formed by listing the values of T in decreasing order i.e. $\nu(T)=\mathrm{sort}(w_\tau)$ for any $\tau\in\mathrm{PSYT}_{\geq 0}(\lambda;T)$. Define $S(T)\in\mathrm{SYT}(\lambda)$ by ordering the boxes of λ according to $\Box_1\leq\Box_2$ if and only if*

- *$T(\Box_1)>T(\Box_2)$ or*
- *$T(\Box_1)=T(\Box_2)$ and $\Box_1$ comes before $\Box_2$ in the column-standard labelling of λ.*

We will often write as a shorthand $\Box_1<_T\Box_2$ whenever $S(T)(\Box_1)<S(T)(\Box_2)$. Define the statistic $b_T\in\mathbb{Z}_{\geq 0}$ by

$$b_T := \sum_{i=1}^{n}\nu(T)_i(c_{S(T)}(i)+i-1).$$

Lastly, define the composition $\mu(T)$ of n as follows. Decompose λ into horizontal strips $h_1,\ldots,h_m$ where T is constant on each strip. We order these strips so that the $\min(T)$ labels in h_i are strictly less than those in h_{i+1} for all i. Note that, unless $T\in\mathrm{RSSYT}_{\geq 0}(\lambda)$, we may have horizontal strips *with the same T-value touching in adjacent rows. We see that each of these horizontal strips h_i has some labels $a_i, \dots, a_i + r_i$.. Then $\mu(T)$ is given as $(r_1, \dots, r_m)$.*

Remark 26. *For every $T \in \mathrm{RYT}_{\geq 0}(\lambda)$ we can recover T from the pair $(S(T), \nu(T))$ by labelling λ with the entries of $\nu(T)$ following the order of $S(T)$. Further, the standard Young tableau $S(T)$ is the largest such tableau following the partial order defined in Definition 3.2.2.*

Below is an example calculation of the various data which we associate to $T \in \mathrm{RYT}_{\geq 0}(\lambda)$.

Example. *For $T \in \mathrm{RYT}_{\geq 0}(6,5,4,2)$ as in Example 3.2 we have that*

$S(T) =$

1	3	5	8	12	17
2	4	6	14	16	
7	10	11	15		
9	13				

$\in \mathrm{SYT}(6,5,4,2)$,

$\nu(T) = (7,6,5,5,5,5,2,2,1,1,1,1,0,0,0,0,0) \in \mathbb{Z}^{17}_{\geq 0}$,

$b_T = 0+0+15+15+30+30+8+20+5+8+10+15+0+0+0+0+0 = 156$,

and $\mu(T) = (1,2,1,1,1,2,1,1,1,2,2,1,1)$.

The next definition will be crucial for many of the results in this chapter.

Definition 3.2.5. *Let $\lambda \in \mathbb{Y}$, with $|\lambda| = n$ and $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda)$ with $T = \mathfrak{p}_\lambda(\tau)$. An ordered pair of boxes $(\square_1, \square_2) \in \lambda \times \lambda$ is called an* ***inversion pair*** *of τ if $S(T)(\square_1) < S(T)(\square_2)$ and $i > j$ where $\tau(\square_1) = iq^a$, $\tau(\square_2) = jq^b$ for some $a, b \geq 0$. The set of all inversion pairs of τ will be denoted by $\mathrm{Inv}(\tau)$. We will use the shorthand $\mathrm{I}(T)$ for the set $\mathrm{Inv}(\min(T))$.*

Example. *In the labelling*

$17q^7$	$12q^5$	$13q^5$	$10q^2$	$6q^1$	$1q^0$
$16q^6$	$14q^5$	$15q^5$	$2q^0$	$3q^0$	
$11q^2$	$7q^1$	$8q^1$	$4q^0$		
$9q^1$	$5q^0$				

we have that the pairs $(17q^7, 12q^5)$, $(14q^5, 13q^5)$, and $(5q^0, 4q^0)$ are all inversions. Here we have referred to boxes according to their labels.

In the following definition our conventions for the Bruhat ordering differ from many other authors and from the conventions seen previously in Chapter 1. These conventions are use to help properly state some triangularity properties later in the chapter. However, one may obtain the below definition from the more standard conventions in [19] by reversing the order of the entries of each vector $(a_1, \dots, a_n) \to (a_n, \dots, a_1)$ and rewriting their Bruhat ordering from this reversed perspective.

Definition 3.2.6. *Define the reversed Bruhat ordering $\preceq$ on $\mathbb{Z}^n_{\geq 0}$ using the following cover relations for $\lambda \in \mathbb{Z}^n_{\geq 0}$:*

- *if $i < j$ with $\lambda_i < \lambda_j$ then $\lambda \prec (i,j)\lambda$*
- *if $i < j$ with $\lambda_i + 1 < \lambda_j$ then $\lambda \succ \lambda + e_i - e_j$.*

Here e_i denotes the i-th standard basis vector of $\mathbb{Z}^n$ and $(i,j) \in \mathfrak{S}_n$ denotes the simple transposition swapping i and j. For $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ we define $\widetilde{\gamma}(\alpha) := (\alpha_2, \dots, \alpha_n, \alpha_1 + 1)$. We will write $\mathrm{sort}(\alpha)$ for the vector formed by listing the entries of α in weakly decreasing order. We define $\mathrm{Stab}(\alpha)$ to be the corresponding stabilizer subgroup of $\mathfrak{S}_n$ for α i.e. the set of all $\sigma \in \mathfrak{S}_n$ with $\sigma(\alpha) = \alpha$.

We require the following simple lemma regarding the interplay between the map $\widetilde{\gamma}$ on $\mathbb{Z}^n_{\geq 0}$ and the ordering $\prec$.

Lemma 3.2.7. *If $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$ satisfy $\alpha \prec \beta$ then $\widetilde{\gamma}(\alpha) \prec \widetilde{\gamma}(\beta)$.*

Proof. We will show that if $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$ and β covers α with respect to the Bruhat order then $\widetilde{\gamma}(\alpha) \prec \widetilde{\gamma}(\beta)$. We will proceed in cases. Let $\lambda \in \mathbb{Z}^n_{\geq 0}$.

First, suppose $1 < i < j$ and $\lambda_i < \lambda_j$. Then

$$\widetilde{\gamma}(\lambda) \prec (i-1, j-1)\widetilde{\gamma}(\lambda) = \widetilde{\gamma}((i,j)\lambda).$$

Now suppose $1 < j$ and $\lambda_1 < \lambda_j$. Then

$$\widetilde{\gamma}((1,j)\lambda) \succ \widetilde{\gamma}((1,j)\lambda) + e_j - e_n \succeq (j,n)(\widetilde{\gamma}((1,j)\lambda) + e_j - e_n) = \widetilde{\gamma}(\lambda).$$

If now we have that $1 < i < j$ and $\lambda_i < \lambda_j - 1$ then

$$\widetilde{\gamma}(\lambda) \succ \widetilde{\gamma}(\lambda) + e_{i-1} - e_{j-1} = \widetilde{\gamma}(\lambda + e_i - e_j).$$

Lastly, consider the case when $1 < j$ and $\lambda_1 < \lambda_j - 1$. If $\lambda_1 + 2 = \lambda_j$ then

$$\widetilde{\gamma}(\lambda) \succ (j-1,n)\widetilde{\gamma}(\lambda) = \widetilde{\gamma}(\lambda + e_1 - e_j).$$

Instead if $\lambda_1 < \lambda_j - 2$ then

$$\widetilde{\gamma}(\lambda) \succ (j-1,n)\widetilde{\gamma}(\lambda) \succ (j-1,n)\widetilde{\gamma}(\lambda) + e_{j-1} - e_n = \widetilde{\gamma}(\lambda + e_1 - e_j).$$

$\square$

Here we review some necessary details about the extended affine symmetric groups.

DEFINITION 3.2.8. *Define $\widehat{\mathfrak{S}}_n$ to be the extended affine symmetric group given by*

$$\widehat{\mathfrak{S}}_n := \mathfrak{S}_n \ltimes \mathbb{Z}^n$$

where $\mathfrak{S}_n$ acts on $\mathbb{Z}^n$ by coordinate permutations. Denote by $t_1, \ldots, t_n$ the standard generators of $\mathbb{Z}^n \subset \widehat{\mathfrak{S}}_n$. Further, we define the special element $\widetilde{\gamma}_n \in \widehat{\mathfrak{S}}_n$ given by

$$\widetilde{\gamma}_n := t_n s_{n-1} \ldots s_1.$$

For any $\beta \in \mathbb{Z}^n$ we will write

$$t_\beta := t_1^{\beta_1} \cdots t_n^{\beta_n}.$$

Define the positive submonoid of $\widehat{\mathfrak{S}}_n$, $\widehat{\mathfrak{S}}_n^+$, as the monoid generated by $\{s_1, \ldots, s_{n-1}, \widetilde{\gamma}_n\}$ (i.e. no $\widetilde{\gamma}_n^{-1}s$).

The length $\ell(\sigma)$ of $\sigma \in \widehat{\mathfrak{S}}_n$ is the minimal number of s_i required to express σ in terms of the generators $\{s_1, \ldots, s_{n-1}, \widetilde{\gamma}_n\}$. We denote by $\widehat{\mathfrak{S}}_n/\mathfrak{S}_n$ the set of minimal length left coset representatives of $\widehat{\mathfrak{S}}_n$ with respect to the subgroup $\mathfrak{S}_n$. We will denote the set of positive minimal length coset representatives of $\widehat{\mathfrak{S}}_n$ with respect to the subgroup $\mathfrak{S}_n$ by $\left(\widehat{\mathfrak{S}}_n/\mathfrak{S}_n\right)^+ := (\widehat{\mathfrak{S}}_n/\mathfrak{S}_n) \cap \widehat{\mathfrak{S}}_n^+$. If $\mu = (\mu_1, \ldots, \mu_r)$ is a composition of $n = \mu_1 + \ldots + \mu_r$ then we will define the Young subgroup $\mathfrak{S}_\mu$ of $\mathfrak{S}_n$ corresponding to μ as $\mathfrak{S}_\mu := \mathfrak{S}_{\mu_1} \times \cdots \times \mathfrak{S}_{\mu_r} \subset \mathfrak{S}_n$. We will write $\mathfrak{S}_n/\mathfrak{S}_\mu$ for the set of minimal length left coset representatives for $\mathfrak{S}_n$ with respect to the subgroup $\mathfrak{S}_\mu$.

For $\beta \in \mathbb{Z}^n$ define $\sigma_\beta \in \widehat{\mathfrak{S}}_n$ by

$$\sigma_\beta := \sigma t_{\mathrm{sort}(\beta)}$$

where σ is the unique minimal length coset representative in $\mathfrak{S}_n/\mathfrak{S}_{\mathrm{Stab}(\mathrm{sort}(\beta))}$ such that $\sigma(\mathrm{sort}(\beta)) = \beta$.

The next two lemmas are standard in the theory of (extended) affine permutations and we leave them to the reader to verify.

LEMMA 3.2.9. *We have that*

$$\widehat{\mathfrak{S}}_n/\mathfrak{S}_n = \{\sigma_\beta | \beta \in \mathbb{Z}^n\}$$

and

$$\left(\widehat{\mathfrak{S}}_n/\mathfrak{S}_n\right)^+ = \{\sigma_\beta | \beta \in \mathbb{Z}^n_{\geq 0}\}.$$

LEMMA 3.2.10. *For all $\alpha \in \mathbb{Z}^n_{\geq 0}$ we have the following:*

- *If α is weakly decreasing then $\sigma_\alpha = t_\alpha$.*
- *If $s_i(\alpha) \succ \alpha$ then $\sigma_{s_i(\alpha)} = s_i\sigma_\alpha$.*
- *If $s_i(\alpha) = \alpha$ then $s_i\sigma_\alpha = \sigma_\alpha s_{\sigma^{-1}(i)}$ where σ is the minimal length permutation with $\sigma(\mathrm{sort}(\alpha)) = \alpha$.*
- *$\sigma_{\widetilde{\gamma}_n(\alpha)} = \widetilde{\gamma}_n(\sigma_\alpha)$.*

Recall that in Definition 3.2.2 we only defined $s_i(\tau)$ for $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda)$ in the situation where swapping the i and $i+1$ labels in the boxes of τ resulted in an element of $\mathrm{PSYT}_{\geq 0}(\lambda)$. We now generalize this notion to elements of $\widehat{\mathfrak{S}}_n^+$.

DEFINITION 3.2.11. *Suppose $z_r \cdots z_1$ is a reduced word in $\widehat{\mathfrak{S}}_n^+$ written in the generators $z_i \in \{s_1, \ldots, s_{n-1}, \widetilde{\gamma}_n\}$. We define inductively on $r \geq 1$ if $z_{r-1} \cdots z_1(\tau) \in \mathrm{PSYT}_{\geq 0}(\lambda)$ the element $z_r \cdots z_1(\tau)$ of $\mathrm{PSYT}_{\geq 0}(\lambda)$ as either*

- *$\Psi(z_{r-1} \cdots z_1(\tau))$ if $z_r = \widetilde{\gamma}_n$*
- *$s_i(z_{r-1} \cdots z_1(\tau))$ if $z_r = s_i$ and swapping the i and $i+1$ labels in the boxes of $z_{r-1} \cdots z_1(\tau)$ results in an element of $\mathrm{PSYT}_{\geq 0}(\lambda)$.*

Otherwise we will leave this symbol undefined. This definition is only dependent on the element $z_r \cdots z_1$ of $\widehat{\mathfrak{S}}_n^+$ in that if $z_r \cdots z_1 = z'_r \cdots z'_1$ is another reduced word then $z_r \cdots z_1(\tau)$ is defined if and only if $z'_r \cdots z'_1(\tau)$ is defined. Thus we will write $\sigma(\tau) = z_r \cdots z_1(\tau)$ unambiguously in this situation if $\sigma = z_r \cdots z_1$.

We will need the following result later in the chapter.

Lemma 3.2.12. *For $T \in \mathrm{RYT}_{\geq 0}(\lambda)$ we have that*

$$\mathrm{top}(T) = \zeta_1^{\nu(T)_1 - \nu(T)_2} \cdots \zeta_n^{\nu(T)_n}(S(T))$$

where for all $1 \leq i \leq n$

$$\zeta_i := (s_i \cdots s_{n-1}\Psi)^i.$$

Proof. One may check by direct computation that if $T \in \mathrm{RYT}_{\geq 0}(\lambda)$ and $1 \leq i \leq n$ then $\zeta_i(\mathrm{top}(T)))$ is well defined according to Definition 3.2.11 and in particular, $\zeta_i(\mathrm{top}(T)) = \mathrm{top}(T')$ where $T'(\square) = T(\square)+1$ for $S(T)(\square) \leq i$ and $T'(\square) = T(\square)$ otherwise. Note that $S(T) = S(T')$ so applying ζ_i does not change the underlying diagram ordering corresponding to the labelling T. Thus given any $T \in \mathrm{RYT}_{\geq 0}(\lambda)$ by applying each ζ_i one at a time we see that $\zeta_1^{\nu(T)_1-\nu(T)_2} \cdots \zeta_n^{\nu(T)_n}(S(T))$ must equal $\mathrm{top}(T)$. $\square$

We will need to identify an explicit bijection between $\mathrm{PSYT}_{\geq 0}(\lambda)$ and $\left(\widehat{\mathfrak{S}}_n/\mathfrak{S}_n\right)^+ \times \mathrm{SYT}(\lambda)$. We already have a map $\mathrm{PSYT}_{\geq 0_{\geq 0}}(\lambda) \to \left(\widehat{\mathfrak{S}}_n/\mathfrak{S}_n\right)^+$ given by $\tau \to \sigma_{w_\tau}$. This is not bijective so we will use elements of $\mathrm{SYT}(\lambda)$ to refine this map to yield a bijection. We now identify the correct choice of $\mathrm{SYT}(\lambda)$ for a given $\tau \in \mathrm{PSYT}_{\geq 0_{\geq 0}}(\lambda)$.

Definition 3.2.13. *For $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda)$ we define $S(\tau) \in \mathrm{SYT}(\lambda)$ by the following recursion:*

- *$S(\mathrm{top}(T)) := S(T)$ as defined in Definition 3.2.4*
- *If $w_\tau(i) < w_\tau(i+1)$ then $S(s_i(\tau)) = S(\tau)$.*
- *$S(\Psi(\tau)) = S(\tau)$*
- *If $w_\tau(i) = w_\tau(i+1)$ and $c_\tau(i) - c_\tau(i+1) > 1$ then $S(s_i(\tau)) = s_j S(\tau)$ where $j = \sigma^{-1}(i)$ and σ is the minimal length permutation with $\sigma(\mathrm{sort}(w_\tau)) = w_\tau$.*

Proposition 3.2.14. *For $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda)$*

$$\tau = \sigma_{w_\tau}(S(\tau)).$$

Proof. Using Lemma 3.2.10 and Lemma 3.2.12 we see that for all $T \in \mathrm{RYT}_{\geq 0}(\lambda)$

$$\sigma_{w_{\mathrm{top}(T)}}(S(\mathfrak{p}_\lambda(\mathrm{top}(T)))) = \sigma_{\nu(T)}(S(T)) = t_{\nu(T)}(S(T)) = \zeta_1^{\nu(T)_1-\nu(T)_2} \cdots \zeta_n^{\nu(T)_n}(S(T)) = \mathrm{top}(T).$$

Let $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda; T)$ and suppose for sake of induction that $\tau = \sigma_{w_\tau}(S(\tau))$. Now let $s_i(\tau) < \tau$. If $w_\tau(i) > w_\tau(i+1)$ then $S(s_i(\tau)) = S(\tau)$ and $\sigma_{w_{s_i(\tau)}} = s_i \sigma_{w_\tau}$ so that

$$\sigma_{w_{s_i(\tau)}}(S(s_i(\tau))) = s_i \sigma_{w_\tau}(S(\tau)) = s_i(\tau).$$

In the case instead that $w_\tau(i) = w_\tau(i+1)$ with $c_\tau(i+1) - c_\tau(i) > 1$ then $S(s_i(\tau)) = s_j(S(\tau))$ and $\sigma_{w_{s_i(\tau)}} = \sigma_{w_\tau}$ where $j = \sigma^{-1}(i)$ and σ is the minimal length permutation with $\sigma(\mathrm{sort}(w_\tau)) = w_\tau$. Then

$$\begin{aligned} &\sigma_{w_{s_i(\tau)}}(S(s_i(\tau))) \\ &= \sigma_{w_\tau}(s_j S(\tau)) \\ &= (\sigma_{w_\tau} s_j)(S(\tau)) \\ &= (s_i \sigma_{w_\tau})(S(\tau)) \\ &= s_i(\tau). \end{aligned}$$

$\square$

We may now obtain the desired bijection.

Proposition 3.2.15. *The map* $\Xi_\lambda : \mathrm{PSYT}_{\geq 0}(\lambda) \to \left(\widehat{\mathfrak{S}}_n / \mathfrak{S}_n\right)^+ \times \mathrm{SYT}(\lambda)$ *given by*

$$\Xi_\lambda(\tau) := (\sigma_{w_\tau}, S(\tau))$$

is a bijection.

Proof. It is immediate from Proposition 3.2.14 that Ξ_λ is bijective onto its image. But it is straightforward to check inductively that given any $\sigma \in \left(\widehat{\mathfrak{S}}_n / \mathfrak{S}_n\right)^+$ and $S \in \mathrm{SYT}(\lambda)$, $\sigma(S)$ is a well defined element of $\mathrm{PSYT}_{\geq 0}(\lambda)$ in the sense of Definition 3.2.11. This shows that Ξ_λ is surjective and thus bijective. $\square$

3.2.1. Intertwiner Relations. We will require the following lemmas regarding the intertwiner relations for the $\theta_i^{(n)}$ operators. These relations involve the following special element.

Definition 3.2.16. *Define* $\gamma_n := X_n T_{n-1} \cdots T_1$.

The element γ_n behaves predictably with the $\theta_i^{(n)}$ operators.

LEMMA 3.2.17. *The following hold:*

- $\theta_i\gamma_n = \gamma_n\theta_{i+1}$ *for* $1 \leq i \leq n-1$
- $\theta_n\gamma_n = \gamma_n q\theta_1$.

PROOF. Let $1 \leq i \leq n-1$. We find that

$$\begin{aligned}
\theta_i\gamma_n &= t^{-(n-i)}T_{i-1}^{-1}\cdots T_1^{-1}\pi_n T_{n-1}\cdots T_i X_n T_{n-1}\cdots T_1\\
&= t^{-(n-i)}T_{i-1}^{-1}\cdots T_1^{-1}\pi_n T_{n-1}X_n T_{n-2}\cdots T_i T_{n-1}\cdots T_1\\
&= t^{-(n-i)}T_{i-1}^{-1}\cdots T_1^{-1}\pi_n t X_{n-1}T_{n-1}^{-1}T_{n-2}\cdots T_i T_{n-1}\cdots T_1\\
&= t^{-(n-(i+1))}T_{i-1}^{-1}\cdots T_1^{-1}X_n\pi_n T_{n-1}^{-1}T_{n-2}\cdots T_i T_{n-1}\cdots T_1\\
&= t^{-(n-(i+1))}X_n T_{i-1}^{-1}\cdots T_1^{-1}\pi_n T_{n-1}^{-1}T_{n-2}\cdots T_i(T_{n-1}\cdots T_1).
\end{aligned}$$

From the braid relations we see that for all $1 \leq j \leq n-2$

$$T_j(T_{n-1}\cdots T_1) = (T_{n-1}\cdots T_1)T_{j+1}$$

and hence

$$\begin{aligned}
&t^{-(n-(i+1))}X_n T_{i-1}^{-1}\cdots T_1^{-1}\pi_n T_{n-1}^{-1}T_{n-2}\cdots T_i(T_{n-1}\cdots T_1)\\
&= t^{-(n-(i+1))}X_n T_{i-1}^{-1}\cdots T_1^{-1}\pi_n T_{n-1}^{-1}(T_{n-1}\cdots T_1)T_{n-1}\cdots T_{i+1}\\
&= t^{-(n-(i+1))}X_n T_{i-1}^{-1}\cdots T_1^{-1}\pi_n T_{n-2}\cdots T_1 T_{n-1}\cdots T_{i+1}\\
&= t^{-(n-(i+1))}X_n T_{i-1}^{-1}\cdots T_1^{-1}T_{n-1}\cdots T_2\pi_n T_{n-1}\cdots T_{i+1}\\
&= t^{-(n-(i+1))}X_n T_{i-1}^{-1}\cdots T_1^{-1}T_{n-1}\cdots T_2 T_1 T_1^{-1}\pi_n T_{n-1}\cdots T_{i+1}\\
&= t^{-(n-(i+1))}X_n T_{n-1}\cdots T_1 T_i^{-1}\cdots T_2^{-1}T_1^{-1}\pi_n T_{n-1}\cdots T_{i+1}\\
&= (X_n T_{n-1}\cdots T_1)(t^{-(n-(i+1))}T_i^{-1}\cdots T_1^{-1}\pi_n T_{n-1}\cdots T_{i+1})\\
&= \gamma_n\theta_{i+1}.
\end{aligned}$$

Now we consider the last case:

$$\begin{aligned}
\theta_n\gamma_n &= T_{n-1}^{-1}\cdots T_1^{-1}\pi_n T_{n-1}\cdots T_1\\
&= T_{n-1}^{-1}\cdots T_1^{-1} qX_1\pi_n T_{n-1}\cdots T_1\\
&= t^{-(n-1)}X_n T_{n-1}\cdots T_1 q\pi_n T_{n-1}\cdots T_1\\
&= (X_nT_{n-1}\cdots T_1)(qt^{-(n-1)}\pi_n T_{n-1}\cdots T_1)\\
&= \gamma_n q\theta_1.
\end{aligned}$$

□

Recall the definition of the intertwiner elements φ_i in Definition 1.4.4. As is standard in DAHA theory we will use the elements $\{\varphi_1,\ldots,\varphi_{n-1},\gamma_n\}$ to define intertwiner operators corresponding to elements of $\widehat{\mathfrak{S}}_n^+$.

Definition 3.2.18. *For any $\sigma\in\widehat{\mathfrak{S}}_n^+$ with $\sigma=(s_{i_1}\cdots s_{i_{j_1}})\widetilde{\gamma}_n\cdots\widetilde{\gamma}_n(s_{i_{j_1+\ldots+j_{r-1}+1}}\cdots s_{i_{j_1+\ldots+j_r}})$ written minimally in terms of the generators $\{s_1,\ldots,s_{n-1},\widetilde{\gamma}\}$ define $\varphi_\sigma\in\mathscr{D}_n$ by*

$$\varphi_\sigma := (\varphi_{i_1}\cdots\varphi_{i_{j_1}})\gamma_n\cdots\gamma_n(\varphi_{i_{j_1+\ldots+j_{r-1}+1}}\cdots\varphi_{i_{j_1+\ldots+j_r}})\in\mathscr{D}_n\,.$$

In particular, we have that $\varphi_{s_i}=\varphi_i$ and $\varphi_{\widetilde{\gamma}_n}=\gamma_n$.

The main utility of considering the intertwiner operators φ_σ comes from the next lemma.

Lemma 3.2.19. *If v is a $\theta^{(n)}$-weight vector in some $\mathscr{D}_n$-module with weight $\alpha=(\alpha_1,\ldots,\alpha_n)$ and $\sigma\in\mathfrak{S}_n$ with $\varphi_\sigma(v)\neq 0$ then $\varphi_\sigma(v)$ is a $\theta^{(n)}$-weight vector with weight α^σ given by the following recursion:*

- $\alpha^{s_i}=(\alpha_1,\ldots,\alpha_{i+1},\alpha_i,\ldots\alpha_n)$
- $\alpha^{\widetilde{\gamma}_n}=(\alpha_2,\ldots,\alpha_n,q\alpha_1)$
- $(\alpha^{\sigma_2})^{\sigma_1}=\alpha^{\sigma_1\sigma_2}$.

Proof. This result follows easily by using induction on $\widehat{\mathfrak{S}}_n^+$ using the relations in Proposition 1.4.5 and Lemma 3.2.17. We leave the details to the reader. □

3.3. DAHA Modules from Young Diagrams

3.3.1. Irreducible Representations of $\mathscr{H}_n$. The following definition gives a description of the irreducible representations of $\mathscr{H}_n$. There are many equivalent methods for defining these representations but we choose to specify eigenvectors for the Jucys-Murphy elements $\overline{\theta}_i$ directly as we will require these eigenvectors throughout this chapter.

Definition 3.3.1. *Let $\lambda \in \mathbb{Y}$ with $|\lambda| = n$. Define S_λ to be the $\mathscr{H}_n$-module spanned by e_τ for $\tau \in \mathrm{SYT}(\lambda)$ defined by the following relations:*

- $\overline{\theta}_i(e_\tau) = t^{c_\tau(i)} e_\tau$
- *If $s_i(\tau) > \tau$ then $\overline{\varphi}_i(e_\tau) = (t^{c_\tau(i)} - t^{c_\tau(i+1)}) e_{s_i(\tau)}$.*
- *If the labels $i, i+1$ are in the same row in τ then $T_i(e_\tau) = e_\tau$.*
- *If the labels $i, i+1$ are in the same column in τ then $T_i(e_\tau) = -te_\tau$.*

Using the relations from Proposition 1.4.2 we can show the following more explicit form for the action of the T_i on the $\mathrm{SYT}(\lambda)$ basis:

- If $s_i(\tau) > \tau$ then
$$T_i(e_\tau) = e_{s_i(\tau)} + \frac{(1-t)t^{c_\tau(i)}}{t^{c_\tau(i)} - t^{c_\tau(i+1)}} e_\tau.$$
- If $s_i(\tau) < \tau$ then
$$T_i(e_\tau) = -\frac{(t^{c_\tau(i+1)+1} - t^{c_\tau(i)})(t^{c_\tau(i)+1} - t^{c_\tau(i+1)})}{(t^{c_\tau(i+1)} - t^{c_\tau(i)})^2} e_{s_i(\tau)} + \frac{(1-t)t^{c_\tau(i)}}{t^{c_\tau(i)} - t^{c_\tau(i+1)}} e_\tau.$$

Proposition 3.3.2. *Definition 3.3.1 is well-posed i.e. the action of the operators T_i on S_λ define an irreducible $\mathscr{H}_n$-module.*

Proof. As this construction is standard we will only give an outline. It follows from standard theory for the finite Hecke algebra $\mathscr{H}_n$ (analogous to that of the symmetric group $\mathfrak{S}_n$ in characteristic 0) that there exists an irreducible representation of $\mathscr{H}_n$, S_λ, corresponding to the partition λ with a basis of weight vectors for the Jucys-Murphy elements $\overline{\theta}_i$, v_τ say, indexed by $\tau \in \mathrm{SYT}(\lambda)$. Further, the weights are given by $\overline{\theta}_i(v_\tau) = t^{c_\tau(i)} v_\tau$. As these weights are all distinct it follows that this basis is unique up to re-normalization by nonzero scalars. The presentation given in Definition 3.3.1 fixes a specific normalization given by choosing first $e_{\tau_\lambda^{rs}} = v_{\tau_\lambda^{rs}}$ and then building the full basis e_τ using the intertwiner $\overline{\varphi_i}$ relations in Proposition 1.4.2 with the choice that whenever $s_i(\tau) > \tau$ we have that $\overline{\varphi_i}(e_\tau) = (t^{c_\tau(i)} - t^{c_\tau(i+1)})e_{s_i(\tau)}$. Up to an initial arbitrary choice for the scalar multiple of $e_{\tau_\lambda^{rs}}$, this uniquely determines the rest of the coefficients of the e_τ. $\square$

REMARK 27. *The set $\{\lambda \in \mathbb{Y} : |\lambda| = n\}$ gives a full set of irreducible $\mathscr{H}_n$-modules up to isomorphism. Note that for $\tau, \tau' \in \mathrm{SYT}(\lambda)$, the $\overline{\theta}$-weights of $e_\tau = e_{\tau'}$ are equal if and only if $\tau = \tau'$.*

In the following lemma we identify a particular map between finite Hecke algebra representations which will be central in the stable-limit construction later in the chapter.

LEMMA 3.3.3. *Let $\lambda \in \mathbb{Y}$ and $n \geq n_\lambda$. Let $\Box_0$ be the unique square in the skew-diagram $\lambda^{(n+1)}/\lambda^{(n)}$. Consider the map $\mathfrak{q}_\lambda^{(n)} : S_{\lambda^{(n+1)}} \to S_{\lambda^{(n)}}$ given for $\tau \in \mathrm{SYT}(\lambda^{(n+1)})$ as*

$$\mathfrak{q}_\lambda^{(n)}(e_\tau) := \begin{cases} e_{\tau|_{\lambda^{(n)}}} & \tau(\Box_0) = n+1 \\ 0 & \tau(\Box_0) \neq n+1. \end{cases}$$

Then $\mathfrak{q}_\lambda^{(n)}$ is a $\mathscr{H}_n$-module map.

PROOF. Let $\tau \in \mathrm{SYT}(\lambda^{(n+1)})$. First, assume that $\tau(\Box_0) \neq n+1$ so that $\mathfrak{q}_\lambda^{(n)}(e_\tau) = 0$. Then for $1 \leq i \leq n-1$, from the relations in Definition 3.3.1, we see that $T_i(e_\tau)$ is either a scalar multiple of e_τ or a linear combination of e_τ and $e_{s_i(\tau)}$. In either case $\mathfrak{q}_\lambda^{(n)}(T_i(e_\tau)) = 0 = T_i\mathfrak{q}_\lambda^{(n)}(e_\tau)$. Now assume $\tau(\Box_0) = n+1$. We will be more detailed about this case as we will need to be careful about the combinatorics regarding the coefficients of expanding $T_i(e_\tau)$ into the $\mathrm{SYT}(\lambda^{(n)})$ basis. For $1 \leq i \leq n-1$ we have the cases

- $T_i(e_\tau) = e_\tau$ if $i, i+1$ are in the same row of τ
- $T_i(e_\tau) = -te_\tau$ if $i, i+1$ are in the same column of τ
- $T_i(e_\tau) = e_{s_i(\tau)} + \frac{(1-t)t^{c_\tau(i)}}{t^{c_\tau(i)} - t^{c_\tau(i+1)}}e_\tau$ if $s_i(\tau) > \tau$
- $T_i(e_\tau) = -\frac{(t^{c_\tau(i+1)+1} - t^{c_\tau(i)})(t^{c_\tau(i)+1} - t^{c_\tau(i+1)})}{(t^{c_\tau(i+1)} - t^{c_\tau(i)})^2}e_{s_i(\tau)} + \frac{(1-t)t^{c_\tau(i)}}{t^{c_\tau(i)} - t^{c_\tau(i+1)}}e_\tau$ if $s_i(\tau) < \tau$.

In any of these cases since $\tau(\Box_0) = n+1$ and $1 \leq i \leq n-1$, we have that $s_i(\tau)(\Box_0) = n+1$ as well. Further, the placement of the boxes labelled $i, i+1$ in the labellings $\tau, s_i(\tau)$ is unchanged when restricted to $\lambda^{(n)}$ i.e. in the labellings $\tau|_{\lambda^{(n)}}, s_i(\tau)|_{\lambda^{(n)}}$. Let $\tau' := \tau|_{\lambda^{(n)}}$. Therefore we have the cases:

- $\mathfrak{q}_\lambda^{(n)}(T_i(e_\tau)) = e_{\tau|_{\lambda^{(n)}}} = T_i\mathfrak{q}_\lambda^{(n)}(e_\tau)$ if $i, i+1$ are in the same row of τ

- $\mathfrak{q}_\lambda^{(n)}(T_i(e_\tau)) = -te_{\tau|_{\lambda(n)}} = T_i\mathfrak{q}_\lambda^{(n)}(e_\tau)$ if $i, i+1$ are in the same column of τ
- $\mathfrak{q}_\lambda^{(n)}(T_i(e_\tau)) = e_{s_i(\tau')} + \frac{(1-t)t^{c_{\tau'}(i)}}{t^{c_{\tau'}(i)}-t^{c_{\tau'}(i+1)}}e_{\tau'} = T_i\mathfrak{q}_\lambda^{(n)}(e_\tau)$ if $s_i(\tau) > \tau$
- $\mathfrak{q}_\lambda^{(n)}(T_i(e_\tau)) = -\frac{(t^{c_{\tau'}(i+1)+1}-t^{c_{\tau'}(i)})(t^{c_{\tau'}(i)+1}-t^{c_{\tau'}(i+1)})}{(t^{c_{\tau'}(i+1)}-t^{c_{\tau'}(i)})^2}e_{s_i(\tau')} + \frac{(1-t)t^{c_{\tau'}(i)}}{t^{c_{\tau'}(i)}-t^{c_{\tau'}(i+1)}}e_{\tau'} = T_i\mathfrak{q}_\lambda^{(n)}(e_\tau)$ if $\tau > s_i(\tau)$.

Thus in all cases we have that $\mathfrak{q}_\lambda^{(n)}(T_i(e_\tau)) = T_i\mathfrak{q}_\lambda^{(n)}(e_\tau)$. Hence, $\mathfrak{q}_\lambda^{(n)}$ is a $\mathscr{H}_n$-module map. □

3.3.2. The $\mathscr{D}_n^+$-module V_λ. We begin by defining a collection of DAHA modules indexed by Young diagrams $\lambda \in \mathbb{Y}$. These modules are the same as those appearing in [12] but we take the approach of using induction from $\mathscr{A}_n$ to $\mathscr{D}_n^+$ for their definition.

DEFINITION 3.3.4. *Let $\lambda \in \mathbb{Y}$ with $|\lambda| = n$. Define the $\mathscr{D}_n^+$-module V_λ to be the induced module*

$$V_\lambda := \mathrm{Ind}_{\mathscr{A}_n}^{\mathscr{D}_n^+} \rho_n^*(S_\lambda).$$

The modules V_λ naturally have the basis given by $X^\alpha \otimes e_\tau$ where X^α is a monomial and $\tau \in \mathrm{SYT}(\lambda)$. We will refer to this as the standard basis of V_λ. Using the theory of intertwiners for DAHA and some combinatorics we are able to show the following structural results. The F_τ appearing below are the version of the non-symmetric vv. Macdonald polynomials from [12] following our conventions. These do **not** align with the non-symmetric vv. Macdonald polynomials of [12].

The next result is fundamental to the rest of this chapter and will be used repeatedly. Recall the definition of the $\theta_i^{(n)}$ elements from Definition 1.4.4.

PROPOSITION 3.3.5. *There exists a basis of V_λ consisting of $\theta^{(n)}$-weight vectors $\{F_\tau : \tau \in \mathrm{PSYT}_{\geq 0}(\lambda)\}$ with distinct $\theta^{(n)}$-weights such that the following hold:*

- $\theta_i^{(n)}(F_\tau) = q^{w_\tau(i)}t^{c_\tau(i)}F_\tau$
- *If $\tau \in \mathrm{SYT}(\lambda)$ then $F_\tau = 1 \otimes e_\tau$.*
- *If $s_i(\tau) > \tau$ then*

$$\left(tT_i^{-1} + \frac{(t-1)q^{w_\tau(i+1)}t^{c_\tau(i+1)}}{q^{w_\tau(i)}t^{c_\tau(i)} - q^{w_\tau(i+1)}t^{c_\tau(i+1)}}\right)(F_\tau) = F_{s_i(\tau)}.$$

- $F_{\Psi(\tau)} = q^{w_1(\tau)}X_n\pi_n^{-1}(F_\tau)$.

PROOF. Using Mackey Decomposition we find

$$
\begin{aligned}
& gr.\operatorname{Res}^{\mathscr{D}_n^+}_{\theta^{(n)}}(V_\lambda) \\
& = gr.\operatorname{Res}^{\mathscr{D}_n^+}_{\theta^{(n)}} \operatorname{Ind}^{\mathscr{D}_n^+}_{\mathscr{A}_n} \rho_n^*(S_\lambda) \\
& = \bigoplus_{\sigma \in \left(\widehat{\mathfrak{S}}_n/\mathfrak{S}_n\right)^+} \left(\operatorname{Res}^{\mathscr{A}_n}_{\theta^{(n)}} \rho_n^*(S_\lambda)\right)^\sigma \\
& = \bigoplus_{\substack{\sigma \in \left(\widehat{\mathfrak{S}}_n/\mathfrak{S}_n\right)^+ \\ \tau \in \mathrm{SYT}(\lambda)}} \mathbb{Q}(q,t)(\varphi_\sigma \otimes e_\tau).
\end{aligned}
$$

As a consequence we find that the set $\{\varphi_\sigma \otimes e_\tau\}_{(\sigma,\tau)\in\left(\widehat{\mathfrak{S}}_n/\mathfrak{S}_n\right)^+ \times \in \mathrm{SYT}(\lambda)}$ is a generalized $\theta^{(n)}$-weight basis for V_λ. We now define

$$F_\tau := g_\tau \varphi_{\sigma_{w_\tau}} \otimes e_{S(\tau)}$$

of V_λ where the scalars g_τ are to be chosen uniquely to satisfy the conditions detailed in this proposition's statement. It is easy to check that since every $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda)$ may be obtained by applying σ_{w_τ} to $S(\tau)$ the scalars g_τ are uniquely determined by setting $g_{\tau_\lambda^{rs}} = 1$. By Proposition 3.2.15 this assignment produces a basis for V_λ labelled by $\mathrm{PSYT}_{\geq 0}(\lambda)$. Further, by induction using Lemma 3.2.19 and Proposition 3.2.14 we see that no matter our choice of nonzero scalars g_τ each F_τ is a $\theta^{(n)}$-weight vector with $\theta_i^{(n)}(F_\tau) = q^{w_\tau(i)} t^{c_\tau(i)} F_\tau$.

The only remaining step to justify is that if $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda)$ then $\gamma_n(F_\tau)$ agrees with $X_n \pi_n^{-1}(F_\tau)$ up to some nonzero scalar. We see that

$$
\begin{aligned}
& \gamma_n(F_\tau) \\
& = X_n T_{n-1} \cdots T_1(F_\tau) \\
& = X_n \pi_n^{-1} \pi_n T_{n-1} \cdots T_1(F_\tau) \\
& = t^{n-1} X_n \pi_n^{-1} \theta_1(F_\tau) \\
& = t^{n-1} q^{w_\tau(1)} t^{c_\tau(1)} X_n \pi_n^{-1}(F_\tau).
\end{aligned}
$$

Therefore, there is no issue in defining the coefficient $g_{\Psi(\tau)}$ so that $F_{\Psi(\tau)} = q^{w_\tau(1)} X_n \pi_n^{-1}(F_\tau)$.

$\square$

EXAMPLE.

$$\begin{aligned} F_{\begin{array}{|c|c|}\hline 1q & 2q \\ \hline 3 \\ \cline{1-1} \end{array}} &= t^{-2} X_1 X_2 \otimes e_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}} + t^{-2}\left(\frac{1-t}{1-qt^2}\right) X_2 X_3 \otimes e_{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}} \\ &+ \frac{t^{-2}}{1+t}\left(\frac{1-t}{1-qt^2}\right) X_2 X_3 \otimes e_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}} - t^{-3}\left(\frac{1-t}{1-qt^2}\right) X_1 X_3 \otimes e_{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}} \\ &+ \frac{t^{-1}}{1+t}\left(\frac{1-t}{1-qt^2}\right) X_1 X_3 \otimes e_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}} \end{aligned}$$

REMARK 28. *Note that from Proposition 3.3.5 we get that*

$$\gamma_n(F_\tau) = t^{n-1+c_\tau(1)} F_{\Psi(\tau)}.$$

By induction we see that

$$\gamma_n^r(F_\tau) = t^{r(n-1)} t^{c_\tau(1)+\ldots+c_\tau(r)} F_{\Psi^r(\tau)}.$$

We now look at the $\mathscr{A}_n$-submodules of V_λ.

PROPOSITION 3.3.6. *The $\mathscr{D}_n^+$-module V_λ has the following decomposition into $\mathscr{A}_n$-submodules:*

$$\operatorname{Res}^{\mathscr{D}_n^+}_{\mathscr{A}_n} V_\lambda = \bigoplus_{T \in \mathrm{RYT}_{\geq 0}(\lambda)} U_T$$

where $U_T := span_{\mathbb{Q}(q,t)}\{F_\tau | \tau \in \mathrm{PSYT}_{\geq 0}(\lambda; T)\}$. Further, each $\mathscr{A}_n$-module U_T is irreducible.

PROOF. Let $T \in \mathrm{RYT}_{\geq 0}(\lambda)$. Note that it follows immediately from Proposition 3.3.5 that each U_T is a $\mathscr{A}_n$-submodule of V_λ. Further, trivially $U_T \cap U_{T'} = \emptyset$ for $T \neq T'$ since the F_τ are a basis for V_λ and the sets $\mathrm{PSYT}_{\geq 0}(\lambda; T)$ partition $\mathrm{PSYT}_{\geq 0}(\lambda)$. Therefore,

$$\operatorname{Res}^{\mathscr{D}_n^+}_{\mathscr{A}_n} V_\lambda = \bigoplus_{T \in \mathrm{RYT}_{\geq 0}(\lambda)} U_T.$$

Now let $T \in \mathrm{RYT}_{\geq 0}(\lambda)$. If $U \subset U_T$ is a nonzero $\mathscr{A}_n$-submodule then U must contain some $\theta^{(n)}$ weight vector as U_T is spanned by $\theta^{(n)}$ weight vectors. Thus there exists some $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda; T)$ with $F_{\tau_0} \in U$. But then it is follows readily from Proposition 3.3.5 that by using intertwiner operators φ_i given any $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda)$ we may find $A \in \mathscr{A}_n$ such that $A(F_{\tau_0}) = F_\tau$. Therefore, $U = U_T$ and hence U_T is irreducible. $\square$

REMARK 29. *It follows by using Frobenius Reciprocity and Proposition 3.3.6 that in fact there are surjective $\mathscr{A}_n$ module maps*

$$\mathrm{Ind}_{\mathscr{A}_{\mu(T)}}^{\mathscr{A}_n} \chi_T \to U_T$$

where χ_T is the 1-dimensional representation of $\mathscr{A}_{\mu(T)}$ determined by the $\theta^{(n)}$-weight of $F_{\min(T)}$ and $T_i \to 1$ for relevant T_i. Thus each U_T is a quotient of an induced module from a parabolic subalgebra of $\mathscr{A}_n$. In the case of $T \in \mathrm{RSSYT}_{\geq 0}(\lambda)$ this map is an isomorphism. We may witness the implied bijection between $\mathrm{PSYT}_{\geq 0}(\lambda; T)$ and $\mathfrak{S}_n/\mathfrak{S}_{\mu(T)}$ combinatorially using the map $\sigma \to \sigma(\min(T))$ for $\sigma \in \mathfrak{S}_n/\mathfrak{S}_{\mu(T)}$. It is straightforward to check by decomposing λ into horizontal strip diagrams where T is constant along rows that this map is actually an isomorphism of posets.

The following lemma exhibits triangularity for the T_i^{-1} operators with respect to the reversed Bruhat order on $\mathbb{Z}^n_{\geq 0}$.

LEMMA 3.3.7. *For $1 \leq i \leq n-1$ and $a \geq 0$,*

$$(tT_i^{-1})X_{i+1}^a = X_i^a(tT_i^{-1}) + (t-1)X_{i+1}\frac{X_i^a - X_{i+1}^a}{X_i - X_{i+1}}.$$

Further, every monomial occurring in the term $X_{i+1}\frac{X_i^a - X_{i+1}^a}{X_i - X_{i+1}}$ is strictly lower than X_i^a with respect to the Bruhat ordering $\preceq$. Consequently, it follows that for any $\alpha \in \mathbb{Z}^n_{\geq 0}$ with $s_i(\alpha) \succeq \alpha$ the following expansion holds for some scalars c_β

$$(tT_i^{-1})X^\alpha = X^{s_i(\alpha)}(tT_i^{-1}) + \sum_{\beta \prec s_i(\alpha)} c_\beta X^\beta.$$

Proof. We start with

$$\begin{aligned}
tT_i^{-1}X_{i+1}^a &= (T_i+t-1)X_{i+1}^a \\
&= T_iX_{i+1}^a + (t-1)X_{i+1}^a \\
&= X_i^aT_i + (1-t)X_i\frac{X_{i+1}^a - X_i^a}{X_i - X_{i+1}} - (1-t)X_{i+1}^a \\
&= X_i^a(tT_i^{-1}+1-t) + (1-t)X_i\frac{X_{i+1}^a - X_i^a}{X_i - X_{i+1}} - (1-t)X_{i+1}^a \\
&= X_i^atT_i^{-1} + (1-t)X_i^a - (1-t)X_{i+1}^a + (1-t)X_i\frac{X_{i+1}^a - X_i^a}{X_i - X_{i+1}} \\
&= X_i^atT_i^{-1} + (t-1)X_{i+1}\frac{X_{i+1}^a - X_i^a}{X_{i+1} - X_i}.
\end{aligned}$$

Further,

$$X_{i+1}\frac{X_{i+1}^a - X_i^a}{X_{i+1} - X_i} = X_{i+1}^a + X_{i+1}^{a-1}X_i + \ldots + X_{i+1}^2X_i^{a-2} + X_{i+1}X_i^{a-1}$$

so that

$$tT_i^{-1}X_{i+1}^a = X_i^atT_i^{-1} + (t-1)(X_{i+1}^a + X_{i+1}^{a-1}X_i + \ldots + X_{i+1}^2X_i^{a-2} + X_{i+1}X_i^{a-1}).$$

Now let $\alpha \in \mathbb{Z}_{\geq 0}^n$ with $s_i(\alpha) \succ \alpha$ i.e. $\alpha_i < \alpha_{i+1}$. Then

$$\begin{aligned}
&tT_i^{-1}X^\alpha \\
&= tT_i^{-1}X_1^{\alpha_1}\cdots X_{i-1}^{\alpha_{i-1}}X_i^{\alpha_i}X_{i+1}^{\alpha_{i+1}}X_{i+2}^{\alpha_{i+2}}\cdots X_n^{\alpha_n} \\
&= X_1^{\alpha_1}\cdots X_{i-1}^{\alpha_{i-1}}X_{i+2}^{\alpha_{i+2}}\cdots X_n^{\alpha_n}tT_i^{-1}X_i^{\alpha_i}X_{i+1}^{\alpha_{i+1}} \\
&= X_1^{\alpha_1}\cdots X_{i-1}^{\alpha_{i-1}}X_i^{\alpha_i}X_{i+1}^{\alpha_i}X_{i+2}^{\alpha_{i+2}}\cdots X_n^{\alpha_n}tT_i^{-1}X_{i+1}^{\alpha_{i+1}-\alpha_i} \\
&= X_1^{\alpha_1}\cdots X_{i-1}^{\alpha_{i-1}}X_i^{\alpha_i}X_{i+1}^{\alpha_i}X_{i+2}^{\alpha_{i+2}}\cdots X_n^{\alpha_n} \\
&\quad\times\left(X_i^{\alpha_{i+1}-\alpha_i}tT_i^{-1} + (t-1)(X_{i+1}^{\alpha_{i+1}-\alpha_i} + X_{i+1}^{\alpha_{i+1}-\alpha_i-1}X_i + \ldots + X_{i+1}X_i^{\alpha_{i+1}-\alpha_i-1})\right) \\
&= X^{s_i(\alpha)}tT_i^{-1} + (t-1)\sum_{j=0}^{\alpha_{i+1}-\alpha_i-1} X^{\alpha+j(e_i-e_{i+1})}.
\end{aligned}$$

Lastly, from Definition 3.2.6 it is clear that for all $0 \leq j \leq \alpha_{i+1} - \alpha_i - 1$, $s_i(\alpha) \succ \alpha + j(e_i - e_{i+1})$. $\square$

Now we show that each F_τ has a triangular monomial expansion of a certain form. It will be important to identify explicitly the vector-valued leading term of the F_τ as this will be crucial when defining the stable-limits of the symmetric v.v. Macdonald polynomials.

COROLLARY 3.3.8. *For $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda)$ each F_τ has a triangular monomial expansion with respect to the reversed Bruhat order on $\mathbb{Z}^n_{\geq 0}$ of the form*

$$F_\tau = X^{w_\tau} \otimes f(\tau) + \sum_{\beta \prec w_\tau} X^\beta \otimes v_\beta$$

for some $v_\beta \in S_\lambda$ where $f(\tau) \in S_\lambda$ is given by the following recurrence relations:

- *If $\tau \in \mathrm{SYT}(\lambda)$ then $f(\tau) = e_\tau$.*
- $f(\Psi(\tau)) = t^{-(n-1)} T_{n-1} \cdots T_1(f(\tau))$
- *If $w_\tau(i) < w_\tau(i+1)$ then $f(s_i(\tau)) = tT_i^{-1} f(\tau)$.*
- *If $w_\tau(i) = w_\tau(i+1)$ and $c_\tau(i) - c_\tau(i+1) > 1$ then*

$$f(s_i(\tau)) = \left(tT_i^{-1} + \frac{(t-1)t^{c_\tau(i+1)}}{t^{c_\tau(i)} - t^{c_\tau(i+1)}} \right) (f(\tau)).$$

PROOF. We will proceed by induction with respect to the partial ordering on $\mathrm{PSYT}_{\geq 0}(\lambda)$ defined in Definition 3.2.2. We will at the same time verify the recurrence relations given for $f(\tau) \in S_\lambda$ given above.

From Proposition 3.3.5 we know that if $\tau \in \mathrm{SYT}(\lambda)$ then $F_\tau = 1 \otimes e_\tau$. Hence, F_τ trivially has a triangular monomial expansion of the correct form in this case and that $f(\tau) = e_\tau$.

In what follows assume that for $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda)$ we have that

$$F_\tau = X^{w_\tau} \otimes f(\tau) + \sum_{\beta \prec w_\tau} X^\beta \otimes v_\beta$$

for some $v_\beta \in S_\lambda$.

First, we see that

$$\begin{aligned}
F_{\Psi(\tau)} &= q^{w_1(\tau)} X_n \pi_n^{-1}(F_\tau) \\
&= q^{w_1(\tau)} X_n \pi_n^{-1} X^{w_\tau} \otimes f(\tau) + \sum_{\beta \prec w_\tau} q^{w_1(\tau)} X_n \pi_n^{-1} X^{\beta} \otimes v_\beta \\
&= q^{w_1(\tau)} q^{-w_1(\tau)} X^{\widetilde{\gamma}(w_\tau)} \pi_n^{-1} \otimes f(\tau) + \sum_{\beta \prec w_\tau} q^{w_1(\tau)} q^{-\beta_1} X^{\widetilde{\gamma}(\beta)} \pi_n^{-1} \otimes v_\beta \\
&= X^{\widetilde{\gamma}(w_\tau)} \otimes \rho_n(\pi_n^{-1}) f(\tau) + \sum_{\beta \prec w_\tau} X^{\widetilde{\gamma}(\beta)} \otimes q^{w_1(\tau)-\beta_1} \rho_n(\pi_n^{-1}) v_\beta \\
&= X^{\widetilde{\gamma}(w_\tau)} \otimes t^{-(n-1)} T_{n-1} \cdots T_1 f(\tau) + \sum_{\beta \prec w_\tau} X^{\widetilde{\gamma}(\beta)} \otimes q^{w_1(\tau)-\beta_1} t^{-(n-1)} T_{n-1} \cdots T_1 v_\beta.
\end{aligned}$$

From Lemma 3.2.7 we know that if $\beta \prec w_\tau$ then $\widetilde{\gamma}(\beta) \prec \widetilde{\gamma}(w_\tau)$. Therefore, we find that $F_{\Psi(\tau)}$ has the expansion

$$F_{\Psi(\tau)} = X^{\widetilde{\gamma}(w_\tau)} \otimes t^{-(n-1)} T_{n-1} \cdots T_1 f(\tau) + \sum_{\beta \prec \widetilde{\gamma}(\tau)} X^{\beta} \otimes v'_\beta$$

for some $v'_\beta \in S_\lambda$. From this we see that $f(\Psi(\tau)) = t^{-(n-1)} T_{n-1} \cdots T_1(f(\tau))$.

Now suppose $s_i(\tau) > \tau$. From Proposition 3.3.5 we get

$$\begin{aligned}
F_{s_i(\tau)} &= \left(tT_i^{-1} + \frac{(t-1) q^{w_\tau(i+1)} t^{c_\tau(i+1)}}{q^{w_\tau(i)} t^{c_\tau(i)} - q^{w_\tau(i+1)} t^{c_\tau(i+1)}} \right) (F_\tau) \\
&= \left(tT_i^{-1} + \frac{(t-1) q^{w_\tau(i+1)} t^{c_\tau(i+1)}}{q^{w_\tau(i)} t^{c_\tau(i)} - q^{w_\tau(i+1)} t^{c_\tau(i+1)}} \right) \left(X^{w_\tau} \otimes f(\tau) + \sum_{\beta \prec w_\tau} X^{\beta} \otimes v_\beta \right) \\
&= tT_i^{-1} \left(X^{w_\tau} \otimes f(\tau) + \sum_{\beta \prec w_\tau} X^{\beta} \otimes v_\beta \right) \\
&\quad + \left(\frac{(t-1) q^{w_\tau(i+1)} t^{c_\tau(i+1)}}{q^{w_\tau(i)} t^{c_\tau(i)} - q^{w_\tau(i+1)} t^{c_\tau(i+1)}} \right) \left(X^{w_\tau} \otimes f(\tau) + \sum_{\beta \prec w_\tau} X^{\beta} \otimes v_\beta \right).
\end{aligned}$$

For any $\beta < w_\tau$ using Lemma 3.3.7 we find that

$$tT_i^{-1} X^{\beta} \otimes v_\beta = \sum_{\beta' \prec s_i(w_\tau)} X^{\beta'} \otimes u_{\beta',\beta}$$

for some $u_{\beta',\beta} \in S_\lambda$; that is to say, each of the monomials $X^{\beta'}$ that appears in the standard basis expansion of $tT_i^{-1}X^\beta \otimes v_\beta$ must have $\beta' \prec s_i(w_\tau)$.

Assume $w_\tau(i) < w_\tau(i+1)$. By Lemma 3.3.7 we see

$$\begin{aligned}(tT_i^{-1})X^{w_\tau} \otimes f(\tau) &= X^{s_i(w_\tau)}(tT_i^{-1}) \otimes f(\tau) + \sum_{\beta \prec s_i(w_\tau)} c_\beta X^\beta \otimes f(\tau) \\ &= X^{s_i(w_\tau)} \otimes (tT_i^{-1})f(\tau) + \sum_{\beta \prec s_i(w_\tau)} c_\beta X^\beta \otimes f(\tau).\end{aligned}$$

Therefore, $F_{s_i(\tau)}$ has the expansion

$$F_{s_i(\tau)} = X^{s_i(w_\tau)} \otimes tT_i^{-1} f(\tau) + \sum_{\beta \prec s_i(w_\tau)} X^\beta \otimes v'_\beta$$

where $v'_\beta \in S_\lambda$. Since $s_i(w_\tau) = w_{s_i(\tau)}$ we have

$$F_{s_i(\tau)} = X^{w_{s_i(\tau)}} \otimes tT_i^{-1} f(\tau) + \sum_{\beta \prec w_{s_i(\tau)}} X^\beta \otimes v'_\beta$$

and $f(s_i(\tau)) = tT_i^{-1} f(\tau)$.

Now assume instead that $w_\tau(i) = w_\tau(i+1)$ and $c_\tau(i) - c_\tau(i+1) > 1$. Then $T_i X^{w_\tau} = X^{w_\tau} T_i$ so

$$\begin{aligned}F_{s_i(\tau)} &= \left(tT_i^{-1} + \frac{(t-1)q^{w_\tau(i+1)}t^{c_\tau(i+1)}}{q^{w_\tau(i)}t^{c_\tau(i)} - q^{w_\tau(i+1)}t^{c_\tau(i+1)}} \right)(F_\tau) \\ &= \left(tT_i^{-1} + \frac{(t-1)t^{c_\tau(i+1)}}{t^{c_\tau(i)} - t^{c_\tau(i+1)}} \right)(F_\tau) \\ &= \left(tT_i^{-1} + \frac{(t-1)t^{c_\tau(i+1)}}{t^{c_\tau(i)} - t^{c_\tau(i+1)}} \right)\left(X^{w_\tau} \otimes f(\tau) + \sum_{\beta \prec w_\tau} X^\beta \otimes v_\beta \right)\end{aligned}$$

$$= \left(tT_i^{-1} + \frac{(t-1)t^{c_\tau(i+1)}}{t^{c_\tau(i)} - t^{c_\tau(i+1)}}\right) X^{w_\tau} \otimes f(\tau) + \left(tT_i^{-1} + \frac{(t-1)t^{c_\tau(i+1)}}{t^{c_\tau(i)} - t^{c_\tau(i+1)}}\right) \sum_{\beta \prec w_\tau} X^\beta \otimes v_\beta$$
$$= X^{w_\tau} \left(tT_i^{-1} + \frac{(t-1)t^{c_\tau(i+1)}}{t^{c_\tau(i)} - t^{c_\tau(i+1)}}\right) \otimes f(\tau) + \left(tT_i^{-1} + \frac{(t-1)t^{c_\tau(i+1)}}{t^{c_\tau(i)} - t^{c_\tau(i+1)}}\right) \sum_{\beta \prec w_\tau} X^\beta \otimes v_\beta$$
$$= X^{w_\tau} \otimes \left(tT_i^{-1} + \frac{(t-1)t^{c_\tau(i+1)}}{t^{c_\tau(i)} - t^{c_\tau(i+1)}}\right) f(\tau) + \left(tT_i^{-1} + \frac{(t-1)t^{c_\tau(i+1)}}{t^{c_\tau(i)} - t^{c_\tau(i+1)}}\right) \sum_{\beta \prec w_\tau} X^\beta \otimes v_\beta.$$

Therefore, since $w_\tau = w_{s_i(\tau)}$ we find that

$$F_{s_i(\tau)} = X^{w_{s_i(\tau)}} \otimes \left(tT_i^{-1} + \frac{(t-1)t^{c_\tau(i+1)}}{t^{c_\tau(i)} - t^{c_\tau(i+1)}}\right) f(\tau) + \sum_{\beta \prec w_{s_i(\tau)}} X^\beta \otimes v'_\beta$$

for some $v'_\beta \in S_\lambda$ and

$$f(s_i(\tau)) = \left(tT_i^{-1} + \frac{(t-1)t^{c_\tau(i+1)}}{t^{c_\tau(i)} - t^{c_\tau(i+1)}}\right)(f(\tau)).$$

□

Using the ζ_i operators on $\mathrm{PSYT}_{\geq 0}(\lambda)$ we may compute $f(\mathrm{top}(T))$ explicitly.

PROPOSITION 3.3.9. *For $T \in \mathrm{RYT}_{\geq 0}(\lambda)$ we have that*

$$f(\mathrm{top}(T)) = \mathscr{C}_1^{\nu(T)_1 - \nu(T)_2} \cdots \mathscr{C}_n^{\nu(T)_n}(e_{S(T)})$$

where define for $1 \leq i \leq n$,

$$\mathscr{C}_i := \left((tT_i^{-1}) \cdots (tT_{n-1}^{-1})(t^{-(n-1)}T_{n-1} \cdots T_1)\right)^i.$$

PROOF. Using the recurrence relations in Corollary 3.3.8 for the elements $f(\tau)$ and Proposition 3.2.14 we see that for any $T \in \mathrm{RSSYT}_{\geq 0}(\lambda)$ since

$$\mathrm{top}(T) = \zeta_1^{\nu(T)_1 - \nu(T)_2} \cdots \zeta_n^{\nu(T)_n}(S(T))$$

with each $\zeta_i := (s_i \cdots s_{n-1}\Psi)^i$ then we have a similar expression for $f(\mathrm{top}(T))$:

$$f(\mathrm{top}(T)) = \mathscr{C}_1^{\nu(T)_1 - \nu(T)_2} \cdots \mathscr{C}_n^{\nu(T)_n}(e_{S(T)})$$

where $\mathscr{C}_i := \big((tT_i^{-1})\cdots(tT_{n-1}^{-1})(t^{-(n-1)}T_{n-1}\cdots T_1)\big)^i$ is obtained by replacing each s_j and Ψ in ζ_i with tT_j^{-1} and $t^{-(n-1)}T_{n-1}\cdots T_1$ respectively. Importantly, when we apply ζ_i to any element of the form $\mathrm{top}(T')$ we never perform any swaps $s_j(\tau) > \tau$ such that $w_\tau(j) = w_\tau(j+1)$ and hence never require the more complicated recurrence relation:

$$f(s_j(\tau)) = \left(tT_j^{-1} + \frac{(t-1)t^{c_\tau(j+1)}}{t^{c_\tau(j)} - t^{c_\tau(j+1)}}\right)(f(\tau)).$$

$\square$

The $\mathscr{C}_i$ operators can be identified concretely using the $\overline{\theta}_j$ elements of the finite Hecke algebra.

Lemma 3.3.10. *For all* $1 \le i \le n$,

$$\mathscr{C}_i = A_i \cdots A_1$$

where $A_j := t^{-(j-1)}\overline{\theta}_j^{-1}$.

Proof. Let $0 \le k \le i-1$. We first show by induction that

$$(t^{i-1}\overline{\theta}_i^{-1})\cdots(t^{i-k-1}\overline{\theta}_{i-k}^{-1}) = (T_{i-1}\cdots T_1)^{k+1}(T_{k+1}\cdots T_{i-1})(T_k\cdots T_{i-2})\cdots(T_1\cdots T_{i-k-1}).$$

To start we see that for $k=0$ we have

$$t^{i-1}\overline{\theta}_i^{-1} = T_{i-1}\cdots T_1^2\cdots T_{i-1} = (T_{i-1}\cdots T_1)^1(T_1\cdots T_{i-1}).$$

Now suppose that for $0 \leq k \leq i-2$ the formula above holds. Then

$$
\begin{aligned}
&(t^{i-1}\overline{\theta}_i^{-1}) \cdots (t^{i-(k+1)}\overline{\theta}_{i-(k+1)}^{-1}) \\
&= (T_{i-1} \cdots T_1)^{k+1}(T_{k+1} \cdots T_{i-1})(T_k \cdots T_{i-2}) \cdots (T_1 \cdots T_{i-k-1})(t^{i-(k+1)}\overline{\theta}_{i-(k+1)}^{-1}) \\
&= (T_{i-1} \cdots T_1)^{k+1}(T_{k+1} \cdots T_{i-1})(T_k \cdots T_{i-2}) \cdots (T_1 \cdots T_{i-k-1})(T_{i-k-2} \cdots T_1^2 \cdots T_{i-k-2}) \\
&= (T_{i-1} \cdots T_1)^{k+1}(T_{k+1} \cdots T_{i-1})(T_k \cdots T_{i-2}) \cdots (T_2 \cdots T_{i-k})(T_1 \cdots T_{i-k-1})(T_{i-k-2} \cdots T_1) \\
&\times (T_1 \cdots T_{i-k-2}) \\
&= (T_{i-1} \cdots T_1)^{k+1}(T_{k+1} \cdots T_{i-1})(T_k \cdots T_{i-2}) \cdots (T_2 \cdots T_{i-k})(T_{i-k-1} \cdots T_2)(T_1 \cdots T_{i-k-1}) \\
&\times (T_1 \cdots T_{i-k-2}) \\
&= (T_{i-1} \cdots T_1)^{k+1}(T_{k+1} \cdots T_{i-1})(T_k \cdots T_{i-2}) \cdots (T_2 \cdots T_{i-k})(T_{i-k-1} \cdots T_1)(T_2 \cdots T_{i-k-1}) \\
&\times (T_1 \cdots T_{i-k-2}) \\
&= (T_{i-1} \cdots T_1)^{k+1}(T_{k+1} \cdots T_{i-1})(T_k \cdots T_{i-2}) \cdots (T_3 \cdots T_{i-k+1})(T_2 \cdots T_{i-k})(T_{i-k-1} \cdots T_1) \\
&\times (T_2 \cdots T_{i-k-1})(T_1 \cdots T_{i-k-2}) \\
&= (T_{i-1} \cdots T_1)^{k+1}(T_{k+1} \cdots T_{i-1})(T_k \cdots T_{i-2}) \cdots (T_3 \cdots T_{i-k+1})(T_{i-k} \cdots T_1)(T_3 \cdots T_{i-k}) \\
&\times (T_2 \cdots T_{i-k-1})(T_1 \cdots T_{i-k-2}) \\
&= \cdots \\
&= (T_{i-1} \cdots T_1)^{k+1}(T_{i-1} \cdots T_1)(T_{k+2} \cdots T_{i-1})(T_{k+1} \cdots T_{i-2}) \cdots (T_1 \cdots T_{i-k-2}) \\
&= (T_{i-1} \cdots T_1)^{k+2}(T_{k+2} \cdots T_{i-1})(T_{k+1} \cdots T_{i-2}) \cdots (T_1 \cdots T_{i-k-2}).
\end{aligned}
$$

By taking $k = i-1$ we find

$$(t^{i-1}\overline{\theta}_i^{-1}) \cdots (t^0\overline{\theta}_1^{-1}) = (T_{i-1} \cdots T_1)^i.$$

Now we see that

$$\begin{aligned}
\mathfrak{C}_i &= \Big((tT_i^{-1})\cdots(tT_{n-1}^{-1})(t^{-(n-1)}T_{n-1}\cdots T_1)\Big)^i \\
&= t^{-i(i-1)}(T_{i-1}\cdots T_1)^i \\
&= t^{-i(i-1)}(t^{i-1}\overline{\theta}_i^{-1})\cdots(t^0\overline{\theta}_1^{-1}) \\
&= t^{-2(i-1)-2(i-2)-\ldots-2(1)-2(0)}(t^{i-1}\overline{\theta}_i^{-1})\cdots(t^0\overline{\theta}_1^{-1}) \\
&= (t^{-(i-1)}\overline{\theta}_i^{-1})\cdots(t^{-0}\overline{\theta}_1^{-1}) \\
&= A_i\cdots A_1
\end{aligned}$$

where $A_j := t^{-(j-1)}\overline{\theta}_j^{-1}$. □

Putting the results of this section together gives the following:

Corollary 3.3.11. *For $T \in \mathrm{RYT}_{\geq 0}(\lambda)$, the triangular expansion of $F_{\mathrm{top}(T)}$ has the form*

$$F_{\mathrm{top}(T)} = t^{-b_T}X^{\nu(T)} \otimes e_{S(T)} + \sum_{\beta \prec \nu(T)} X^{\beta} \otimes v_{\beta}$$

for some $v_\beta \in S_\lambda$.

Proof. First, notice that for $T \in \mathrm{RYT}_{\geq 0}(\lambda)$ $w_{\mathrm{top}(T)} = \nu(T)$. From Proposition 3.3.9 and Lemma 3.2.12

$$\begin{aligned}
f(\mathrm{top}(T)) &= \mathscr{C}_1^{\nu(T)_1-\nu(T)_2}\cdots\mathscr{C}_n^{\nu(T)_n}(e_{S(T)}) \\
&= A_1^{\nu(T)_1-\nu(T)_2}(A_1A_2)^{\nu(T)_2-\nu(T)_3}\cdots(A_1\cdots A_n)^{\nu(T)_n}(e_{S(T)}) \\
&= A_1^{(\nu(T)_1-\nu(T)_2)+\ldots+(\nu(T)_{n-1}-\nu(T)_n)+\nu(T)_n}\cdots A_{n-1}^{(\nu(T)_{n-1}-\nu(T)_n)+\nu(T)_n}A_n^{\nu(T)_n}(e_{S(T)}) \\
&= (\overline{\theta}_1^{-1})^{\nu(T)_1}\cdots(\overline{\theta}_n^{-1})^{\nu(T)_n}(e_{S(T)}) \\
&= t^{-\nu(T)_1(c_{S(T)}(1)-(1-1))}\cdots t^{-\nu(T)_n(c_{S(T)}(n)-(n-1))}e_{S(T)} \\
&= t^{-\sum_{i=1}^n \nu(T)_i(c_{S(T)}(i)+i-1)}e_{S(T)} \\
&= t^{-b_T}e_{S(T)}.
\end{aligned}$$

Therefore, the leading term of $F_{\mathrm{top}(T)}$ is

$$X^{w_{\mathrm{top}(T)}} \otimes f(\mathrm{top}(T)) = t^{-b_T} X^{\nu(T)} \otimes e_{S(T)}.$$

□

3.3.3. Connecting Maps Between $V_{\lambda(n)}$. We now construct special maps between the v.v. polynomial DAHA modules which satisfy particular stability properties.

Definition 3.3.12. *Let $\lambda \in \mathbb{Y}$. For $n \geq n_\lambda$ define $\Phi_\lambda^{(n)} : V_{\lambda(n+1)} \to V_{\lambda(n)}$ as the $\mathbb{Q}(q,t)$-linear map given on any element $X^\alpha \otimes v \in V_{\lambda(n+1)}$ by*

$$\Phi_\lambda^{(n)}(X^\alpha \otimes v) = \mathbb{1}(\alpha_{n+1} = 0) X_1^{\alpha_1} \cdots X_n^{\alpha_n} \otimes \mathfrak{q}_\lambda^{(n)}(v).$$

Proposition 3.3.13. *The map $\Phi_\lambda^{(n)}$ satisfies the following relations:*

- $\Phi_\lambda^{(n)} T_i = T_i \Phi_\lambda^{(n)}$ *for* $1 \leq i \leq n-1$
- $\Phi_\lambda^{(n)} X_i = X_i \Phi_\lambda^{(n)}$ *for* $1 \leq i \leq n$
- $\Phi_\lambda^{(n)} X_{n+1} = 0$
- $\Phi_\lambda^{(n)} t^{-n} \pi_{n+1} T_n = t^{-(n-1)} \pi_n \Phi_\lambda^{(n)}$
- $\Phi_\lambda^{(n)} \theta_i^{(n+1)} = \theta_i^{(n)} \Phi_\lambda^{(n)}$ *for* $1 \leq i \leq n$
- $\Phi_\lambda^{(n)} (\theta_{n+1}^{(n+1)} - t^{n-|\lambda|}) = 0.$

Proof. From Lemma 3.3.3 and Definition 3.3.12 it follows immediately for all $1 \leq i \leq n-1$ and $1 \leq j \leq n$ that $\Phi_\lambda^{(n)} T_i = T_i \Phi_\lambda^{(n)}$, $\Phi_\lambda^{(n)} X_j = X_j \Phi_\lambda^{(n)}$, and $\Phi_\lambda^{(n)} X_{n+1} = 0$.

Let $X^\alpha \otimes v \in V_{\lambda(n+1)}$. By direct calculation we find

$$
\begin{aligned}
&\Phi_\lambda^{(n)} t^{-n}\pi_{n+1}T_n(X_1^{\alpha_1}\cdots X_{n+1}^{\alpha_{n+1}}\otimes v)\\
&= \Phi_\lambda^{(n)} t^{-n}\pi_{n+1}X_1^{\alpha_1}\cdots X_{n-1}^{\alpha_{n-1}}T_n(X_n^{\alpha_n}X_{n+1}^{\alpha_{n+1}}\otimes v)\\
&= \Phi_\lambda^{(n)} t^{-n}X_2^{\alpha_1}\cdots X_n^{\alpha_{n-1}}\pi_{n+1}T_n(X_n^{\alpha_n}X_{n+1}^{\alpha_{n+1}}\otimes v)\\
&= t^{-n}X_2^{\alpha_1}\cdots X_n^{\alpha_{n-1}}\Phi_\lambda^{(n)}\pi_{n+1}T_n(X_n^{\alpha_n}X_{n+1}^{\alpha_{n+1}}\otimes v)\\
&= t^{-n}X_2^{\alpha_1}\cdots X_n^{\alpha_{n-1}}\Phi_\lambda^{(n)}\pi_{n+1}\left(X_{n+1}^{\alpha_n}X_n^{\alpha_{n+1}}T_n\otimes v+(1-t)X_n\frac{X_n^{\alpha_n}X_{n+1}^{\alpha_{n+1}}-X_{n+1}^{\alpha_n}X_n^{\alpha_{n+1}}}{X_n-X_{n+1}}\otimes v\right)\\
&= t^{-n}X_2^{\alpha_1}\cdots X_n^{\alpha_{n-1}}\\
&\times\Phi_\lambda^{(n)}\left(q^{\alpha_n}X_1^{\alpha_n}X_{n+1}^{\alpha_{n+1}}\pi_{n+1}T_n\otimes v+(1-t)X_{n+1}\pi_{n+1}\frac{X_n^{\alpha_n}X_{n+1}^{\alpha_{n+1}}-X_{n+1}^{\alpha_n}X_n^{\alpha_{n+1}}}{X_n-X_{n+1}}\otimes v\right)\\
&= \mathbb{1}(\alpha_{n+1}=0)t^{-n}q^{\alpha_n}X_1^{\alpha_n}X_2^{\alpha_1}\cdots X_n^{\alpha_{n-1}}\Phi_\lambda^{(n)}(1\otimes\rho_{n+1}(\pi_{n+1}T_n)v)\\
&= \mathbb{1}(\alpha_{n+1}=0)t^{-n}q^{\alpha_n}X_1^{\alpha_n}X_2^{\alpha_1}\cdots X_n^{\alpha_{n-1}}\Phi_\lambda^{(n)}(1\otimes t^nT_1^{-1}\cdots T_{n-1}^{-1}v)\\
&= \mathbb{1}(\alpha_{n+1}=0)q^{\alpha_n}X_1^{\alpha_n}X_2^{\alpha_1}\cdots X_n^{\alpha_{n-1}}\otimes T_1^{-1}\cdots T_{n-1}^{-1}\mathfrak{q}_\lambda^{(n)}(v).
\end{aligned}
$$

On the other hand we see

$$
\begin{aligned}
&t^{-(n-1)}\pi_n\Phi_\lambda^{(n)}(X_1^{\alpha}\cdots X_{n+1}^{\alpha_{n+1}}\otimes v)\\
&= \mathbb{1}(\alpha_{n+1}=0)t^{-(n-1)}\pi_n(X_1^{\alpha_1}\cdots X_n^{\alpha_n}\otimes\mathfrak{q}_\lambda^{(n)}(v))\\
&= \mathbb{1}(\alpha_{n+1}=0)t^{-(n-1)}q^{\alpha_n}X_1^{\alpha_n}X_2^{\alpha_1}\cdots X_n^{\alpha_{n-1}}\otimes\rho_n(\pi_n)(\mathfrak{q}_\lambda^{(n)}(v))\\
&= \mathbb{1}(\alpha_{n+1}=0)q^{\alpha_n}X_1^{\alpha_n}X_2^{\alpha_1}\cdots X_n^{\alpha_{n-1}}\otimes T_1^{-1}\cdots T_{n-1}^{-1}\mathfrak{q}_\lambda^{(n)}(v).
\end{aligned}
$$

Therefore, $\Phi_\lambda^{(n)}t^{-n}\pi_{n+1}T_n=t^{-(n-1)}\pi_n\Phi_\lambda^{(n)}$ as desired.

Now let $1 \leq i \leq n$. We see that

$$\begin{aligned}
\Phi_\lambda^{(n)}\theta_i^{(n+1)} &= \Phi_\lambda^{(n)} t^{-(n-i+1)} T_{i-1}^{-1} \cdots T_1^{-1} \pi_{n+1} T_n \cdots T_i \\
&= t^{i-1} T_{i-1}^{-1} \cdots T_1^{-1} (\Phi_\lambda^{(n)} t^{-n} \pi_{n+1} T_n) T_{n-1} \cdots T_i \\
&= t^{i-1} T_{i-1}^{-1} \cdots T_1^{-1} (t^{-(n-1)} \pi_n \Phi_\lambda^{(n)}) T_{n-1} \cdots T_i \\
&= t^{-(n-i)} T_{i-1}^{-1} \cdots T_1^{-1} \pi_n T_{n-1} \cdots T_i \Phi_\lambda^{(n)} \\
&= \theta_i^{(n)} \Phi_\lambda^{(n)}.
\end{aligned}$$

Now let $\alpha \in \mathbb{Z}_{\geq 0}^{n+1}$ and $\tau \in \mathrm{SYT}(\lambda^{(n+1)})$. We find

$$\begin{aligned}
&\Phi_\lambda^{(n)}\theta_{n+1}^{(n+1)}(X^\alpha \otimes e_\tau) \\
&= \Phi_\lambda^{(n)} T_n^{-1} \cdots T_1^{-1} \pi_{n+1}(X^\alpha \otimes e_\tau) \\
&= \Phi_\lambda^{(n)} T_n^{-1} \cdots T_1^{-1} q^{\alpha_{n+1}} X_1^{\alpha_{n+1}} X_2^{\alpha_1} \cdots X_{n+1}^{\alpha_n} \otimes \rho_{n+1}(\pi_{n+1}) e_\tau \\
&= q^{\alpha_{n+1}} \Phi_\lambda^{(n)} T_n^{-1} \cdots T_1^{-1} X_1^{\alpha_{n+1}} X_2^{\alpha_1} \cdots X_{n+1}^{\alpha_n} (1 \otimes t^n T_1^{-1} \cdots T_n^{-1}(e_\tau)).
\end{aligned}$$

Now if $\alpha_{n+1} > 0$ then this evaluates to 0 since

$$\Phi_\lambda^{(n)} T_n^{-1} \cdots T_1^{-1} X_1 = t^{-n} \Phi_\lambda^{(n)} X_{n+1} T_n \cdots T_1 = 0.$$

Hence,

$$\Phi_\lambda^{(n)}\theta_{n+1}^{(n+1)}(X^\alpha \otimes e_\tau) = \mathbb{1}(\alpha_{n+1} = 0) \Phi_\lambda^{(n)} T_n^{-1} \cdots T_1^{-1} X_2^{\alpha_1} \cdots X_{n+1}^{\alpha_n} (1 \otimes t^n T_1^{-1} \cdots T_n^{-1}(e_\tau)).$$

Now we by repeatedly applying Lemma 3.3.7 we see that as maps $V_{\lambda^{(n+1)}} \to V_{\lambda^{(n)}}$

$$
\begin{aligned}
&\Phi_\lambda^{(n)} T_n^{-1} \cdots T_2^{-1} (T_1^{-1} X_2^{\alpha_1}) X_3^{\alpha_2} \cdots X_{n+1}^{\alpha_n} \\
&= \Phi_\lambda^{(n)} T_n^{-1} \cdots T_2^{-1} \left(X_1^{\alpha_1} T_1^{-1} + (1-t^{-1}) X_2 \frac{X_1^{\alpha_1} - X_2^{\alpha_1}}{X_1 - X_2} \right) X_3^{\alpha_2} \cdots X_{n+1}^{\alpha_n} \\
&= X_1^{\alpha_1} \Phi_\lambda^{(n)} T_n^{-1} \cdots T_1^{-1} X_3^{\alpha_2} \cdots X_{n+1}^{\alpha_n} + (1-t^{-1}) \Phi_\lambda^{(n)} T_n^{-1} \cdots T_2^{-1} X_2 \frac{X_1^{\alpha_1} - X_2^{\alpha_1}}{X_1 - X_2} X_3^{\alpha_2} \cdots X_{n+1}^{\alpha_n} \\
&= X_1^{\alpha_1} \Phi_\lambda^{(n)} T_n^{-1} \cdots T_1^{-1} X_3^{\alpha_2} \cdots X_{n+1}^{\alpha_n} \\
&+ (1-t^{-1}) t^{-(n-2)} \Phi_\lambda^{(n)} X_{n+1} T_{n-1} \cdots T_2 \frac{X_1^{\alpha_1} - X_2^{\alpha_1}}{X_1 - X_2} X_3^{\alpha_2} \cdots X_{n+1}^{\alpha_n} \\
&= X_1^{\alpha_1} \Phi_\lambda^{(n)} T_n^{-1} \cdots T_1^{-1} X_3^{\alpha_2} \cdots X_{n+1}^{\alpha_n} + 0 \\
&= X_1^{\alpha_1} \Phi_\lambda^{(n)} T_n^{-1} \cdots T_3^{-1} (T_2^{-1} X_3^{\alpha_2}) T_1^{-1} X_4^{\alpha_3} \cdots X_{n+1}^{\alpha_n} \\
&= \dots \\
&= X_1^{\alpha_1} X_2^{\alpha_2} \Phi_\lambda^{(n)} T_n^{-1} \cdots T_1^{-1} X_4^{\alpha_3} \cdots X_{n+1}^{\alpha_n} \\
&= \dots \\
&= X_1^{\alpha_1} \cdots X_n^{\alpha_n} \Phi_\lambda^{(n)} T_n^{-1} \cdots T_1^{-1}.
\end{aligned}
$$

As usual let $\square_0$ denote the unique square of the skew diagram $\lambda^{(n+1)}/\lambda^{(n)}$. Returning to our main calculation now shows

$$
\begin{aligned}
&\Phi_\lambda^{(n)} \theta_{n+1}^{(n+1)} (X^\alpha \otimes e_\tau) \\
&= \mathbb{1}(\alpha_{n+1} = 0) \Phi_\lambda^{(n)} T_n^{-1} \cdots T_1^{-1} X_2^{\alpha_1} \cdots X_{n+1}^{\alpha_n} (1 \otimes t^n T_1^{-1} \cdots T_n^{-1} (e_\tau)) \\
&= \mathbb{1}(\alpha_{n+1} = 0) X_1^{\alpha_1} \cdots X_n^{\alpha_n} \Phi_\lambda^{(n)} T_n^{-1} \cdots T_1^{-1} (1 \otimes t^n T_1^{-1} \cdots T_n^{-1} (e_\tau)) \\
&= \mathbb{1}(\alpha_{n+1} = 0) X_1^{\alpha_1} \cdots X_n^{\alpha_n} \Phi_\lambda^{(n)} (1 \otimes t^n T_n^{-1} \cdots T_1^{-1} T_1^{-1} \cdots T_n^{-1} (e_\tau)) \\
&= \mathbb{1}(\alpha_{n+1} = 0) X_1^{\alpha_1} \cdots X_n^{\alpha_n} \Phi_\lambda^{(n)} (1 \otimes \overline{\theta}_{n+1}^{(n+1)} (e_\tau))
\end{aligned}
$$

$$= \mathbb{1}(\alpha_{n+1} = 0) X_1^{\alpha_1} \cdots X_n^{\alpha_n} \Phi_\lambda^{(n)}(1 \otimes t^{c_\tau(n+1)} e_\tau)$$

$$= t^{c_\tau(n+1)} \mathbb{1}(\alpha_{n+1} = 0) X_1^{\alpha_1} \cdots X_n^{\alpha_n} \otimes \mathfrak{q}_\lambda^{(n)}(e_\tau)$$

$$= t^{c(\Box_0)} \mathbb{1}(\alpha_{n+1} = 0) \mathbb{1}(\tau(\Box_0) = n+1) X_1^{\alpha_1} \cdots X_n^{\alpha_n} \otimes e_{\tau|\lambda^{(n)}}$$

$$= t^{n-|\lambda|} \mathbb{1}(\alpha_{n+1} = 0) \mathbb{1}(\tau(\Box_0) = n+1) X_1^{\alpha_1} \cdots X_n^{\alpha_n} \otimes e_{\tau|\lambda^{(n)}}$$

$$= \Phi_\lambda^{(n)}(t^{n-|\lambda|} X^\alpha \otimes e_\tau).$$

Therefore,

$$\Phi_\lambda^{(n)}(\theta_{n+1}^{(n+1)} - t^{n-|\lambda|}) = 0.$$

$\Box$

Corollary 3.3.14. *Let $n \geq n_\lambda$ and $\Box_0 = \lambda^{(n+1)}/\lambda^{(n)}$. For $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda^{(n+1)})$ we have*

$$\Phi_\lambda^{(n)}(F_\tau) := \begin{cases} F_{\tau|_{\lambda^{(n)}}} & \tau(\Box_0) = n+1 \\ 0 & \tau(\Box_0) \neq n+1. \end{cases}$$

Proof. We will first deal with the case when $\tau(\Box_0) = n+1$. Let $T \in \mathrm{RYT}_{\geq 0}(\lambda^{(n)})$ and let $T' \in \mathrm{RYT}_{\geq 0}(\lambda^{(n+1)})$ with $T'(\Box_0) = 0$ and $T'|_{\lambda^{(n)}} = T|_{\lambda^{(n)}}$. By looking at the eigenvalues of $\theta_1^{(n+1)}, \ldots, \theta_n^{(n+1)}$ on $F_{\mathrm{top}(T')}$ and the eigenvalues of $\theta_1^{(n)}, \ldots, \theta_n^{(n)}$ on $F_{\mathrm{top}(T)}$ we see that $\Phi_\lambda^{(n)}(F_{\mathrm{top}(T')}) = \beta F_{\mathrm{top}(T)}$ for some scalar β. We will now show that $\beta = 1$. From Corollary 3.3.11 we know that

$$F_{\mathrm{top}(T')} = t^{-b_{T'}} X^{\nu(T')} \otimes e_{S(T')} + \sum_{\beta \prec \nu(T')} X^\beta \otimes v'_\beta$$

and

$$F_{\mathrm{top}(T)} = t^{-b_T} X^{\nu(T)} \otimes e_{S(T)} + \sum_{\beta \prec \nu(T)} X^\beta \otimes v_\beta$$

for some $v_\beta \in S_{\lambda^{(n)}}$ and $v'_\beta \in S_{\lambda^{(n+1)}}$. Since $T'(\Box_0) = 0$ and $T'|_{\lambda^{(n)}} = T|_{\lambda^{(n)}}$, it follows that $b_{T'} = b_T$, $\nu(T') = \nu(T) * 0$, and $\mathfrak{q}_\lambda^{(n)}(e_{S(T')}) = e_{S(T)}$. Therefore,

$$\Phi_\lambda^{(n)}(t^{-b_{T'}} X^{\nu(T')} \otimes e_{S(T')}) = t^{-b_T} X^{\nu(T)} \otimes e_{S(T)}.$$

Now if $\beta \prec \nu(T')$ then $\Phi_\lambda^{(n)}(X^\beta \otimes v'_\beta) = \mathbb{1}(\beta_{n+1} = 0) X_1^{\beta_1} \cdots X_n^{\beta_n} \otimes \mathfrak{q}_\lambda^{(n)}(v'_\beta)$ cannot be of the form $X^{\nu(T)} \otimes w$ for any $w \in S_{\lambda^{(n)}}$. As such the coefficient of $X^{\nu(T)} \otimes e_{S(T)}$ in the standard basis expansion of $\Phi_\lambda^{(n)}(F_{\mathrm{top}(T')})$ is t^{-b_T}. Since this agrees with the same coefficient in the expansion of $F_{\mathrm{top}(T)}$ we know that $\beta = 1$ and thus $\Phi_\lambda^{(n)}(F_{\mathrm{top}(T')}) = F_{\mathrm{top}(T)}$.

Now consider any $\tau' \in \mathrm{PSYT}_{\geq 0}(\lambda^{(n+1)})$ with $\tau'(\Box_0) = n+1$. Let $T' := \mathfrak{p}_{\lambda^{(n+1)}}(\tau) \in \mathrm{RYT}_{\geq 0}(\lambda^{(n+1)})$. Then $T'(\Box_0) = 0$ so if we set $T := T'|_{\lambda^{(n)}}$ we have that $\Phi_\lambda^{(n)}(F_{\mathrm{top}(T')}) = F_{\mathrm{top}(T)}$. Write $\tau := \tau'|_{\lambda^{(n)}}$. As seen before there exists a sequence $\tau < s_{i_1}(\tau) < \ldots < s_{i_r} \cdots s_{i_1}(\tau) = \mathrm{top}(T)$. Since $\tau'(\Box_0) = n+1$, we see that $\tau' < s_{i_1}(\tau') < \ldots < s_{i_r} \cdots s_{i_1}(\tau') = \mathrm{top}(T')$ as well. For each $1 \leq j \leq r$ we will consider using the intertwiner operators from Proposition 3.3.5 to obtain $F_{s_{i_j} s_{i_{j-1}} \cdots s_{i_1}(\tau)}$ from $F_{s_{i_{j-1}} \cdots s_{i_1}(\tau)}$. We have that

$$\left(tT_{i_j}^{-1} + \frac{(t-1) q^{w_{s_{i_{j-1}} \cdots s_{i_1}(\tau)}(i_j+1)} t^{c_{s_{i_{j-1}} \cdots s_{i_1}(\tau)}(i_j+1)}}{q^{w_{s_{i_{j-1}} \cdots s_{i_1}(\tau)}(i_j)} t^{c_{s_{i_{j-1}} \cdots s_{i_1}(\tau)}(i_j)} - q^{w_{s_{i_{j-1}} \cdots s_{i_1}(\tau)}(i_j+1)} t^{c_{s_{i_{j-1}} \cdots s_{i_1}(\tau)}(i_j+1)}} \right) (F_{s_{i_{j-1}} \cdots s_{i_1}(\tau)})$$
$$= F_{s_{i_j}(s_{i_{j-1}} \cdots s_{i_1}(\tau))}.$$

Now the same exact formula holds with τ replaced by τ'. Importantly, we have that $w_{s_{i_{j-1}} \cdots s_{i_1}(\tau)}(i_j + 1) = w_{s_{i_{j-1}} \cdots s_{i_1}(\tau')}(i_j + 1)$ and $c_{s_{i_{j-1}} \cdots s_{i_1}(\tau)}(i_j + 1) = c_{s_{i_{j-1}} \cdots s_{i_1}(\tau')}(i_j + 1)$. Therefore, we may write

$$D_j(F_{s_{i_{j-1}} \cdots s_{i_1}(\tau)}) = F_{s_{i_j} s_{i_{j-1}} \cdots s_{i_1}(\tau)}$$

and

$$D_j(F_{s_{i_{j-1}} \cdots s_{i_1}(\tau')}) = F_{s_{i_j} s_{i_{j-1}} \cdots s_{i_1}(\tau')}$$

for $D_j \in \mathscr{A}_n \subset \mathscr{A}_{(n,1)}$ of the form $D_j = T_{i_j} + \alpha_j$ where $\alpha_j \in \mathbb{Q}(q,t)$. Here we have used $tT_{i_j}^{-1} = T_{i_j} + t - 1$. By using the quadratic relation for T_{i_j} we may locally invert the operator D_j in the sense that there exists operators $C_j \in \mathscr{A}_n$ with

$$F_{s_{i_{j-1}} \cdots s_{i_1}(\tau)} = C_j(F_{s_{i_j} s_{i_{j-1}} \cdots s_{i_1}(\tau)})$$

and

$$F_{s_{i_{j-1}} \cdots s_{i_1}(\tau')} = C_j(F_{s_{i_j} s_{i_{j-1}} \cdots s_{i_1}(\tau')}).$$

Therefore, if we assume that $\Phi_\lambda^{(n)}(F_{s_{i_j}s_{i_{j-1}}\cdots s_{i_1}(\tau')}) = F_{s_{i_j}s_{i_{j-1}}\cdots s_{i_1}(\tau)}$ then

$$\begin{aligned}
&\Phi_\lambda^{(n)}(F_{s_{i_{j-1}}\cdots s_{i_1}(\tau')}) \\
&= \Phi_\lambda^{(n)}(C_j(F_{s_{i_j}s_{i_{j-1}}\cdots s_{i_1}(\tau')})) \\
&= C_j\Phi_\lambda^{(n)}(F_{s_{i_j}s_{i_{j-1}}\cdots s_{i_1}(\tau')}) \\
&= C_j F_{s_{i_j}s_{i_{j-1}}\cdots s_{i_1}(\tau)} \\
&= F_{s_{i_{j-1}}\cdots s_{i_1}(\tau)}.
\end{aligned}$$

Thus by induction, since we know $\Phi_\lambda^{(n)}(F_{\text{top}(T')}) = F_{\text{top}(T)}$, it follows that $\Phi_\lambda^{(n)}(F_{\tau'}) = F_\tau$.

Lastly, we consider the case of $\tau(\square_0) \neq n+1$. Then $\tau(\square_0) = iq^a$ with either $i \neq n+1$ or $a \geq 0$. If $a > 0$ then $\tau = \Psi(\tau')$ for some τ' and thus from Proposition 3.3.5 we know X_{n+1} divides F_τ. Since $\Phi_\lambda^{(n)} X_{n+1} = 0$ it follows that $\Phi_\lambda^{(n)}(F_\tau) = 0$. Now suppose $a = 0$ and $i \neq n+1$. Notice for any $m \geq n_\lambda$ that the largest power of t occurring in the $\theta^{(m)}$-weight of any $F_{\tau'}$ with $\tau' \in \text{PSYT}_{\geq 0}(\lambda^{(m)})$ is exactly $t^{m-|\lambda|-1}$. Since $i \neq n+1$ we know that if $\Phi_\lambda^{(n)}(F_\tau) = \beta F_{\tau'}$ for some nonzero scalar β and $\tau' \in \text{PSYT}_{\geq 0}(\lambda^{(n+1)})$ then the maximal power of t occurring in the $\theta^{(n)}$-weight of $F_{\tau'}$ is $t^{n-|\lambda|}$ coming from

$$\begin{aligned}
&\theta_i(F_{\tau'}) \\
&= \theta_i(\Phi_\lambda^{(n)}(F_\tau)) \\
&= \Phi_\lambda^{(n)}(\theta_i(F_\tau))) \\
&= \Phi_\lambda^{(n)}(t^{c(\square_0)}F_\tau) \\
&= \Phi_\lambda^{(n)}(t^{n-|\lambda|}F_\tau) \\
&= t^{n-|\lambda|}F_{\tau'}.
\end{aligned}$$

Thus $\Phi_\lambda^{(n)}(F_\tau)$ cannot be a $\theta^{(n)}$-weight vector in $V_{\lambda^{(n)}}$ and so $\beta = 0$. $\square$

The maps $\Phi_\lambda^{(n)}$ possess another important stability property.

PROPOSITION 3.3.15. *For all $\ell \in \mathbb{Z} \setminus \{0\}$ and $n \geq n_\lambda$,*

$$\Phi_\lambda^{(n)} \left(\sum_{j=1}^{n+1} (\theta_j^{(n+1)})^\ell - \sum_{\square \in \lambda^{(n+1)}} t^{\ell c(\square)} \right) = \left(\sum_{j=1}^{n} (\theta_j^{(n)})^\ell - \sum_{\square \in \lambda^{(n)}} t^{\ell c(\square)} \right) \Phi_\lambda^{(n)}.$$

PROOF. Let $\ell \in \mathbb{Z} \setminus \{0\}$ and $n \geq n_\lambda$. As usual let $\square_0$ denote the unique square of the skew diagram $\lambda^{(n+1)}/\lambda^{(n)}$. Directly from Proposition 3.3.13 we see

$$\begin{aligned}
&\Phi_\lambda^{(n)} \left(\sum_{j=1}^{n+1} (\theta_j^{(n+1)})^\ell - \sum_{\square \in \lambda^{(n+1)}} t^{\ell c(\square)} \right) \\
&= \Phi_\lambda^{(n)} \left(\sum_{j=1}^{n} (\theta_j^{(n+1)})^\ell - \sum_{\square \in \lambda^{(n)}} t^{\ell c(\square)} \right) + \Phi_\lambda^{(n)} \left((\theta_{n+1}^{(n+1)})^\ell - t^{\ell c(\square_0)} \right) \\
&= \left(\sum_{j=1}^{n} (\theta_j^{(n)})^\ell - \sum_{\square \in \lambda^{(n)}} t^{\ell c(\square)} \right) \Phi_\lambda^{(n)} + \Phi_\lambda^{(n)} \left((\theta_{n+1}^{(n+1)})^\ell - t^{\ell (n-|\lambda|)} \right).
\end{aligned}$$

It follows from the relation $\Phi_\lambda^{(n)} \left(\theta_{n+1}^{(n+1)} - t^{n-|\lambda|} \right) = 0$ and the fact that $\theta_{n+1}^{(n+1)}$ is invertible on $V_{\lambda^{(n+1)}}$ that

$$\Phi_\lambda^{(n)} \left((\theta_{n+1}^{(n+1)})^\ell - t^{\ell (n-|\lambda|)} \right) = 0.$$

Therefore,

$$\Phi_\lambda^{(n)} \left(\sum_{j=1}^{n+1} (\theta_j^{(n+1)})^\ell - \sum_{\square \in \lambda^{(n+1)}} t^{\ell c(\square)} \right) = \left(\sum_{j=1}^{n} (\theta_j^{(n)})^\ell - \sum_{\square \in \lambda^{(n)}} t^{\ell c(\square)} \right) \Phi_\lambda^{(n)}.$$

$\square$

3.4. Positive EHA Representations from Young Diagrams

3.4.1. The $\mathscr{D}_n^{\mathbf{sph}}$-modules $W_{\lambda^{(n)}}$. We now turn to the corresponding spherical DAHA modules and symmetric v.v. polynomials to the positive DAHA modules V_λ and the non-symmetric v.v. polynomials F_τ considered in the prior sections.

DEFINITION 3.4.1. *For $\lambda \in \mathbb{Y}$ with $|\lambda| = n$ define the $\mathscr{D}_n^{sph}$-module $W_\lambda := \epsilon^{(n)}(V_\lambda)$.*

The F_τ expansions of any symmetrized element of any $\mathscr{A}_n$ submodule U_T satisfy a simple set of recurrence relations.

Lemma 3.4.2. *Let $T \in \mathrm{RSSYT}_{\geq 0}(\lambda)$ and $v \in \epsilon^{(n)}(U_T)$. Suppose that v has the following expansion into the F_τ basis:*

$$v = \sum_{\tau \in \mathrm{PSYT}_{\geq 0}(\lambda;T)} \kappa_\tau F_\tau.$$

Then for each $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda;T)$ with $1 \leq i \leq n-1$ such that $s_i(\tau) > \tau$ we have the relation

$$\kappa_{s_i(\tau)} = \left(\frac{q^{w_\tau(i)} t^{c_\tau(i)} - q^{w_\tau(i+1)} t^{c_\tau(i+1)}}{q^{w_\tau(i)} t^{c_\tau(i)} - q^{w_\tau(i+1)} t^{c_\tau(i+1)+1}} \right) \kappa_\tau.$$

As a consequence, if $\kappa_{\mathrm{top}(T)} \neq 0$ then each coefficient κ_τ is also nonzero.

Proof. Let $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda;T)$ and $1 \leq i \leq n-1$ with $s_i(\tau) > \tau$. Note that $\mathbb{Q}(q,t)\{F_\tau, F_{s_i(\tau)}\}$ is a 2-dimensional submodule for $\mathbb{Q}(q,t)[T_i]$. The T_i-invariant subspace of $\mathbb{Q}(q,t)\{F_\tau, F_{s_i(\tau)}\}$ is given by $\mathbb{Q}(q,t)(1+tT_i^{-1})F_\tau$. From Proposition 3.3.5 we find

$$\begin{aligned}
(1+tT_i^{-1})F_\tau &= F_\tau + tT_i^{-1}F_\tau \\
&= F_\tau + F_{s_i(\tau)} + \frac{(1-t)q^{w_\tau(i+1)} t^{c_\tau(i+1)}}{q^{w_\tau(i)} t^{c_\tau(i)} - q^{w_\tau(i+1)} t^{c_\tau(i+1)}} F_\tau \\
&= F_{s_i(\tau)} + \frac{q^{w_\tau(i)} t^{c_\tau(i)} - q^{w_\tau(i+1)} t^{c_\tau(i+1)+1}}{q^{w_\tau(i)} t^{c_\tau(i)} - q^{w_\tau(i+1)} t^{c_\tau(i+1)}} F_\tau.
\end{aligned}$$

Since $v = \sum_{\tau \in \mathrm{PSYT}_{\geq 0}(\lambda;T)} \kappa_\tau F_\tau$ is T_i-invariant then we know that in particular $\kappa_\tau F_\tau + \kappa_{s_i(\tau)} F_{s_i(\tau)}$ is also T_i-invariant and therefore must be a scalar multiple of $(1+tT_i^{-1})F_\tau$. Therefore,

$$\kappa_\tau F_\tau + \kappa_{s_i(\tau)} F_{s_i(\tau)} = \kappa_{s_i(\tau)} F_{s_i(\tau)} + \frac{q^{w_\tau(i)} t^{c_\tau(i)} - q^{w_\tau(i+1)} t^{c_\tau(i+1)+1}}{q^{w_\tau(i)} t^{c_\tau(i)} - q^{w_\tau(i+1)} t^{c_\tau(i+1)}} \kappa_{s_i(\tau)} F_\tau$$

and so

$$\kappa_{s_i(\tau)} = \left(\frac{q^{w_\tau(i)} t^{c_\tau(i)} - q^{w_\tau(i+1)} t^{c_\tau(i+1)}}{q^{w_\tau(i)} t^{c_\tau(i)} - q^{w_\tau(i+1)} t^{c_\tau(i+1)+1}} \right) \kappa_\tau.$$

□

Using the recurrence relations in Lemma 3.4.2 and the irreducibility of each of the $\mathscr{A}_n$ submodules of V_λ we may determine which $T \in \mathrm{RYT}_{\geq 0}(\lambda)$ have a non-zero space of $\mathscr{H}_n$-invariants $\epsilon^{(n)}(U_T)$.

Proposition 3.4.3. *For $\lambda \in \mathbb{Y}$ with $|\lambda| = n$ and $T \in \mathrm{RYT}_{\geq 0}(\lambda)$,*

$$\dim_{\mathbb{Q}(q,t)} \epsilon^{(n)}(U_T) = \begin{cases} 1 & T \in \mathrm{RSSYT}_{\geq 0}(\lambda) \\ 0 & T \notin \mathrm{RSSYT}_{\geq 0}(\lambda). \end{cases}$$

Proof. By Proposition 3.3.6 each $\mathscr{A}_n$-module U_T is irreducible with simple $\theta^{(n)}$ spectrum. This implies that $\dim_{\mathbb{Q}(q,t)} \epsilon^{(n)}(U_T) \leq 1$ for any $T \in \mathrm{RYT}_{\geq 0}(\lambda)$. Further, we have that $\epsilon^{(n)}(U_T)$ is zero if for any $\theta^{(n)}$-weight vector v in U_T, $\epsilon^{(n)}(v)$ is zero. If $T \in \mathrm{RYT}_{\geq 0}(\lambda) \setminus \mathrm{RSSYT}_{\geq 0}(\lambda)$ then there exists a pair of boxes $\square_1, \square_2 \in \lambda$ with $\square_1$ directly above $\square_2$ such that $T(\square_1) = T(\square_2) = a$. Hence, $\mathrm{top}(T)(\square_1) = iq^a$ and $\mathrm{top}(T)(\square_2) = (i+1)q^a$ for some $1 \leq i \leq n-1$. Then $T_i(F_{\mathrm{top}(T)}) = -tF_{\mathrm{top}(T)}$ which implies that $\epsilon^{(n)}(F_{\mathrm{top}(T)}) = 0$. Thus $\epsilon^{(n)}(U_T) = 0$.

Alternatively, now suppose $T \in \mathrm{RSSYT}_{\geq 0}(\lambda)$. Following Lemma 3.4.2 we construct a vector $v \in U_T$ of the form

$$v = \sum_{\tau \in \mathrm{PSYT}_{\geq 0}(\lambda; T)} \kappa_\tau F_\tau$$

where $\kappa_{\mathrm{top}(T)} = 1$ and if $s_i(\tau) > \tau$ then

$$\kappa_{s_i(\tau)} = \left(\frac{q^{w_\tau(i)} t^{c_\tau(i)} - q^{w_\tau(i+1)} t^{c_\tau(i+1)}}{q^{w_\tau(i)} t^{c_\tau(i)} - q^{w_\tau(i+1)} t^{c_\tau(i+1)+1}} \right) \kappa_\tau.$$

These coefficients κ_τ have the property that if $s_i(\tau) > \tau$ then

$$T_i(\kappa_\tau F_\tau + \kappa_{s_i(\tau)} F_{s_i(\tau)}) = \kappa_\tau F_\tau + \kappa_{s_i(\tau)} F_{s_i(\tau)}.$$

By construction $v \neq 0$ since $\frac{q^{w_\tau(i)} t^{c_\tau(i)} - q^{w_\tau(i+1)} t^{c_\tau(i+1)}}{q^{w_\tau(i)} t^{c_\tau(i)} - q^{w_\tau(i+1)} t^{c_\tau(i+1)+1}} \neq 0$ whenever $s_i(\tau) > \tau$. We will show that $T_i(v) = v$ for all $1 \leq i \leq n-1$ and thus $\epsilon^{(n)}(U_T) \neq 0$.

We find that

$$
\begin{aligned}
&T_i(v)\\
&= \sum_{\tau \in \mathrm{PSYT}_{\geq 0}(\lambda;T)} \kappa_\tau T_i(F_\tau)\\
&= \sum_{\substack{(\tau, s_i(\tau))\ \mathrm{PSYT}_{\geq 0}(\lambda)^2 \\ s_i(\tau) > \tau}} T_i\left(\kappa_\tau(F_\tau) + \kappa_{s_i(\tau)}(F_{s_i(\tau)})\right) + \sum_{\substack{\tau \in \mathrm{PSYT}_{\geq 0}(\lambda) \\ i,i+1 \text{ same row of } \tau}} \kappa_\tau T_i(F_\tau)\\
&+ \sum_{\substack{\tau \in \mathrm{PSYT}_{\geq 0}(\lambda) \\ i,i+1 \text{ same column of } \tau}} \kappa_\tau T_i(F_\tau)\\
&= \sum_{\substack{(\tau, s_i(\tau))\ \mathrm{PSYT}_{\geq 0}(\lambda)^2 \\ s_i(\tau) > \tau}} \left(\kappa_\tau(F_\tau) + \kappa_{s_i(\tau)}(F_{s_i(\tau)})\right) + \sum_{\substack{\tau \in \mathrm{PSYT}_{\geq 0}(\lambda) \\ i,i+1 \text{ same row of } \tau}} \kappa_\tau F_\tau\\
&+ \sum_{\substack{\tau \in \mathrm{PSYT}_{\geq 0}(\lambda) \\ i,i+1 \text{ same column of } \tau}} (-t)\kappa_\tau F_\tau.
\end{aligned}
$$

Thus

$$T_i(v) - v = \sum_{\substack{\tau \in \mathrm{PSYT}_{\geq 0}(\lambda) \\ i,i+1 \text{ same column of } \tau}} (1+t)\kappa_\tau F_\tau.$$

Lastly, since $T \in \mathrm{RSSYT}_{\geq 0}(\lambda)$ there cannot be any $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda; T)$ with $i, i+1$ occurring in the same column as necessarily this would imply that T would have redundant values in those boxes contradicting the fact that T is reverse semi-standard. Hence, the above sum vanishes and we find $T_i(v) = v$. $\square$

Finally, we are able to define the symmetric v.v. Macdonald polynomials following the conventions of this chapter.

Definition 3.4.4. *Let $T \in \mathrm{RSSYT}_{\geq 0}(\lambda)$. Define $P_T \in \epsilon^{(n)}(U_T)$ to be the unique element of the form*

$$P_T = F_{\mathrm{top}(T)} + \sum_y \kappa_y F_y$$

where the sum above ranges over $y \in \mathrm{PSYT}_{\geq 0}(\lambda)$ with $\mathfrak{p}_\lambda(y) = T$ and $y < \mathrm{top}(T)$.

Now we are able to explicitly compute the F_τ expansion of each P_T using the recurrence relations found in Lemma 3.4.2.

Corollary 3.4.5. *For all* $T \in \mathrm{RSSYT}_{\geq 0}(\lambda)$,

$$P_T = \sum_{\tau \in \mathrm{PSYT}_{\geq 0}(\lambda;T)} \prod_{(\square_1,\square_2)\in \mathrm{Inv}(\tau)} \left(\frac{q^{T(\square_1)}t^{c(\square_1)+1} - q^{T(\square_2)}t^{c(\square_2)}}{q^{T(\square_1)}t^{c(\square_1)} - q^{T(\square_2)}t^{c(\square_2)}} \right) F_\tau.$$

Proof. For $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda;T)$ let

$$\kappa_\tau = \prod_{(\square_1,\square_2)\in \mathrm{Inv}(\tau)} \left(\frac{q^{T(\square_1)}t^{c(\square_1)+1} - q^{T(\square_2)}t^{c(\square_2)}}{q^{T(\square_1)}t^{c(\square_1)} - q^{T(\square_2)}t^{c(\square_2)}} \right).$$

From Lemma 3.4.2 it suffices to show that

- $\kappa_{\mathrm{top}(T)} = 1$
- If $s_i(\tau) > \tau$ then $\kappa_{s_i(\tau)} = \left(\frac{q^{w_\tau(i)}t^{c_\tau(i)} - q^{w_\tau(i+1)}t^{c_\tau(i+1)}}{q^{w_\tau(i)}t^{c_\tau(i)} - q^{w_\tau(i+1)}t^{c_\tau(i+1)+1}} \right) \kappa_\tau$.

It is easy to see that $\mathrm{Inv}(\mathrm{top}(T)) = \emptyset$ so $\kappa_{\mathrm{top}(T)} = 1$. Now suppose $s_i(\tau) > \tau$. Let $\square^{(i)}, \square^{(i+1)} \in \lambda$ denote the boxes of λ with $\tau(\square^{(i)}) = iq^a$ and $\tau(\square^{(i+1)}) = (i+1)q^b$ for some $a, b \geq 0$. It is straightforward to check that

$$\mathrm{Inv}(s_i(\tau)) = \{(\square^{(i)}, \square^{(i+1)})\} \sqcup \mathrm{Inv}(\tau).$$

Therefore,

$$\begin{aligned}
&\kappa_{s_i(\tau)} \\
&= \prod_{(\square_1,\square_2)\in \mathrm{Inv}(s_i(\tau))} \left(\frac{q^{T(\square_1)}t^{c(\square_1)+1} - q^{T(\square_2)}t^{c(\square_2)}}{q^{T(\square_1)}t^{c(\square_1)} - q^{T(\square_2)}t^{c(\square_2)}} \right) \\
&= \left(\frac{q^{T(\square^{(i)})}t^{c(\square^{(i)})+1} - q^{T(\square^{(i+1)})}t^{c(\square^{(i+1)})}}{q^{T(\square^{(i)})}t^{c(\square^{(i)})} - q^{T(\square^{(i+1)})}t^{c(\square^{(i+1)})}} \right) \prod_{(\square_1,\square_2)\in \mathrm{Inv}(\tau)} \left(\frac{q^{T(\square_1)}t^{c(\square_1)+1} - q^{T(\square_2)}t^{c(\square_2)}}{q^{T(\square_1)}t^{c(\square_1)} - q^{T(\square_2)}t^{c(\square_2)}} \right) \\
&= \left(\frac{q^{w_\tau(i)}t^{c_\tau(i)} - q^{w_\tau(i+1)}t^{c_\tau(i+1)}}{q^{w_\tau(i)}t^{c_\tau(i)} - q^{w_\tau(i+1)}t^{c_\tau(i+1)+1}} \right) \kappa_\tau.
\end{aligned}$$

$\square$

We look now at the action of the special spherical DAHA elements $P_{(0,\ell)}^{(n)}$.

Proposition 3.4.6. *Let $|\lambda| = n$. The set $\{P_T : T \in \mathrm{RSSYT}_{\geq 0}(\lambda)\}$ is a $\mathbb{Q}(q,t)[\theta_1^{\pm 1}, \ldots, \theta_n^{\pm 1}]^{\mathfrak{S}_n}$-weight basis for W_λ. Further, for $\ell \in \mathbb{Z} \setminus \{0\}$*

$$P_{(0,\ell)}^{(n)}(P_T) = \left(\sum_{\square \in \lambda} q^{\ell T(\square)} t^{\ell c(\square)} \right) P_T.$$

Consequently, $P_{(0,1)}^{(n)}$ acts on W_λ with simple spectrum

$$\left\{ \sum_{\square \in \lambda} q^{T(\square)} t^{c(\square)} : T \in \mathrm{RSSYT}_{\geq 0}(\lambda) \right\}.$$

Proof. It follows directly from Proposition 3.3.6 and Proposition 3.4.3 that the set $\{P_T : T \in \mathrm{RSSYT}_{\geq 0}(\lambda)\}$ is a linear basis for W_λ. We need to show that the P_T are $\mathbb{Q}(q,t)[\theta_1^{\pm 1}, \ldots, \theta_n^{\pm 1}]^{\mathfrak{S}_n}$-weight vectors. Let $T \in \mathrm{RSSYT}_{\geq 0}(\lambda)$. Then from Definition 3.4.4 we know that

$$P_T = \beta \epsilon^{(n)}(F_{\mathrm{top}(T)})$$

for some nonzero scalar β depending on T. Then for any $\ell \in \mathbb{Z} \setminus \{0\}$ we have that

$$\begin{aligned}
&\left(\sum_{j=1}^{n} (\theta_j^{(n)})^\ell \right) (P_T) \\
&= \left(\sum_{j=1}^{n} (\theta_j^{(n)})^\ell \right) (\beta \epsilon^{(n)}(F_{\mathrm{top}(T)})) \\
&= \beta \epsilon^{(n)} \left(\left(\sum_{j=1}^{n} (\theta_j^{(n)})^\ell \right) (F_{\mathrm{top}(T)}) \right) \\
&= \beta \epsilon^{(n)} \left(\left(\sum_{j=1}^{n} q^{\ell w_{\mathrm{top}(T)}(j)} t^{\ell c_{\mathrm{top}(T)}(j)} \right) F_{\mathrm{top}(T)} \right) \\
&= \beta \epsilon^{(n)} \left(\left(\sum_{\square \in \lambda} q^{\ell T(\square)} t^{\ell c(\square)} \right) F_{\mathrm{top}(T)} \right) \\
&= \left(\sum_{\square \in \lambda} q^{\ell T(\square)} t^{\ell c(\square)} \right) \beta \epsilon^{(n)}(F_{\mathrm{top}(T)}) \\
&= \left(\sum_{\square \in \lambda} q^{\ell T(\square)} t^{\ell c(\square)} \right) P_T.
\end{aligned}$$

Hence, P_T is a $\mathbb{Q}(q,t)[\theta_1^{\pm 1}, \ldots, \theta_n^{\pm 1}]^{\mathfrak{S}_n}$-weight vector.

Now let $S \in \mathrm{RSSYT}_{\geq 0}(\lambda)$ and suppose that

$$\sum_{\Box \in \lambda} q^{T(\Box)} t^{c(\Box)} = \sum_{\Box \in \lambda} q^{S(\Box)} t^{c(\Box)}.$$

Fix any $d \in \mathbb{Z}$. Since q and t are algebraically independent over $\mathbb{Q}$,

$$\sum_{\substack{\Box \in \lambda \\ c(\Box) = d}} q^{T(\Box)} = \sum_{\substack{\Box \in \lambda \\ c(\Box) = d}} q^{S(\Box)}.$$

Since the labelling T is reverse semi-standard, the values of $T(\Box)$ for $\Box \in \lambda$ with $c(\Box) = d$ are all distinct and strictly decreasing down the d-diagonal. Of course, the same is true for S. Therefore, the values of T and S agree along the d-diagonal of λ. As $d \in \mathbb{Z}$ was general it follows that $T = S$. Thus the spectrum of the operator $P_{0,1}^{(n)}$ on W_λ is simple.

$\Box$

As mentioned previously, the non-symmetric v.v. Macdonald polynomials do not align with those of Dunkl-Luque. However, we are able to show that, once symmetrized, the symmetrized v.v. polynomials agree.

Corollary 3.4.7. *The symmetric vector valued Macdonald polynomials of Dunkl-Luque [12] agree with the P_T of this chapter up to nonzero scalars.*

Proof. The $\mathscr{D}_n^+$-modules V_λ in this thesis are isomorphic (after aligning conventions) to the $\mathscr{D}_n^+$-modules $\mathcal{M}_\lambda$ in Dunkl-Luque's paper. Dunkl and Luque characterize the symmetric vector valued Macdonald polynomials as eigenvectors with distinct eigenvalues for the operator $Y_1^{(n)} + \ldots + Y_n^{(n)}$ acting on $\epsilon^{(n)}(\mathcal{M}_\lambda)$. Here $Y_i^{(n)}$ are the standard Cherednik elements given in our conventions by $Y_i^{(n)} = t^{n-i+1} T_{i-1} \cdots T_1 \pi_n T_{n-1}^{-1} \cdots T_i^{-1}$. A simple calculation shows that the spherical DAHA elements $\epsilon^{(n)}(Y_1^{(n)} + \ldots + Y_n^{(n)})\epsilon^{(n)}$ and $\epsilon^{(n)}(\theta_1^{(n)} + \ldots + \theta_n^{(n)})\epsilon^{(n)}$ are both nonzero scalar multiples of $\epsilon^{(n)}\pi_n\epsilon^{(n)}$. Since the spectrum of $\epsilon^{(n)}\pi_n\epsilon^{(n)}$ acting on W_λ is simple, it follows that the P_T are eigenvectors for $\epsilon^{(n)}(Y_1^{(n)} + \ldots + Y_n^{(n)})\epsilon^{(n)}$ and hence agree with the symmetric vector valued Macdonald polynomials of Dunkl-Luque up to re-normalization. $\Box$

3.4.2. Stable Limit of the $W_{\lambda^{(n)}}$. Finally, we identify a special stability property for the P_T elements.

Corollary 3.4.8. *For $T \in \mathrm{RSSYT}_{\geq 0}(\lambda^{(n)})$ let $T' \in \mathrm{RSSYT}_{\geq 0}(\lambda^{(n+1)})$ be such that $T(\Box) = T'(\Box)$ for $\Box \in \lambda^{(n)}$ and $T'(\Box_0) = 0$ for $\Box_0 \in \lambda^{(n+1)}/\lambda^{(n)}$. Then*

$$\Phi_\lambda^{(n)}(P_{T'}) = P_T.$$

Proof. Note that restriction from $\lambda^{(n+1)}$ to $\lambda^{(n)}$ identifies $\mathrm{PSYT}_{\geq 0}(\lambda^{(n)}; T)$ as the subset of $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda^{(n+1)}; T')$ with $\tau(\Box_0) = n+1$. Thus by using Corollary 3.3.14 in conjunction with Corollary 3.4.5 we find that

$$\begin{aligned}
&\Phi_\lambda^{(n)}(P_{T'}) \\
&= \sum_{\tau \in \mathrm{PSYT}_{\geq 0}(\lambda^{(n+1)}; T')} \prod_{(\Box_1, \Box_2) \in \mathrm{Inv}(\tau)} \left(\frac{q^{T(\Box_1)} t^{c(\Box_1)+1} - q^{T(\Box_2)} t^{c(\Box_2)}}{q^{T(\Box_1)} t^{c(\Box_1)} - q^{T(\Box_2)} t^{c(\Box_2)}} \right) \Phi_\lambda^{(n)}(F_\tau) \\
&= \sum_{\substack{\tau \in \mathrm{PSYT}_{\geq 0}(\lambda^{(n+1)}; T') \\ \tau(\Box_0) = n+1}} \prod_{(\Box_1, \Box_2) \in \mathrm{Inv}(\tau)} \left(\frac{q^{T(\Box_1)} t^{c(\Box_1)+1} - q^{T(\Box_2)} t^{c(\Box_2)}}{q^{T(\Box_1)} t^{c(\Box_1)} - q^{T(\Box_2)} t^{c(\Box_2)}} \right) F_{\tau|_{\lambda^{(n)}}} \\
&= \sum_{\substack{\tau \in \mathrm{PSYT}_{\geq 0}(\lambda^{(n+1)}; T') \\ \tau(\Box_0) = n+1}} \prod_{(\Box_1, \Box_2) \in \mathrm{Inv}(\tau|_{\lambda^{(n)}})} \left(\frac{q^{T(\Box_1)} t^{c(\Box_1)+1} - q^{T(\Box_2)} t^{c(\Box_2)}}{q^{T(\Box_1)} t^{c(\Box_1)} - q^{T(\Box_2)} t^{c(\Box_2)}} \right) F_{\tau|_{\lambda^{(n)}}} \\
&= \sum_{\tau \in \mathrm{PSYT}_{\geq 0}(\lambda^{(n)}; T)} \prod_{(\Box_1, \Box_2) \in \mathrm{Inv}(\tau)} \left(\frac{q^{T(\Box_1)} t^{c(\Box_1)+1} - q^{T(\Box_2)} t^{c(\Box_2)}}{q^{T(\Box_1)} t^{c(\Box_1)} - q^{T(\Box_2)} t^{c(\Box_2)}} \right) F_\tau \\
&= P_T.
\end{aligned}$$

$\Box$

This stability allows for the following definition.

Definition 3.4.9. *Let $\lambda \in \mathbb{Y}$. Define the infinite diagram $\lambda^{(\infty)} := \bigcup_{n \geq n_\lambda} \lambda^{(n)}$. Define $\Omega(\lambda)$ to be the set of all labellings $T : \lambda^{(\infty)} \to \mathbb{Z}_{\geq 0}$ such that*

- $|\{\Box \in \lambda^{(\infty)} : T(\Box) \neq 0\}| < \infty$
- *T decreases weakly across rows*
- *T decreases strictly down columns.*

For $T \in \Omega(\lambda)$ we define the degree of T as $\deg(T) := \sum_{\square \in \lambda^{(\infty)}} T(\square)$. *Define the rank of T,* $\mathrm{rk}(T)$, *to be the minimal $n \geq n_\lambda$ such that $T|_{\lambda^{(\infty)} \setminus \lambda^{(n)}} = 0$.*

Define the space $W_\lambda^{(\infty)}$ to be the inverse limit $\varprojlim W_{\lambda^{(n)}}$ with respect to the maps $\Phi_\lambda^{(n)}$. Let $\widetilde{W}_\lambda$ be the subspace of all bounded X-degree elements of $W_\lambda^{(\infty)}$. For $T \in \Omega_\lambda$ define the ***generalized Macdonald function***

$$\mathfrak{P}_T := \lim_n P_{T|_{\lambda^{(n)}}} \in \widetilde{W}_\lambda.$$

Example. *For $\lambda = (3,2,1)$*

$$\lambda^{(\infty)} = \begin{array}{|c|c|c|c|c|c|c|c|c|l}
\hline
 & & & & & & & & & \cdots \\
\hline
 & & & \multicolumn{1}{c}{} \\
\cline{1-3}
 & & \multicolumn{1}{c}{} \\
\cline{1-2}
 & \multicolumn{1}{c}{} \\
\cline{1-1}
\end{array} \quad \textit{and}$$

$$\begin{array}{|c|c|c|c|c|c|c|c|l}
\hline
5 & 3 & 3 & 2 & 1 & 0 & 0 & 0 & \cdots \\
\hline
3 & 2 & 0 & \multicolumn{1}{c}{} \\
\cline{1-3}
1 & 1 & \multicolumn{1}{c}{} \\
\cline{1-2}
0 & \multicolumn{1}{c}{} \\
\cline{1-1}
\end{array} \quad \in \Omega(\lambda).$$

Remark 30. *The degree of each $\mathfrak{P}_T$ is given simply as*

$$\deg(\mathfrak{P}_T) = \deg(T) = \sum_{\square \in \lambda^{(\infty)}} T(\square).$$

It is clear from definition that the set of all $\mathfrak{P}_T$ for $T \in \Omega(\lambda)$ gives a $\mathbb{Q}(q,t)$-basis of $\widetilde{W}_\lambda$.

Using the stability of the action of the $P_{(0,\ell)}^{(n)}$ operators we may define the following operators.

Definition 3.4.10. *For $\ell \in \mathbb{Z} \setminus \{0\}$ define the operator $\widetilde{\Delta}_r^{(\infty)} : W_\lambda^{(\infty)} \to W_\lambda^{(\infty)}$ to be the stable-limit*

$$\widetilde{\Delta}_\ell^{(\infty)} := \lim_n \left(\sum_{j=1}^{n} (\theta_j^{(n)})^\ell - \sum_{\square \in \lambda^{(n)}} t^{\ell c(\square)} \right).$$

A simple calculation shows the following:

Lemma 3.4.11. *For all $\ell \in \mathbb{Z} \setminus \{0\}$ and $T \in \Omega(\lambda)$,*

$$\widetilde{\Delta}_\ell^{(\infty)}(\mathfrak{P}_T) = \left(\sum_{\square \in \lambda^{(\infty)}} (q^{\ell T(\square)} - 1) t^{\ell c(\square)} \right) \mathfrak{P}_T \,.$$

Proof. Let $\ell \in \mathbb{Z} \setminus \{0\}$ and $T \in \Omega(\lambda)$. Then

$$\begin{aligned}
&\widetilde{\Delta}_\ell^{(\infty)}(\mathfrak{P}_T) \\
&= \lim_n \left(\sum_{j=1}^n (\theta_j^{(n)})^\ell - \sum_{\square \in \lambda^{(n)}} t^{\ell c(\square)} \right) \left(\lim_n P_{T|_{\lambda^{(n)}}} \right) \\
&= \lim_n \left(\sum_{j=1}^n (\theta_j^{(n)})^\ell - \sum_{\square \in \lambda^{(n)}} t^{\ell c(\square)} \right) (P_{T|_{\lambda^{(n)}}}) \\
&= \lim_n \left(\sum_{\square \in \lambda^{(n)}} q^{\ell T(\square)} t^{\ell c(\square)} - \sum_{\square \in \lambda^{(n)}} t^{\ell c(\square)} \right) P_{T|_{\lambda^{(n)}}} \\
&= \lim_n \left(\sum_{\square \in \lambda^{(n)}} (q^{\ell T(\square)} - 1) t^{\ell c(\square)} \right) P_{T|_{\lambda^{(n)}}} .
\end{aligned}$$

Importantly, for $n \geq \mathrm{rk}(T)$

$$\sum_{\square \in \lambda^{(n)}} (q^{\ell T(\square)} - 1) t^{\ell c(\square)} = \sum_{\square \in \lambda^{(\infty)}} (q^{\ell T(\square)} - 1) t^{\ell c(\square)}.$$

Therefore,

$$\begin{aligned}
&\lim_n \left(\sum_{\square \in \lambda^{(n)}} (q^{\ell T(\square)} - 1) t^{\ell c(\square)} \right) P_{T|_{\lambda^{(n)}}} \\
&= \lim_n \left(\sum_{\square \in \lambda^{(\infty)}} (q^{\ell T(\square)} - 1) t^{\ell c(\square)} \right) P_{T|_{\lambda^{(n)}}} \\
&= \left(\sum_{\square \in \lambda^{(\infty)}} (q^{\ell T(\square)} - 1) t^{\ell c(\square)} \right) \lim_n P_{T|_{\lambda^{(n)}}} \\
&= \left(\sum_{\square \in \lambda^{(\infty)}} (q^{\ell T(\square)} - 1) t^{\ell c(\square)} \right) \mathfrak{P}_T \,.
\end{aligned}$$

$\square$

COROLLARY 3.4.12. *For $\ell \in \mathbb{Z} \setminus \{0\}$ the operator $\widetilde{\Delta}_\ell^{(\infty)}$ restricts to an operator on $\widetilde{W}_\lambda$.*

PROOF. Let $\ell \in \mathbb{Z} \setminus \{0\}$. We know that the set $\{\mathfrak{P}_T \,|T \in \Omega(\lambda)\}$ is a basis for $\widetilde{W}_\lambda$. From Lemma 3.4.11 we further know that $\widetilde{\Delta}_\ell^{(\infty)}$ acts diagonally on this basis. Therefore, $\widetilde{\Delta}_\ell^{(\infty)}$ restricts to an operator on $\widetilde{W}_\lambda$. $\square$

EXAMPLE. *For $T \in \Omega(3,2,1)$ as is Example 3.4.2,*

$$\widetilde{\Delta}_1^{(\infty)}(\mathfrak{P}_T) = \big((q^5-1) + (q^3-1)(t^{-1}+t^1+t^2) + (q^2-1)(t^0+t^3) + (q-1)(t^{-2}+t^{-1}+t^4)\big)\, \mathfrak{P}_T \,.$$

3.4.3. Positive Elliptic Hall Algebra Action on $\widetilde{W}_\lambda$. Combining every result of this chapter thus far we are able to define a novel family of positive EHA representations.

THEOREM 3.4.13. *For $\lambda \in \mathbb{Y}$, $\widetilde{W}_\lambda$ is a graded $\mathscr{E}^+$-module with action determined for $\ell \in \mathbb{Z} \setminus \{0\}$ and $r > 0$ by*

- $P_{r,0} \to q^r p_r^\bullet$
- $P_{0,\ell} \to \widetilde{\Delta}_\ell^{(\infty)}$.

Further, $\widetilde{W}_\lambda$ is spanned by a basis of eigenvectors $\{\mathfrak{P}_T\}_{T \in \Omega(\lambda)}$ with distinct eigenvalues for the Macdonald operator $\Delta = \widetilde{\Delta}_1^{(\infty)}$.

Proof. It suffices to establish that the map $\mathscr{E}^+ \to \mathrm{End}_{\mathbb{Q}(q,t)}(\widetilde{W}_\lambda)$ satisfies the generating relations of $\mathscr{E}^+$. Any such relation is a non-commutative polynomial expression in $\mathscr{E}^+$ of the form

$$F(P_{0,-r},\ldots,P_{0,-1}P_{0,1},\ldots,P_{0,r},P_{1,0},\ldots,P_{s,0}) = 0$$

for some $r > 0$ and $s > 0$. By an argument of Schiffmann-Vasserot (Lemma 1.3 in [34]), there are automorphisms $\Gamma^{(n)}$ of $\mathscr{D}_n^{\text{sph}}$ such that for all $\ell \in \mathbb{Z}\backslash\{0\}$ and $s > 0$, $\Gamma^{(n)}(P_{0,\ell}^{(n)}) = P_{0,\ell}^{(n)} - \sum_{\square\in\lambda^{(n)}} t^{\ell c(\square)}$ and $\Gamma^{(n)}(P_{s,0}^{(n)}) = P_{s,0}^{(n)}$. By applying the canonical quotient maps $\Pi_n : \widetilde{W}_\lambda \to W_{\lambda^{(n)}}$ we see using Cor. 3.3.15 that as maps

$$\begin{aligned}
&\Pi_n F(P_{0,-r},\ldots,P_{0,-1},P_{0,1},\ldots,P_{0,r},P_{1,0},\ldots,P_{s,0})\\
&= F(\Gamma^{(n)}(P_{0,-r}^{(n)}),\ldots,\Gamma^{(n)}(P_{0,-1}^{(n)}),\Gamma^{(n)}(P_{0,1}^{(n)}),\ldots,\Gamma^{(n)}(P_{0,r}^{(n)}),\Gamma^{(n)}(P_{1,0}^{(n)}),\ldots,\Gamma^{(n)}(P_{s,0}^{(n)}))\Pi_n\\
&= \Gamma^{(n)}(F(P_{0,-r}^{(n)},\ldots,P_{0,-1}^{(n)},P_{0,1}^{(n)},\ldots,P_{0,r}^{(n)},P_{1,0}^{(n)},\ldots,P_{s,0}^{(n)}))\Pi_n = 0.
\end{aligned}$$

As this holds for all $n \geq n_\lambda$, it follows that $F(P_{0,-r},\ldots,P_{0,-1},P_{0,1},\ldots,P_{0,r},P_{1,0},\ldots,P_{s,0}) = 0$ in $\mathrm{End}_{\mathbb{Q}(q,t)}(\widetilde{W}_\lambda)$ as desired. The last statement regarding the spectrum of Δ follows directly from Prop. 3.4.6 and Cor. 3.4.8. $\square$

Remark 31. *For $T \in \Omega(\lambda)$ and $\ell \in \mathbb{Z}\setminus\{0\}$,*

$$P_{0,\ell}(\mathfrak{P}_T) = \left(\sum_{\square\in\lambda^{(\infty)}} \left(q^{\ell T(\square)} - 1\right) t^{\ell c(\square)}\right)\mathfrak{P}_T\,.$$

Remark 32. *For $\lambda = \emptyset$, $\widetilde{W}_\emptyset = \Lambda_{q,t}$ recovers the standard representation of $\mathscr{E}^+$. In this case, $\Omega(\emptyset) = \mathbb{Y}$ and $\mathfrak{P}_\mu = P_\mu[X;q^{-1},t]$ (up to nonzero scalar).*

Now we identify a special element of each $\widetilde{W}_\lambda$.

Definition 3.4.14. *For any $\lambda \in \mathbb{Y}$ define the labelling T_λ^{min} of $\lambda^{(\infty)}$ by*

$$T_\lambda^{min}(\square) = \#\{\square' \in \lambda^{(\infty)} | \square' \text{ strictly below } \square\}.$$

Lemma 3.4.15. *The labelling T_λ^{min} is the unique element of $\Omega(\lambda)$ of lowest degree $d_\lambda := \sum_{i\geq 1} i\lambda_i$.*

PROOF. It is immediate that since λ is a partition $T_\lambda^{\min} \in \Omega(\lambda)$. Further, by construction each entry of $T_\lambda^{\min}$ is chosen minimally in that for any $T \in \Omega(\lambda)$ and $\Box \in \lambda^{(\infty)}$, $T_\lambda^{\min}(\Box) \leq T(\Box)$. To see this simply note that if $T \in \Omega(\lambda)$ and $\Box \in \lambda^{(\infty)}$ then if $\Box'$ is the box directly below $\Box$ then $T(\Box) > T(\Box')$. Hence, $T(\Box)$ must be at least as large as the number of boxes strictly below $\Box$. Therefore, $T_\lambda^{\min}$ has the minimal degree among all elements of $\Omega(\lambda)$. Lastly, the number of boxes $\Box \in \lambda^{(\infty)}$ with $T_\lambda^{\min}(\Box) = i$ is λ_i so $\deg(T_\lambda^{\min}) = d_\lambda$ as defined above. $\Box$

PROPOSITION 3.4.16. *For $\lambda, \mu \in \mathbb{Y}$ distinct, $\widetilde{W}_\lambda \ncong \widetilde{W}_\mu$ as graded $\mathscr{E}^+$ modules.*

PROOF. Let $\lambda, \mu \in \mathbb{Y}$ and suppose that $f : \widetilde{W}_\lambda \to \widetilde{W}_\mu$ is a graded $\mathscr{E}^+$ module isomorphism. Then by Lemma 3.4.15 we know that

$$f(\mathfrak{P}_{T_\lambda^{\min}}) = \alpha\, \mathfrak{P}_{T_\mu^{\min}}$$

for some nonzero scalar $\alpha \in \mathbb{Q}(q,t)$. Further,

$$\begin{aligned} P_{0,1}(f(\mathfrak{P}_{T_\lambda^{\min}})) &= f(P_{0,1}(\mathfrak{P}_{T_\lambda^{\min}})) \\ &= f((\sum_{\Box \in \lambda^{(\infty)}} (q^{T_\lambda^{\min}(\Box)} - 1)t^{c(\Box)})\, \mathfrak{P}_{T_\lambda^{\min}}) \\ &= \left(\sum_{\Box \in \lambda^{(\infty)}} (q^{T_\lambda^{\min}(\Box)} - 1)t^{c(\Box)} \right) f(\mathfrak{P}_{T_\lambda^{\min}}) \\ &= \left(\sum_{\Box \in \lambda^{(\infty)}} (q^{T_\lambda^{\min}(\Box)} - 1)t^{c(\Box)} \right) \alpha\, \mathfrak{P}_{T_\mu^{\min}}\, . \end{aligned}$$

On the other hand,

$$P_{0,1}(\alpha\, \mathfrak{P}_{T_\mu^{\min}}) = \left(\sum_{\Box \in \mu^{(\infty)}} (q^{T_\mu^{\min}(\Box)} - 1)t^{c(\Box)} \right) \alpha\, \mathfrak{P}_{T_\mu^{\min}}\, .$$

By assumption $\alpha \neq 0$ so

$$\sum_{\Box \in \lambda^{(\infty)}} (q^{T_\lambda^{\min}(\Box)} - 1)t^{c(\Box)} = \sum_{\Box \in \mu^{(\infty)}} (q^{T_\mu^{\min}(\Box)} - 1)t^{c(\Box)}.$$

This gives

$$\sum_{\square\in\lambda^{(n_\lambda)}}(q^{T_\lambda^{\min}(\square)}-1)t^{c(\square)}=\sum_{\square\in\mu^{(n_\mu)}}(q^{T_\mu^{\min}(\square)}-1)t^{c(\square)}$$

which after limiting $q\to 0$ gives

$$\sum_{\square\in\lambda}t^{c(\square)}=\sum_{\square\in\mu}t^{c(\square)}.$$

By comparing the coefficient of t^d for all $d\in\mathbb{Z}$ on both sides of the above equality we see that λ and μ have the same number of boxes on each diagonal and are therefore equal.

$\square$

3.5. Pieri Rule for Generalized Macdonald Functions

The goal of this section is to derive and utilize an explicit combinatorial formula for the action of the multiplication operators $e_r[X]^\bullet$ on $\widetilde{W}_\lambda$. We will show investigate the e_1 Pieri coefficients in more depth and show that they satisfy a simple non-vanishing conditions. We will use this non-vanishing to prove that the $\widetilde{W}_\lambda$ modules are cyclic.

3.5.1. Pieri Rule Preliminaries. We begin first by establishing some useful lemmas.

Lemma 3.5.1. *For $T\in\mathrm{RYT}_{\geq 0}(\lambda)$*

$$\epsilon^{(n)}(F_{\min(T)})=\frac{[\mu(T)]_t!}{[n]_t!}\sum_{\sigma\in\mathfrak{S}_n/\mathfrak{S}_{\mu(T)}}t^{\left(\binom{n}{2}-\binom{\mu(T)}{2}\right)-\ell(\sigma)}T_\sigma(F_{\min(T)}).$$

Proof. The result follows from the following simple calculation:

$$
\begin{aligned}
&\epsilon^{(n)}(F_{\min(T)}) \\
&= \frac{1}{[n]_t!} \sum_{\sigma \in \mathfrak{S}_n} t^{\binom{n}{2} - \ell(\sigma)} T_\sigma(F_{\min(T)}) \\
&= \frac{1}{[n]_t!} \sum_{\sigma \in \mathfrak{S}_n / \mathfrak{S}_{\mu(T)}} \sum_{\gamma \in \mathfrak{S}_{\mu(T)}} t^{\binom{n}{2} - \ell(\sigma\gamma)} T_{\sigma\gamma}(F_{\min(T)}) \\
&= \frac{1}{[n]_t!} \sum_{\sigma \in \mathfrak{S}_n / \mathfrak{S}_{\mu(T)}} \sum_{\gamma \in \mathfrak{S}_{\mu(T)}} t^{\binom{n}{2} - \ell(\sigma) - \ell(\gamma)} T_\sigma T_\gamma(F_{\min(T)}) \\
&= \frac{1}{[n]_t!} \sum_{\sigma \in \mathfrak{S}_n / \mathfrak{S}_{\mu(T)}} \sum_{\gamma \in \mathfrak{S}_{\mu(T)}} t^{\left(\binom{n}{2} - \binom{\mu(T)}{2}\right) - \ell(\sigma)} t^{\binom{\mu(T)}{2} - \ell(\sigma)} T_\sigma(F_{\min(T)}) \\
&= \frac{1}{[n]_t!} \sum_{\sigma \in \mathfrak{S}_n / \mathfrak{S}_{\mu(T)}} t^{\left(\binom{n}{2} - \binom{\mu(T)}{2}\right) - \ell(\sigma)} T_\sigma(F_{\min(T)}) \left(\sum_{\gamma \in \mathfrak{S}_{\mu(T)}} t^{\binom{\mu(T)}{2} - \ell(\sigma)} \right) \\
&= \frac{[\mu(T)]_t!}{[n]_t!} \sum_{\sigma \in \mathfrak{S}_n / \mathfrak{S}_{\mu(T)}} t^{\left(\binom{n}{2} - \binom{\mu(T)}{2}\right) - \ell(\sigma)} T_\sigma(F_{\min(T)}).
\end{aligned}
$$

□

Lemma 3.5.2. *For* $\mathrm{RSSYT}_{\geq 0}(\lambda)$ *and* $\sigma \in \mathfrak{S}_n/\mathfrak{S}_{\mu(T)}$

$$
T_\sigma(F_{\min(T)}) = F_{\sigma(\min(T))} + \sum_{\tau < \sigma(\min(T))} \kappa_\tau F_\tau
$$

for some scalars κ_τ.

Proof. We will proceed by induction using the fact that $\mathrm{PSYT}_{\geq 0}(\lambda; T)$ is isomorphic to $\mathfrak{S}_n/\mathfrak{S}_{\mu(T)}$ as posets which we saw in Remark 29. Certainly, the statement holds trivially for $\tau = \min(T)$. Take some $\sigma(\min(T)) = \tau \in \mathrm{PSYT}_{\geq 0}(\lambda; T)$ with $s_i(\tau) > \tau$ and suppose that

$$
T_\sigma(F_{\min(T)}) = F_{\sigma(\min(T))} + \sum_{\tau' < \sigma(\min(T))} \kappa_{\tau'} F_{\tau'}
$$

for some scalars κ_τ. Then using Proposition 3.3.5

$$\begin{aligned}
&T_{s_i\sigma}(F_{\min(T)})\\
&= T_i T_\sigma(F_{\min(T)})\\
&= T_i F_\tau + \sum_{\tau' < \sigma(\min(T))} \kappa_{\tau'} T_i F_{\tau'}\\
&= F_{s_i(\tau)} + \frac{(1-t)q^{w_\tau(i)}t^{c_\tau(i)}}{q^{w_\tau(i)}t^{c_\tau(i)} - q^{w_\tau(i+1)}t^{c_\tau(i+1)}} F_\tau\\
&+ \sum_{\tau' < \sigma(\min(T))} \kappa_{\tau'} \left(F_{s_i(\tau')} + \frac{(1-t)q^{w_{\tau'}(i)}t^{c_{\tau'}(i)}}{q^{w_{\tau'}(i)}t^{c_{\tau'}(i)} - q^{w_{\tau'}(i+1)}t^{c_{\tau'}(i+1)}} F_{\tau'} \right)\\
&= F_{(s_i\sigma)(\min(T))} + \sum_{\tau' < (s_i\sigma)(\min(T))} \kappa'_{\tau'} F_{\tau'}.
\end{aligned}$$

□

The above lemmas may now be used to compute the symmetrization of each F_τ in terms of the P_T basis.

Proposition 3.5.3. *For $T \in \mathrm{RSSYT}_{\geq 0}(\lambda)$*

$$\epsilon^{(n)}(F_{\min(T)}) = \frac{[\mu(T)]_t!}{[n]_t!} P_T.$$

Proof. Recall from Definition 3.4.4 that the coefficient of $F_{\mathrm{top}(T)}$ in P_T is 1. We know that from the proof of Proposition 3.4.3 that since $T \in \mathrm{RSSYT}_{\geq 0}(\lambda)$,

$$\epsilon^{(n)}(F_{\min(T)}) = \alpha P_T$$

for some nonzero scalar α. Let σ_0 denote the longest element of $\mathfrak{S}_n/\mathfrak{S}_{\mu(T)}$. Note that $\sigma_0(\min(T)) = \mathrm{top}(T)$. We now use Lemmas 3.5.1 and 3.5.2 to compute the coefficient of $F_{\mathrm{top}(T)}$ in $\epsilon^{(n)}(F_{\min(T)})$

determining α :

$$\begin{aligned}
&\epsilon^{(n)}(F_{\min(T)})\\
&= \frac{[\mu(T)]_t!}{[n]_t!} \sum_{\sigma\in\mathfrak{S}_n/\mathfrak{S}_{\mu(T)}} t^{\left(\binom{n}{2}-\binom{\mu(T)}{2}\right)-\ell(\sigma)} T_\sigma(F_{\min(T)})\\
&= \frac{[\mu(T)]_t!}{[n]_t!} \sum_{\sigma\in\mathfrak{S}_n/\mathfrak{S}_{\mu(T)}} t^{\left(\binom{n}{2}-\binom{\mu(T)}{2}\right)-\ell(\sigma)} \left(F_{\sigma(\min(T))} + \sum_{\tau<\sigma(\min(T))} \kappa_\tau^\sigma F_\tau\right)\\
&= \frac{[\mu(T)]_t!}{[n]_t!} F_{\sigma_0(\min(T))} t^{\left(\binom{n}{2}-\binom{\mu(T)}{2}\right)-\ell(\sigma_0)} + \sum_{\tau<\sigma_0(\min(T))} \kappa'_\tau F_\tau\\
&= \frac{[\mu(T)]_t!}{[n]_t!} F_{\mathrm{top}(T)} + \sum_{\tau<\mathrm{top}(T)} \kappa'_\tau F_\tau.
\end{aligned}$$

Therefore, $\alpha = \frac{[\mu(T)]_t!}{[n]_t!}$. $\square$

LEMMA 3.5.4. *For $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda)$ with $\mathfrak{p}_\lambda(\tau) = T \in \mathrm{RSSYT}_{\geq 0}(\lambda)$*

$$\epsilon^{(n)}(F_\tau) = \prod_{(\square_1,\square_2)\in \mathrm{Inv}(\tau)} \left(\frac{q^{T(\square_1)}t^{c(\square_1)} - q^{T(\square_2)}t^{c(\square_2)}}{q^{T(\square_1)}t^{c(\square_1)} - q^{T(\square_2)}t^{c(\square_2)+1}} \right) \epsilon^{(n)}(F_{\mathrm{top}(T)}).$$

PROOF. Let $T \in \mathrm{RSSYT}_{\geq 0}(\lambda)$ and $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda; T)$ with $s_i(\tau) > \tau$. Then using Proposition 3.3.5 we see

$$\begin{aligned}
&\epsilon^{(n)}(F_{s_i(\tau)})\\
&= \epsilon^{(n)}\left(\left(T_i + \frac{(t-1)q^{w_\tau(i)}t^{c_\tau(i)}}{q^{w_\tau(i)}t^{c_\tau(i)} - q^{w_\tau(i+1)}t^{c_\tau(i+1)}}\right) F_\tau\right)\\
&= \left(1 + \frac{(t-1)q^{w_\tau(i)}t^{c_\tau(i)}}{q^{w_\tau(i)}t^{c_\tau(i)} - q^{w_\tau(i+1)}t^{c_\tau(i+1)}}\right) \epsilon^{(n)}(F_\tau)\\
&= \left(\frac{q^{w_\tau(i+1)}t^{c_\tau(i+1)} - q^{w_\tau(i)}t^{c_\tau(i)}}{q^{w_\tau(i+1)}t^{c_\tau(i+1)} - q^{w_\tau(i)}t^{c_\tau(i)}}\right) \epsilon^{(n)}(F_\tau).
\end{aligned}$$

Now using an induction argument nearly identical to the proof of Corollary 3.4.5 we see that for any $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda;T)$

$$\epsilon^{(n)}(F_\tau) = \prod_{(\square_1,\square_2)\in \mathrm{Inv}(\tau)} \left(\frac{q^{T(\square_1)}t^{c(\square_1)} - q^{T(\square_2)}t^{c(\square_2)}}{q^{T(\square_1)}t^{c(\square_1)} - q^{T(\square_2)}t^{c(\square_2)+1}} \right) \epsilon^{(n)}(F_{\mathrm{top}(T)}).$$

$\square$

Corollary 3.5.5. *For $\mathfrak{p}_\lambda(\tau) = T \in \mathrm{RSSYT}_{\geq 0}(\lambda)$*

$$\epsilon^{(n)}(F_\tau) = K_T(q,t) \prod_{(\square_1,\square_2)\in \mathrm{Inv}(\tau)} \left(\frac{q^{T(\square_1)}t^{c(\square_1)} - q^{T(\square_2)}t^{c(\square_2)}}{q^{T(\square_1)}t^{c(\square_1)} - q^{T(\square_2)}t^{c(\square_2)+1}} \right) P_T$$

where

$$K_T(q,t) := \frac{[\mu(T)]_t!}{[n]_t!} \prod_{(\square_1,\square_2)\in \mathrm{Inv}(\min(T))} \left(\frac{q^{T(\square_1)}t^{c(\square_1)} - q^{T(\square_2)}t^{c(\square_2)+1}}{q^{T(\square_1)}t^{c(\square_1)} - q^{T(\square_2)}t^{c(\square_2)}} \right).$$

Proof. We begin by noting that from Lemma 3.5.4 applied to $\min(T)$:

$$\epsilon^{(n)}(F_{\min}) = \prod_{(\square_1,\square_2)\in \mathrm{Inv}(\tau)} \left(\frac{q^{T(\square_1)}t^{c(\square_1)} - q^{T(\square_2)}t^{c(\square_2)}}{q^{T(\square_1)}t^{c(\square_1)} - q^{T(\square_2)}t^{c(\square_2)+1}} \right) \epsilon^{(n)}(F_{\mathrm{top}(T)}).$$

But from Proposition 3.5.3 we know that $\epsilon^{(n)}(F_{\min}) = \frac{[\mu(T)]_t!}{[n]_t!} P_T$ so

$$\prod_{(\square_1,\square_2)\in \mathrm{Inv}(\tau)} \left(\frac{q^{T(\square_1)}t^{c(\square_1)} - q^{T(\square_2)}t^{c(\square_2)}}{q^{T(\square_1)}t^{c(\square_1)} - q^{T(\square_2)}t^{c(\square_2)+1}} \right) \epsilon^{(n)}(F_{\mathrm{top}(T)}) = \frac{[\mu(T)]_t!}{[n]_t!} P_T.$$

Thus

$$\epsilon^{(n)}(F_{\mathrm{top}(T)}) = K_T(q,t) P_T$$

as defined in the corollary statement above.

Lastly, we can now use Lemma 3.5.4 to finish the proof. $\square$

The last lemma of this section relates the action of $e_r[X]^\bullet$ to the action of γ_n^r on symmetrized elements.

Lemma 3.5.6. *For $1 \leq r \leq n$, $\epsilon^{(n)} e_r[X_1 + \ldots + X_n]\epsilon^{(n)} = t^{-((n-1)+\ldots+(n-r))} e_r[\frac{1-t^n}{1-t}]\epsilon^{(n)}\gamma_n^r \epsilon^{(n)}$.*

Proof. First, we will show by induction that for $1 \leq r \leq n$

$$\gamma_n^r = t^{(n-1)+\ldots+(n-r)} (T_{n-1}^{-1}\cdots T_1^{-1})(T_{n-1}^{-1}\cdots T_2^{-1}T_1)\cdots(T_{n-1}^{-1}\cdots T_r^{-1}T_{r-1}\cdots T_1)X_1\cdots X_r.$$

For $r = 1$ we see that

$$\gamma_n = X_n T_{n-1} \cdots T_1 = t^{n-1} T_{n-1}^{-1} \cdots T_1^{-1} X_1.$$

Now suppose this equation holds for some $1 \leq r \leq n-1$. Then we have

$$\gamma_n^{r+1}$$

$$= \gamma_n^r \gamma_n$$

$$= t^{(n-1)+\ldots+(n-r)} (T_{n-1}^{-1} \cdots T_1^{-1})(T_{n-1}^{-1} \cdots T_2^{-1} T_1) \cdots (T_{n-1}^{-1} \cdots T_r^{-1} T_{r-1} \cdots T_1)$$

$$\times X_1 \cdots X_r t^{n-1} T_{n-1}^{-1} \cdots T_1^{-1} X_1$$

$$= t^{(n-1)+\ldots+(n-r)} t^{n-1} (T_{n-1}^{-1} \cdots T_1^{-1})(T_{n-1}^{-1} \cdots T_2^{-1} T_1) \cdots (T_{n-1}^{-1} \cdots T_r^{-1} T_{r-1} \cdots T_1)$$

$$\times X_1 \cdots X_r T_{n-1}^{-1} \cdots T_1^{-1} X_1$$

$$= t^{(n-1)+\ldots+(n-r)} t^{n-1} (T_{n-1}^{-1} \cdots T_1^{-1})(T_{n-1}^{-1} \cdots T_2^{-1} T_1) \cdots (T_{n-1}^{-1} \cdots T_r^{-1} T_{r-1} \cdots T_1) T_{n-1}^{-1} \cdots T_{r+1}^{-1}$$

$$\times X_1 \cdots X_r T_r^{-1} \cdots T_1^{-1} X_1.$$

A simple calculation verifies that

$$X_1 \cdots X_r T_r^{-1} \cdots T_1^{-1} = t^{-r} T_r \cdots T_1 X_2 \cdots X_{r+1}.$$

Therefore,

$$\gamma_n^{r+1}$$

$$= t^{(n-1)+\ldots+(n-r)} t^{n-1} (T_{n-1}^{-1} \cdots T_1^{-1})(T_{n-1}^{-1} \cdots T_2^{-1} T_1) \cdots (T_{n-1}^{-1} \cdots T_r^{-1} T_{r-1} \cdots T_1) T_{n-1}^{-1} \cdots T_{r+1}^{-1}$$

$$\times t^{-r} T_r \cdots T_1 X_1 X_2 \cdots X_{r+1}$$

$$= t^{(n-1)+\ldots+(n-r)+(n-(r+1))} (T_{n-1}^{-1} \cdots T_1^{-1})(T_{n-1}^{-1} \cdots T_2^{-1} T_1) \cdots (T_{n-1}^{-1} \cdots T_r^{-1} T_{r-1} \cdots T_1)$$

$$\times (T_{n-1}^{-1} \cdots T_{r+1}^{-1} T_r \cdots T_1) X_1 \cdots X_{r+1}$$

which is of the correct form.

Now we see that for any $1 \leq r \leq n$,

$$\begin{aligned}
&\epsilon^{(n)}\gamma_n^r\epsilon^{(n)} \\
&= \epsilon^{(n)}t^{(n-1)+\ldots+(n-r)}(T_{n-1}^{-1}\cdots T_1^{-1})(T_{n-1}^{-1}\cdots T_2^{-1}T_1)\cdots(T_{n-1}^{-1}\cdots T_r^{-1}T_{r-1}\cdots T_1)X_1\cdots X_r\epsilon^{(n)} \\
&= t^{(n-1)+\ldots+(n-r)}\epsilon^{(n)}X_1\cdots X_r\epsilon^{(n)}.
\end{aligned}$$

Suppose that $1 = i_0 \leq i_1 < \ldots < i_r \leq i_{r+1} = n$ with $i_j < i_{j+1} - 1$ for some $0 \leq j \leq r$. Then

$$\begin{aligned}
&X_{i_1}\cdots X_{i_{j-1}}X_{i_j+1}X_{i_{j+1}}X_{i_{j+2}}\cdots X_{i_r} \\
&= X_{i_1}\cdots X_{i_{j-1}}(t^{-1}T_{i_j}^{-1}X_{i_j}T_{i_j}^{-1})X_{i_{j+1}}X_{i_{j+2}}\cdots X_{i_r} \\
&= tT_{i_j}^{-1}X_{i_1}\cdots X_{i_{j-1}}X_{i_j}X_{i_{j+1}}X_{i_{j+2}}\cdots X_{i_r}T_{i_j}^{-1} \\
&= tT_{i_j}^{-1}X_{i_1}\cdots X_{i_r}T_{i_j}^{-1}
\end{aligned}$$

which shows that

$$\epsilon^{(n)}X_{i_1}\cdots X_{i_{j-1}}X_{i_j+1}X_{i_{j+1}}X_{i_{j+2}}\cdots X_{i_r}\epsilon^{(n)} = t\epsilon^{(n)}X_{i_1}\cdots X_{i_r}\epsilon^{(n)}.$$

It follows that for any $1 \leq i_1 < \ldots < i_r \leq n$

$$\epsilon^{(n)}X_{i_1}\cdots X_{i_r}\epsilon^{(n)} = t^{(i_r-r)+\ldots+(i_1-1)}\epsilon^{(n)}X_1\cdots X_r\epsilon^{(n)}.$$

Now we see

$$
\begin{aligned}
&\epsilon^{(n)} e_r[X_1 + \ldots + X_n]\epsilon^{(n)} \\
&= \epsilon^{(n)} \left(\sum_{1\le i_1<\ldots<i_r\le n} X_{i_1} \cdots X_{i_r} \right) \epsilon^{(n)} \\
&= \sum_{1\le i_1<\ldots<i_r\le n} \epsilon^{(n)} X_{i_1} \cdots X_{i_r} \epsilon^{(n)} \\
&= \sum_{1\le i_1<\ldots<i_r\le n} t^{(i_r-r)+\ldots+(i_1-1)} \epsilon^{(n)} X_1 \cdots X_r \epsilon^{(n)} \\
&= \sum_{1\le i_1<\ldots<i_r\le n} t^{(i_r-r)+\ldots+(i_1-1)} t^{-((n-1)+\ldots+(n-r))} \epsilon^{(n)} \gamma_n^r \epsilon^{(n)} \\
&= t^{-((n-1)+\ldots+(n-r))} \left(\sum_{1\le i_1<\ldots<i_r\le n} t^{(i_1-1)+\ldots+(i_r-r)} \right) \epsilon^{(n)} \gamma_n^r \epsilon^{(n)} \\
&= t^{-((n-1)+\ldots+(n-r))} e_r(1, \ldots, t^{n-1}) \epsilon^{(n)} \gamma_n^r \epsilon^{(n)} \\
&= t^{-((n-1)+\ldots+(n-r))} e_r \left[\frac{1-t^n}{1-t} \right] \epsilon^{(n)} \gamma_n^r \epsilon^{(n)}.
\end{aligned}
$$

$\square$

3.5.2. Pieri Rule. Using the above lemmas, we may derive an explicit formula for the action of $e_r[X]^\bullet$ on the symmetric v.v. Macdonald polynomials in the finite variable situation. We will then use the stability of the P_T to derive a similar formula for the $\mathfrak{P}_T$.

THEOREM 3.5.7. *For $T \in \mathrm{RSSYT}_{\geq 0}(\lambda)$ and $1 \leq r \leq n$ we have the expansion*

$$e_r[X_1 + \ldots + X_n]P_T = \sum_S d_{S,T}^{(r)} P_S$$

where

$$\frac{d_{S,T}^{(r)}}{t^{\binom{r}{2}} e_r\left[\frac{1-t^n}{1-t}\right] K_S(q,t)} = \sum_{\substack{\tau \in \mathrm{PSYT}_{\geq 0}(\lambda;T) \\ s.t. \\ \Psi^r(\tau) \in \mathrm{PSYT}_{\geq 0}(\lambda;S)}} t^{c_\tau(1)+\ldots+c_\tau(r)} H(\tau, \Psi^r(\tau))$$

where $H(\tau, \Psi^r(\tau))$ is given by

$$\prod_{(\Box_1,\Box_2)\in \mathrm{Inv}(\tau)} \left(\frac{q^{T(\Box_1)}t^{c(\Box_1)+1} - q^{T(\Box_2)}t^{c(\Box_2)}}{q^{T(\Box_1)}t^{c(\Box_1)} - q^{T(\Box_2)}t^{c(\Box_2)}} \right) \prod_{(\Box_1,\Box_2)\in \mathrm{Inv}(\Psi^r(\tau))} \left(\frac{q^{S(\Box_1)}t^{c(\Box_1)} - q^{S(\Box_2)}t^{c(\Box_2)}}{q^{S(\Box_1)}t^{c(\Box_1)} - q^{S(\Box_2)}t^{c(\Box_2)+1}} \right)$$

and T' ranges over all $T' \in \mathrm{RSSYT}_{\geq 0}(\lambda)$ one can obtain from T by adding r to the boxes of T with at most one 1 being added to each box.

PROOF. Using Lemma 3.4.5 and Remark 28 we find

$$\begin{aligned}
&e_r[X_1 + \ldots + X_n]P_T \\
&= \epsilon^{(n)} e_r[X_1 + \ldots + X_n] \epsilon^{(n)}(P_T) \\
&= t^{-((n-1)+\ldots+(n-r))} e_r\left[\frac{1-t^n}{1-t}\right] \epsilon^{(n)} \gamma_n^r \epsilon^{(n)}(P_T) \\
&= t^{-((n-1)+\ldots+(n-r))} e_r\left[\frac{1-t^n}{1-t}\right] \epsilon^{(n)} \gamma_n^r (P_T) \\
&= t^{-((n-1)+\ldots+(n-r))} e_r\left[\frac{1-t^n}{1-t}\right] \\
&\times \epsilon^{(n)} \gamma_n^r \sum_{\tau \in \mathrm{PSYT}_{\geq 0}(\lambda;T)} \prod_{(\Box_1,\Box_2)\in \mathrm{Inv}(\tau)} \left(\frac{q^{T(\Box_1)}t^{c(\Box_1)+1} - q^{T(\Box_2)}t^{c(\Box_2)}}{q^{T(\Box_1)}t^{c(\Box_1)} - q^{T(\Box_2)}t^{c(\Box_2)}} \right) F_\tau
\end{aligned}$$

$$= t^{-((n-1)+\ldots+(n-r))} e_r \left[\frac{1-t^n}{1-t}\right]$$

$$\times \sum_{\tau\in \mathrm{PSYT}_{\geq 0}(\lambda;T)} \prod_{(\square_1,\square_2)\in \mathrm{Inv}(\tau)} \left(\frac{q^{T(\square_1)}t^{c(\square_1)+1}-q^{T(\square_2)}t^{c(\square_2)}}{q^{T(\square_1)}t^{c(\square_1)}-q^{T(\square_2)}t^{c(\square_2)}}\right) \epsilon^{(n)}\gamma_n^r(F_\tau)$$

$$= t^{-((n-1)+\ldots+(n-r))} e_r \left[\frac{1-t^n}{1-t}\right] \sum_{\tau\in \mathrm{PSYT}_{\geq 0}(\lambda;T)} \prod_{(\square_1,\square_2)\in \mathrm{Inv}(\tau)} \left(\frac{q^{T(\square_1)}t^{c(\square_1)+1}-q^{T(\square_2)}t^{c(\square_2)}}{q^{T(\square_1)}t^{c(\square_1)}-q^{T(\square_2)}t^{c(\square_2)}}\right)$$

$$\times \epsilon^{(n)}(t^{r(n-1)}t^{c_\tau(1)+\ldots+c_\tau(r)}F_{\Psi^r(\tau)})$$

$$= t^{\binom{r}{2}} e_r \left[\frac{1-t^n}{1-t}\right] \sum_{\tau\in \mathrm{PSYT}_{\geq 0}(\lambda;T)} t^{c_\tau(1)+\ldots+c_\tau(r)} \prod_{(\square_1,\square_2)\in \mathrm{Inv}(\tau)} \left(\frac{q^{T(\square_1)}t^{c(\square_1)+1}-q^{T(\square_2)}t^{c(\square_2)}}{q^{T(\square_1)}t^{c(\square_1)}-q^{T(\square_2)}t^{c(\square_2)}}\right)$$

$$\times \epsilon^{(n)}(F_{\Psi^r(\tau)}).$$

From Corollary 3.5.5,

$$\epsilon^{(n)}(F_{\Psi^r(\tau)})$$

$$= \mathbb{1}\left(\mathfrak{p}_\lambda(\Psi^r(\tau)) \in \mathrm{RSSYT}_{\geq 0}(\lambda)\right) K_{\mathfrak{p}_\lambda(\Psi^r(\tau))}(q,t)$$

$$\times \prod_{(\square_1,\square_2)\in \mathrm{Inv}(\Psi^r(\tau))} \left(\frac{q^{T(\square_1)}t^{c(\square_1)}-q^{T(\square_2)}t^{c(\square_2)}}{q^{T(\square_1)}t^{c(\square_1)}-q^{T(\square_2)}t^{c(\square_2)+1}}\right) P_{\mathfrak{p}_\lambda(\Psi^r(\tau))}.$$

Hence, by collecting coefficients around each P_S for $S \in \mathrm{RSSYT}_{\geq 0}(\lambda)$ we see that

$$e_r[X_1+\ldots+X_n]P_T = \sum_S d_{S,T}^{(r)} P_S$$

where $d_{S,T}^{(r)}$ are as given in the theorem statement above.

Lastly, if $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda;T)$ then the boxes of λ containing the labels $1,\ldots,r$ (with some powers of q given by T) are exactly those boxes $\square \in \lambda$ with $\mathfrak{p}_\lambda(\Psi^r(\tau))(\square) = T(\square)+1$. Thus if $S = \mathfrak{p}_\lambda(\Psi^r(\tau)) \in \mathrm{RSSYT}_{\geq 0}(\lambda)$ then we may obtain S from T by adding the value 1 to r boxes of T with at most one 1 added to each box.

$\square$

DEFINITION 3.5.8. *For $S,T \in \Omega(\lambda)$ and $r \geq 1$ define $\mathfrak{d}^{(r)}_{S,T} \in \mathbb{Q}(q,t)$ by*

$$e_r[X]\,\mathfrak{P}_T = \sum_{S \in \Omega(\lambda)} \mathfrak{d}^{(r)}_{S,T}\,\mathfrak{P}_S\,.$$

REMARK 33. *Note that from Theorem 3.5.7 it is clear for $T \in \Omega(\lambda)$ and $r \geq 1$ that each $S \in \Omega(\lambda)$ such that $\mathfrak{d}^{(r)}_{S,T} \neq 0$ will necessarily be obtained from T by adding r to the boxes of T with at most one 1 being added to each box. As such the set of such S is finite. Further, any such S has* $\mathrm{rk}(S) \leq \mathrm{rk}(T) + r$.

As an immediate consequence of Theorem 3.5.7 and the definition of $\mathfrak{P}_T$ from Definition 3.4.9 we obtain the following result.

COROLLARY 3.5.9 (Pieri Rule). *Let $S,T \in \Omega(\lambda)$ and $r \geq 1$. For all $n \geq \mathrm{rk}(T) + r$*

$$\mathfrak{d}^{(r)}_{S,T} = d^{(r)}_{S|_{\lambda(n)},T|_{\lambda(n)}}\,.$$

3.5.3. Non-vanishing for e_1 Pieri Coefficients. In this section we will prove that if T,S satisfy a simple combinatorial relationship then $\mathfrak{d}^{(1)}_{T',T} \neq 0$. This will be instrumental in the proof that the modules $\widetilde{W}_\lambda$ are cyclic.

DEFINITION 3.5.10. *Let $\lambda \in \mathbb{Y}$ and $T \in \mathrm{RSSYT}_{\geq 0}(\lambda)$. A box $\square_0$ in λ is T-**raisable** if the labelling S defined by*

$$S(\square) = \begin{cases} T(\square) & \square \neq \square_0 \\ T(\square)+1 & \square = \square_0. \end{cases}$$

*is also in $\mathrm{RSSYT}_{\geq 0}(\lambda)$. We say that S is obtained by raising the box $\square_0$ of T. Further, we say that $\square_0$ is a S-**lowerable** box in λ.*

We will write $T \uparrow S$ if S may be obtained by raising one box of T.

REMARK 34. *We may define a partial order $\sqsubseteq$ on the set $\mathrm{RSSYT}_{\geq 0}(\lambda)$ simply by*

$$T \sqsubseteq S \leftrightarrow \forall\, \square \in \lambda^{(\infty)},\ T(\square) \leq S(\square).$$

Then the relation $T \uparrow S$ defined in Definition 3.5.10 is simply the cover relation of $\sqsubseteq$. Lastly, we may extend the definitions of raisable/lowerable boxes and of the relation $T \uparrow S$ to $\Omega(\lambda)$ analogously in the obvious way.

We require the following lemmas.

LEMMA 3.5.11. *Let $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda; T)$ for $T \in \mathrm{RSSYT}_{\geq 0}(\lambda)$. If $(\square_1, \square_2) \in \mathrm{Inv}(\tau)$ with $T(\square_1) = T(\square_2)$ then $c(\square_2) - c(\square_1) \geq 2$.*

PROOF. Since $T \in \mathrm{RSSYT}_{\geq 0}(\lambda)$, for all $n \geq 0$ the set of boxes $\{\square \in \lambda | T(\square) = n\}$ is a skew-diagram consisting of a union of disjoint horizontal strips. Suppose $(\square_1, \square_2) \in \mathrm{Inv}(\tau)$ with $T(\square_1) = T(\square_2) = n$. Then $\square_1$ and $\square_2$ cannot be in the same horizontal strip component of $\{\square \in \lambda | T(\square) = n\}$. Further, $\square_1$ must be to the left of $\square_2$. Hence, $c(\square_2) - c(\square_1) \geq 2$. $\square$

LEMMA 3.5.12. *Let $T \in \mathrm{RSSYT}_{\geq 0}(\lambda)$. Given a T-raisable box of λ, $\square_0$, there exists a unique $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda; T)$ such that*

- $\tau(\square_0) = 1q^a$ *for some* $a \geq 0$
- $\mathrm{inv}(\tau) = S(T)(\square_0) - 1$.

PROOF. Since the count $\mathrm{inv}(\tau) = S(T)(\square_0) - 1$ is tight there exists at most one such labelling. We may simply define $\tau \in \mathrm{PSYT}_{\geq 0}$ by labelling the boxes $\square \in \lambda$ with $\square <_T \square_0$ with the labels $\{2, \ldots, S(T)(\square_0) - 1$ following the box ordering $S(T)$. We then fill the boxes $\square >_T \square_0$ with the values $\{S(T)(\square_0), \ldots, n\}$ following the box ordering $S(T)$. Thus τ has exactly $S(T)(\square_0) - 1$ inversion pairs. $\square$

LEMMA 3.5.13. *Let $T, T' \in \mathrm{RSSYT}_{\geq 0}(\lambda)$ with $T \uparrow T'$. Let $\square_0$ be the box of λ on which T and T' differ. Let $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda; T)$ with $\Psi(\tau) \in \mathrm{PSYT}_{\geq 0}(\lambda; T')$. Then we have the following:*

- $\mathrm{Inv}(\tau) = \{(\square_1, \square_2) \in \mathrm{Inv}(\tau) | \square_i \neq \square_0\} \sqcup \{(\square, \square_0) | \square <_T \square_0\}$
- $\mathrm{Inv}(\Psi(\tau)) = \{(\square_1, \square_2) \in \mathrm{Inv}(\Psi(\tau)) | \square_i \neq \square_0\} \sqcup \{(\square_0, \square) | \square_0 <_{T'} \square\}$
- $\{(\square_1, \square_2) \in \mathrm{Inv}(\tau) | \square_i \neq \square_0\} = \{(\square_1, \square_2) \in \mathrm{Inv}(\Psi(\tau)) | \square_i \neq \square_0\}$.

PROOF. This result follows by simple case work which we leave to the reader. $\square$

Putting these lemmas together we may show the following:

Theorem 3.5.14. *Let $\lambda \in \mathbb{Y}$ and $T, T' \in \mathrm{RSSYT}_{\geq 0}(\lambda)$ with $T \uparrow T'$. Then $d^{(1)}_{T',T} \neq 0$.*

Proof. Let $\Box_0$ be the T-raisable box on which T and T' differ. From 3.5.7 we see that

$$\frac{d^{(1)}_{T',T}}{(\frac{1-t^n}{1-t})K_{T'}(q,t)} = \sum_{\substack{\tau\in\mathrm{PSYT}_{\geq 0}(\lambda;T)\\ \text{s.t.}\\ \Psi(\tau)\in\mathrm{PSYT}_{\geq 0}(\lambda;T')}} t^{c_\tau(1)} \prod_{(\Box_1,\Box_2)\in\mathrm{Inv}(\tau)} \left(\frac{q^{T(\Box_1)}t^{c(\Box_1)+1}-q^{T(\Box_2)}t^{c(\Box_2)}}{q^{T(\Box_1)}t^{c(\Box_1)}-q^{T(\Box_2)}t^{c(\Box_2)}}\right) \times$$
$$\prod_{(\Box_1,\Box_2)\in\mathrm{Inv}(\Psi(\tau))} \left(\frac{q^{T'(\Box_1)}t^{c(\Box_1)}-q^{T'(\Box_2)}t^{c(\Box_2)}}{q^{T'(\Box_1)}t^{c(\Box_1)}-q^{T'(\Box_2)}t^{c(\Box_2)+1}}\right).$$

Therefore, it suffices to show that the sum on the right hand side of the above equation is nonzero. If $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda;T)$ with $\Psi(\tau) \in \mathrm{PSYT}_{\geq 0}(\lambda;T')$ then $c_\tau(1) = c(\Box_0)$. Hence, we may factor out the term $t^{c_\tau(1)} = t^{c(\Box_0)}$ outside the sum. From Lemma 3.5.13 we have the following for any $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda;T)$ with $\Psi(\tau) \in \mathrm{PSYT}_{\geq 0}(\lambda;T')$:

$$\prod_{(\Box_1,\Box_2)\in\mathrm{Inv}(\tau)} \left(\frac{q^{T(\Box_1)}t^{c(\Box_1)+1}-q^{T(\Box_2)}t^{c(\Box_2)}}{q^{T(\Box_1)}t^{c(\Box_1)}-q^{T(\Box_2)}t^{c(\Box_2)}}\right) \prod_{(\Box_1,\Box_2)\in\mathrm{Inv}(\Psi(\tau))} \left(\frac{q^{T'(\Box_1)}t^{c(\Box_1)}-q^{T'(\Box_2)}t^{c(\Box_2)}}{q^{T'(\Box_1)}t^{c(\Box_1)}-q^{T'(\Box_2)}t^{c(\Box_2)+1}}\right)$$
$$= \prod_{\Box <_T \Box_0} \left(\frac{q^{T(\Box)}t^{c(\Box)+1}-q^{T(\Box_0)}t^{c(\Box_0)}}{q^{T(\Box)}t^{c(\Box)}-q^{T(\Box_0)}t^{c(\Box_0)}}\right) \prod_{\Box_0 <_{T'} \Box} \left(\frac{q^{T(\Box_0)+1}t^{c(\Box_0)}-q^{T(\Box)}t^{c(\Box)}}{q^{T(\Box_0)+1}t^{c(\Box_0)}-q^{T(\Box)}t^{c(\Box)+1}}\right)$$
$$\times \prod_{\substack{(\Box_1,\Box_2)\in\mathrm{Inv}(\tau)\\ \Box_i\neq\Box_0}} \left(\frac{q^{T(\Box_1)}t^{c(\Box_1)+1}-q^{T(\Box_2)}t^{c(\Box_2)}}{q^{T(\Box_1)}t^{c(\Box_1)}-q^{T(\Box_2)}t^{c(\Box_2)+1}}\right).$$

The first two products above are nonzero and do not depend on τ and can therefore be brought outside the summation

$$\sum_{\substack{\tau\in\mathrm{PSYT}_{\geq 0}(\lambda;T)\\ \text{s.t.}\\ \Psi(\tau)\in\mathrm{PSYT}_{\geq 0}(\lambda;T')}} .$$

Hence, it suffices to show that

$$\sum_{\substack{\tau\in\mathrm{PSYT}_{\geq 0}(\lambda;T)\\ \text{s.t.}\\ \Psi(\tau)\in\mathrm{PSYT}_{\geq 0}(\lambda;T')}} \prod_{\substack{(\Box_1,\Box_2)\in\mathrm{Inv}(\tau)\\ \Box_i\neq\Box_0}} \left(\frac{q^{T(\Box_1)}t^{c(\Box_1)+1}-q^{T(\Box_2)}t^{c(\Box_2)}}{q^{T(\Box_1)}t^{c(\Box_1)}-q^{T(\Box_2)}t^{c(\Box_2)+1}}\right) \neq 0.$$

Notice that we can rewrite the above product terms in the following way:

$$\prod_{\substack{(\Box_1,\Box_2)\in\mathrm{Inv}(\tau)\\ \Box_i\neq\Box_0}}\left(\frac{q^{T(\Box_1)}t^{c(\Box_1)+1}-q^{T(\Box_2)}t^{c(\Box_2)}}{q^{T(\Box_1)}t^{c(\Box_1)}-q^{T(\Box_2)}t^{c(\Box_2)+1}}\right)$$
$$=t^{\mathrm{inv}(\tau)-S(T)(\Box_0)+1}\prod_{\substack{(\Box_1,\Box_2)\in\mathrm{Inv}(\tau)\\ \Box_i\neq\Box_0}}\left(\frac{1-q^{T(\Box_2)-T(\Box_1)}t^{c(\Box_2)-c(\Box_1)-1}}{1-q^{T(\Box_2)-T(\Box_1)}t^{c(\Box_2)-c(\Box_1)+1}}\right)$$

Therefore,

$$\sum_{\substack{\tau\in\mathrm{PSYT}_{\geq 0}(\lambda;T)\\ \text{s.t.}\\ \Psi(\tau)\in\mathrm{PSYT}_{\geq 0}(\lambda;T')}}\prod_{\substack{(\Box_1,\Box_2)\in\mathrm{Inv}(\tau)\\ \Box_i\neq\Box_0}}\left(\frac{q^{T(\Box_1)}t^{c(\Box_1)+1}-q^{T(\Box_2)}t^{c(\Box_2)}}{q^{T(\Box_1)}t^{c(\Box_1)}-q^{T(\Box_2)}t^{c(\Box_2)+1}}\right)$$
$$=\sum_{\substack{\tau\in\mathrm{PSYT}_{\geq 0}(\lambda;T)\\ \text{s.t.}\\ \Psi(\tau)\in\mathrm{PSYT}_{\geq 0}(\lambda;T')}}t^{\mathrm{inv}(\tau)-S(T)(\Box_0)+1}\prod_{\substack{(\Box_1,\Box_2)\in\mathrm{Inv}(\tau)\\ \Box_i\neq\Box_0}}\left(\frac{1-q^{T(\Box_2)-T(\Box_1)}t^{c(\Box_2)-c(\Box_1)-1}}{1-q^{T(\Box_2)-T(\Box_1)}t^{c(\Box_2)-c(\Box_1)+1}}\right)$$

Now we have by definition for any inversion pair $(\Box_1,\Box_2)$ that $T(\Box_2)-T(\Box_1)\leq 0$. Therefore, by limiting $q\to\infty$ we see that

$$\lim_{q\to\infty}\sum_{\substack{\tau\in\mathrm{PSYT}_{\geq 0}(\lambda;T)\\ \text{s.t.}\\ \Psi(\tau)\in\mathrm{PSYT}_{\geq 0}(\lambda;T')}}t^{\mathrm{inv}(\tau)-S(T)(\Box_0)+1}\prod_{\substack{(\Box_1,\Box_2)\in\mathrm{Inv}(\tau)\\ \Box_i\neq\Box_0}}\left(\frac{1-q^{T(\Box_2)-T(\Box_1)}t^{c(\Box_2)-c(\Box_1)-1}}{1-q^{T(\Box_2)-T(\Box_1)}t^{c(\Box_2)-c(\Box_1)+1}}\right)$$
$$=\sum_{\substack{\tau\in\mathrm{PSYT}_{\geq 0}(\lambda;T)\\ \text{s.t.}\\ \Psi(\tau)\in\mathrm{PSYT}_{\geq 0}(\lambda;T')}}t^{\mathrm{inv}(\tau)-S(T)(\Box_0)+1}\prod_{\substack{(\Box_1,\Box_2)\in\mathrm{Inv}(\tau)\\ \Box_i\neq\Box_0\\ T(\Box_1)=T(\Box_2)}}\left(\frac{1-t^{c(\Box_2)-c(\Box_1)-1}}{1-t^{c(\Box_2)-c(\Box_1)+1}}\right).$$

Using Lemma 3.5.11 we see that for each of the inversion pairs $(\Box_1,\Box_2)\in\mathrm{Inv}(\tau)$ for τ in $\mathrm{PSYT}_{\geq 0}(\lambda;T)$ with $\Psi(\tau)\in\mathrm{PSYT}_{\geq 0}(\lambda;T')$ and $T(\Box_1)=T(\Box_2)$ that $c(\Box_2)-c(\Box_1)-1\geq 1$. Therefore, if we limit $t\to 0$

$$\lim_{t\to 0} \sum_{\substack{\tau\in \mathrm{PSYT}_{\geq 0}(\lambda;T)\\ \text{s.t.}\\ \Psi(\tau)\in \mathrm{PSYT}_{\geq 0}(\lambda;T')}} t^{\mathrm{inv}(\tau)-S(T)(\square_0)+1} \prod_{\substack{(\square_1,\square_2)\in \mathrm{Inv}(\tau)\\ \square_i\neq \square_0\\ T(\square_1)=T(\square_2)}} \left(\frac{1-t^{c(\square_2)-c(\square_1)-1}}{1-t^{c(\square_2)-c(\square_1)+1}}\right)$$

$$= \sum_{\substack{\tau\in \mathrm{PSYT}_{\geq 0}(\lambda;T)\\ \text{s.t.}\\ \Psi(\tau)\in \mathrm{PSYT}_{\geq 0}(\lambda;T')}} \mathbb{1}\left(\mathrm{inv}(\tau)=S(T)(\square_0)-1\right) \lim_{t\to 0} \prod_{\substack{(\square_1,\square_2)\in \mathrm{Inv}(\tau)\\ \square_i\neq \square_0\\ T(\square_1)=T(\square_2)}} \left(\frac{1-t^{c(\square_2)-c(\square_1)-1}}{1-t^{c(\square_2)-c(\square_1)+1}}\right)$$

$$= \sum_{\substack{\tau\in \mathrm{PSYT}_{\geq 0}(\lambda;T)\\ \text{s.t.}\\ \Psi(\tau)\in \mathrm{PSYT}_{\geq 0}(\lambda;T')}} \mathbb{1}\left(\mathrm{inv}(\tau)=S(T)(\square_0)-1\right) \prod_{\substack{(\square_1,\square_2)\in \mathrm{Inv}(\tau)\\ \square_i\neq \square_0\\ T(\square_1)=T(\square_2)}} (1)$$

$$= \#\{\tau\in \mathrm{PSYT}_{\geq 0}(\lambda;T) | \Psi(\tau)\in \mathrm{PSYT}_{\geq 0}(\lambda;T'), \mathrm{inv}(\tau)=S(T)(\square_0)-1\}.$$

By Lemma 3.5.12, $\#\{\tau\in \mathrm{PSYT}_{\geq 0}(\lambda;T) | \Psi(\tau)\in \mathrm{PSYT}_{\geq 0}(\lambda;T'), \mathrm{inv}(\tau)=S(T)(\square_0)-1\} = 1$ which in particular is not 0. Therefore, $d^{(1)}_{T',T}\neq 0$. $\square$

Using stability we find the following:

Corollary 3.5.15. *Let $\lambda\in\mathbb{Y}$ and $T,T'\in\Omega(\lambda)$ with $T\uparrow T'$. Then $\mathfrak{d}^{(1)}_{T',T}\neq 0$.*

Proof. From Corollary 3.5.9 we know that for all $n\geq \mathrm{rk}(T)+1$

$$\mathfrak{d}^{(1)}_{T',T} = d^{(1)}_{T'|_{\lambda^{(n)}},T|_{\lambda^{(n)}}}.$$

Since T' is obtained from T by increasing the value of a single box of T by 1 we know that the same must be true for $T'|_{\lambda^{(n)}}$ and $T|_{\lambda^{(n)}}$ for all $n\geq \mathrm{rk}(T)+1$. Therefore, from Theorem 3.5.14 we conclude that $\mathfrak{d}^{(1)}_{T',T} = d^{(1)}_{T'|_{\lambda^{(n)}},T|_{\lambda^{(n)}}}\neq 0$. $\square$

The non-vanishing of the e_1 Pieri coefficients is sufficient to prove that the $\widetilde{W}_\lambda$ are cyclic $\mathscr{E}^+$-modules.

Corollary 3.5.16. *For $\lambda\in\mathbb{Y}$, $\widetilde{W}_\lambda$ is a cyclic $\mathscr{E}^+$-module.*

Proof. We will show that $\mathfrak{P}_{T_\lambda^{\min}}$ is a cyclic vector for $\widetilde{W}_\lambda$ i.e. $\mathscr{E}^+ \mathfrak{P}_{T_\lambda^{\min}} = \widetilde{W}_\lambda$. It suffices to show that for every $T \in \Omega(\lambda)$ there exists some $X \in \mathscr{E}^+$ with $X(\mathfrak{P}_{T_\lambda^{\min}}) = \mathfrak{P}_T$. Notice that given any $T \in \Omega(\lambda)$ we may choose any lowerable box $\square_1$ of T and obtain a labelling $T_1 \in \Omega(\lambda)$ by subtracting the value of 1 from $\square_1$ in the labelling T. Continuing in this process yields a sequence of labellings $T_1, T_2, \dots$ with $T_{i+1} \uparrow T_i$ which must eventually terminate as $\deg(T_i) = \deg(T) - i$. It is easy to verify that the only element of $\Omega(\lambda)$ without any lowerable boxes is $T_\lambda^{\min}$ so the sequence $T_1, T_2, \dots$ must end at $T_\lambda^{\min}$. Reversing this process shows that any $T \in \Omega(\lambda)$ may be obtained from $T_\lambda^{\min}$ by a sequence $T_\lambda^{\min} = T_1, \dots, T_n = T$ with $T_i \uparrow T_{i+1}$. Hence, by induction it suffices to show that if $T \uparrow T'$ then there exists $X \in \mathscr{E}^+$ such that $X(\mathfrak{P}_T) = \mathfrak{P}_{T'}$.

Let $T, T' \in \Omega(\lambda)$ with $T \uparrow T'$. Consider the element $X \in \mathscr{E}^+$ defined by

$$X := \prod_{\substack{T \uparrow S \\ S \neq T'}} \left(\frac{P_{0,1} - \sum_{\square \in \lambda^{(\infty)}} (q^{S(\square)} - 1) t^{c(\square)}}{\sum_{\square \in \lambda^{(\infty)}} (q^{T'(\square)} - q^{S(\square)}) t^{c(\square)}} \right).$$

The denominator of the above product is nonzero since $P_{0,1}$ acts with simple spectrum on $\widetilde{W}_\lambda$. Further, as mentioned before the set of $S \in \Omega(\lambda)$ with $T \uparrow S$ is finite so the above product is finite. We have that for $T \uparrow V$

$$\begin{aligned} X(\mathfrak{P}_V) &= \prod_{\substack{T \uparrow S \\ S \neq T'}} \left(\frac{P_{0,1} - \sum_{\square \in \lambda^{(\infty)}} (q^{S(\square)} - 1) t^{c(\square)}}{\sum_{\square \in \lambda^{(\infty)}} (q^{T'(\square)} - q^{S(\square)}) t^{c(\square)}} \right) (\mathfrak{P}_V) \\ &= \prod_{\substack{T \uparrow S \\ S \neq T'}} \left(\frac{\sum_{\square \in \lambda^{(\infty)}} (q^{V(\square)} - q^{S(\square)}) t^{c(\square)}}{\sum_{\square \in \lambda^{(\infty)}} (q^{T'(\square)} - q^{S(\square)}) t^{c(\square)}} \right) \mathfrak{P}_V \\ &= \delta_{V,T'} \mathfrak{P}_V. \end{aligned}$$

From Corollary 3.5.15 we know that $\mathfrak{d}^{(1)}_{T',T} \neq 0$. Therefore, we may consider the element $X' \in \mathscr{E}^+$ defined by

$$X' := \frac{q^{-1}}{\mathfrak{d}^{(1)}_{T',T}} X P_{1,0}.$$

We find that

$$\begin{aligned}
X'(\mathfrak{P}_T) &= \frac{q^{-1}}{\mathfrak{d}^{(1)}_{T',T}} X P_{1,0}(\mathfrak{P}_T) \\
&= \frac{q^{-1}}{\mathfrak{d}^{(1)}_{T',T}} X q e_1^{\bullet}(\mathfrak{P}_T) \\
&= \frac{1}{\mathfrak{d}^{(1)}_{T',T}} X (\sum_{T \uparrow S} \mathfrak{d}^{(1)}_{S,T} \mathfrak{P}_S) \\
&= \frac{1}{\mathfrak{d}^{(1)}_{T',T}} \sum_{T \uparrow S} \mathfrak{d}^{(1)}_{S,T} \delta_{S,T'} \mathfrak{P}_S \\
&= \mathfrak{P}_{T'} .
\end{aligned}$$

$\square$

3.6. Family of (q,t) Product-Sum Identities

In the final section of this chapter we will investigate an interesting family of (q,t) product-sum identities which are derived using the combinatorics underpinning the structure of the generalized symmetric Macdonald functions $\mathfrak{P}_T$ along with some elementary non-archimedean analysis.

Definition 3.6.1. *A non-negative asymptotic periodic standard Young tableau with base shape $\lambda \in \mathbb{Y}$ is a labelling $\tau : \lambda^{(\infty)} \to \{iq^a : i \geq 1, a \geq 0\}$ such that*

- *τ is strictly increasing along rows and columns*
- *The set of boxes $\square \in \lambda^{(\infty)}$ such that $\tau(\square) = iq^a$ for some $i \geq 1$ and $a > 0$ is finite*
- *For all $i \geq 1$ there exists a unique $\square \in \lambda^{(\infty)}$ such that $\tau(\square) = iq^a$ for some $a \geq 0$.*

We will write $\mathrm{APSYT}_{\geq 0}(\lambda)$ for the set of all non-negative asymptotic periodic standard Young tableaux with base shape $\lambda \in \mathbb{Y}$. If $\tau \in \mathrm{APSYT}_{\geq 0}(\lambda)$ has that for every $\square \in \lambda$, $\tau(\square) = iq^0$ for some $i \geq 1$ then we will call τ an asymptotic standard Young tableau with base shape λ. We will write $\mathrm{ASYT}(\lambda)$ for the set of asymptotic standard Young tableau with base shape λ. As an abuse of notation will write $\mathfrak{p}_\lambda : \mathrm{APSYT}_{\geq 0}(\lambda) \to \Omega(\lambda)$ for the map given on $\tau \in \mathrm{APSYT}_{\geq 0}(\lambda)$ by $\mathfrak{p}_\lambda(\tau)(\square) = a$ whenever $\tau(\square) = iq^a$ for some $i \geq 1$. We will let $\mathrm{APSYT}_{\geq 0}(\lambda; T)$ denote the set of all $\tau \in \mathrm{APSYT}_{\geq 0}(\lambda)$ with $\mathfrak{p}_\lambda(\tau) = T$.

Definition 3.6.2. *For $T \in \Omega(\lambda)$ define $S(T) \in \mathrm{ASYT}(\lambda)$ by ordering the boxes of $\lambda^{(\infty)}$ according to $\square_1 \le \square_2$ if and only if*

- $T(\square_1) > T(\square_2)$ *or*
- $T(\square_1) = T(\square_2)$ *and $\square_1$ comes before $\square_2$ in the column-standard labelling of $\lambda^{(\infty)}$.*

Let $\tau \in \mathrm{APSYT}_{\ge 0}(\lambda; T)$. An ordered pair of boxes $(\square_1, \square_2) \in \lambda^{(\infty)} \times \lambda^{(\infty)}$ is called an inversion pair of τ if $S(T)(\square_1) < S(T)(\square_2)$ and $\tau(\square_1) = iq^a$ $\tau(\square_2) = jq^b$ for some $i > j$ and $a, b \ge 0$. The set of all inversion pairs of τ will be denoted by $\mathrm{Inv}(\tau)$. We will write $\mathrm{inv}(\tau) = |\mathrm{Inv}(\tau)|$. Define $\mathrm{rk}(\tau)$ to be the minimal $n \ge n_\lambda$ such that $\tau|_{\lambda^{(\infty)}/\lambda^{(n)}}$ has consecutive labels.

Example. *Consider $T \in \Omega(3,2,1)$ from Example 3.4.2. Then*

$$\tau = \begin{array}{|c|c|c|c|c|c|c|c|c|l}
\hline
1q^3 & 2q^3 & 3q^3 & 5q^2 & 7q^2 & 12q^1 & 13q^0 & 14q^0 & 15q^0 & \ldots \\
\hline
4q^2 & 6q^2 & 11q^1 \\
\cline{1-3}
8q^1 & 9q^1 \\
\cline{1-2}
10q^0 \\
\cline{1-1}
\end{array} \in \mathrm{APSYT}_{\ge 0}(3,2,1;T),$$

$$S(T) = \begin{array}{|c|c|c|c|c|c|c|c|c|l}
\hline
1 & 2 & 3 & 6 & 7 & 11 & 13 & 14 & 15 & \ldots \\
\hline
4 & 5 & 10 \\
\cline{1-3}
8 & 9 \\
\cline{1-2}
12 \\
\cline{1-1}
\end{array},$$

and $rk(T) = 12$.

Recall Corollary 3.5.5 for the definition of $K_T(q,t)$.

Proposition 3.6.3. *For $T \in \mathrm{RSSYT}_{\ge 0}(\lambda)$*

$$\frac{1}{K_T(q,t)} = \sum_{\tau \in \mathrm{PSYT}_{\ge 0}(\lambda;T)} t^{\mathrm{inv}(\tau)} \prod_{(\square_1,\square_2) \in \mathrm{Inv}(\tau)} \left(\frac{1 - q^{T(\square_2)-T(\square_1)} t^{c(\square_2)-c(\square_1)-1}}{1 - q^{T(\square_2)-T(\square_1)} t^{c(\square_2)-c(\square_1)+1}} \right).$$

Proof. Using Corollary 3.4.5 and Corollary 3.5.5 we find

$$
\begin{aligned}
P_T &= \epsilon^{(n)}(P_T) \\
&= \epsilon^{(n)}\left(\sum_{\tau\in \mathrm{PSYT}_{\geq 0}(\lambda;T)} \prod_{(\square_1,\square_2)\in \mathrm{Inv}(\tau)} \left(\frac{q^{T(\square_1)}t^{c(\square_1)+1}-q^{T(\square_2)}t^{c(\square_2)}}{q^{T(\square_1)}t^{c(\square_1)}-q^{T(\square_2)}t^{c(\square_2)}}\right) F_\tau\right) \\
&= \sum_{\tau\in \mathrm{PSYT}_{\geq 0}(\lambda;T)} \prod_{(\square_1,\square_2)\in \mathrm{Inv}(\tau)} \left(\frac{q^{T(\square_1)}t^{c(\square_1)+1}-q^{T(\square_2)}t^{c(\square_2)}}{q^{T(\square_1)}t^{c(\square_1)}-q^{T(\square_2)}t^{c(\square_2)}}\right) \epsilon^{(n)}(F_\tau)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\tau\in \mathrm{PSYT}_{\geq 0}(\lambda;T)} \prod_{(\square_1,\square_2)\in \mathrm{Inv}(\tau)} \left(\frac{q^{T(\square_1)}t^{c(\square_1)+1}-q^{T(\square_2)}t^{c(\square_2)}}{q^{T(\square_1)}t^{c(\square_1)}-q^{T(\square_2)}t^{c(\square_2)}}\right) K_T(q,t) \\
&\times \prod_{(\square_1,\square_2)\in \mathrm{Inv}(\tau)} \left(\frac{q^{T(\square_1)}t^{c(\square_1)}-q^{T(\square_2)}t^{c(\square_2)}}{q^{T(\square_1)}t^{c(\square_1)}-q^{T(\square_2)}t^{c(\square_2)+1}}\right) P_T \\
&= K_T(q,t)\left(\sum_{\tau\in \mathrm{PSYT}_{\geq 0}(\lambda;T)} \prod_{(\square_1,\square_2)\in \mathrm{Inv}(\tau)} \left(\frac{q^{T(\square_1)}t^{c(\square_1)+1}-q^{T(\square_2)}t^{c(\square_2)}}{q^{T(\square_1)}t^{c(\square_1)}-q^{T(\square_2)}t^{c(\square_2)+1}}\right)\right) P_T.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{1}{K_T(q,t)} &= \sum_{\tau\in \mathrm{PSYT}_{\geq 0}(\lambda;T)} \prod_{(\square_1,\square_2)\in \mathrm{Inv}(\tau)} \left(\frac{q^{T(\square_1)}t^{c(\square_1)+1}-q^{T(\square_2)}t^{c(\square_2)}}{q^{T(\square_1)}t^{c(\square_1)}-q^{T(\square_2)}t^{c(\square_2)+1}}\right) \\
&= \sum_{\tau\in \mathrm{PSYT}_{\geq 0}(\lambda;T)} t^{\mathrm{inv}(\tau)} \prod_{(\square_1,\square_2)\in \mathrm{Inv}(\tau)} \left(\frac{1-q^{T(\square_2)-T(\square_1)}t^{c(\square_2)-c(\square_1)-1}}{1-q^{T(\square_2)-T(\square_1)}t^{c(\square_2)-c(\square_1)+1}}\right).
\end{aligned}
$$

□

Our goal now is to compute the limit of both sides of the equation in Proposition 3.6.3 along sequences of the form $(\lambda^{(n)})_{n\geq n_\lambda}$. One side gives an infinite product and the other a power series which are dealt with separately. We require the following straightforward lemmas.

LEMMA 3.6.4. *For $\tau \in \mathrm{APSYT}_{\geq 0}(\lambda;T)$, $\mathrm{rk}(\tau)-\mathrm{rk}(T)\leq \mathrm{inv}(\tau)\leq \binom{\mathrm{rk}(\tau)}{2}$.*

Proof. Any inversion pair $(\square_1, \square_2) \in \mathrm{Inv}(\tau)$ has $\square_1, \square_2 \in \lambda^{(\mathrm{rk}(\tau))}$. Therefore, trivially $\mathrm{inv}(\tau) \leq \binom{\mathrm{rk}(\tau)}{2}$.

For the other side of the inequality, we only need to consider the case when $\mathrm{rk}(\tau) > \mathrm{rk}(T)$ since $\mathrm{inv}(\tau) \geq 0$. Let $\square_0$ be the unique square of $\lambda^{(\mathrm{rk}(\tau))}/\lambda^{(\mathrm{rk}(\tau))}$. Then by of the definition of rank $\tau(\square_0) \neq \mathrm{rk}(\tau)$. Further, for any $\square \in \lambda^{(\mathrm{rk}(\tau))}/\lambda^{(\mathrm{rk}(T))}$ we must have that $\tau(\square) \neq \mathrm{rk}(\tau)$ as τ must be strictly increasing to the right along the horizontal strip $\lambda^{(\mathrm{rk}(\tau))}/\lambda^{(\mathrm{rk}(T))}$. Therefore, if $\square_1$ is the box of $\lambda^{(\mathrm{rk}(T))}$ with $\tau(\square_1) = \mathrm{rk}(\tau)q^a$ for some $a \geq 0$ then for all $\square \in \lambda^{(\mathrm{rk}(\tau))}/\lambda^{(\mathrm{rk}(T))}$ we find that $(\square_1, \square) \in \mathrm{Inv}(\tau)$. Therefore, $\mathrm{inv}(\tau) \geq \mathrm{rk}(\tau) - \mathrm{rk}(T)$. $\square$

Lemma 3.6.5. *For $k \geq 0$ there are only finitely many $\tau \in \mathrm{APSYT}_{\geq 0}(\lambda; T)$ with $\mathrm{rk}(\tau) \leq k$.*

Proof. The map $\{\tau \in \mathrm{APSYT}_{\geq 0}(\lambda;T) | \, \mathrm{rk}(\tau) \leq k\} \to \mathrm{PSYT}_{\geq 0}(\lambda^{(k)};T)$ given by $\tau \to \tau|_{\lambda^{(k)}}$ is easily seen to be a bijection. Since $\mathrm{PSYT}_{\geq 0}(\lambda^{(k)};T)$ is a finite set we are done. $\square$

Corollary 3.6.6. *For $k \geq 0$ there are only finitely many $\tau \in \mathrm{APSYT}_{\geq 0}(\lambda; T)$ with $\mathrm{inv}(\tau) \leq k$.*

Proof. If $\mathrm{inv}(\tau) \leq k$ then by Lemma 3.6.4 we know that $\mathrm{rk}(\tau) \leq k + \mathrm{rk}(T)$. Thus by Lemma 3.6.5

$$\#\{\tau | \, \mathrm{inv}(\tau) \leq k\} \leq \#\{\tau | \, \mathrm{rk}(\tau) \leq k + \mathrm{rk}(T)\} < \infty.$$

$\square$

Lemma 3.6.7. *For $T \in \mathrm{RSSYT}_{\geq 0}(\lambda)$, the set $\mathrm{I}(T) = \mathrm{Inv}(\min(T))$ consists of all pairs of boxes $(\square_1, \square_2) \in \lambda \times \lambda$ with $\square_1 <_T \square_2$ except those pairs with $T(\square_1) = T(\square_2)$ and $\square_1$ before $\square_2$ in the same row.*

Proof. This follows immediately from the definition of $\min(T)$. $\square$

Now we deal with the limit of products.

Proposition 3.6.8. *Let $T \in \Omega(\lambda)$. The sequence $(K_{T|_{\lambda^{(n)}}}(q,t))_{n \geq n_\lambda}$ converges with respect to the t-adic topology on $\mathbb{Q}(q)((t))$ to*

$$\frac{(1-t)^{\mathrm{rk}(T)}[\mu(T|_{\lambda^{(\mathrm{rk}(T))}})]_t!}{\prod_{\square \in \lambda^{(\mathrm{rk}(T))}} \left(1 - q^{-T(\square)} t^{\mathrm{rk}(T) - |\lambda| - c(\square)}\right)} \prod_{(\square_1, \square_2) \in \mathrm{I}(T|_{\lambda^{(\mathrm{rk}(T))}})} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1)}} \right).$$

Proof. Let $n \geq \mathrm{rk}(T)$. From Lemma 3.6.7 we know

$$\mathrm{I}(T|_{\lambda^{(n)}}) = \mathrm{I}(T|_{\lambda^{(\mathrm{rk}(T))}}) \sqcup \{(\Box_1, \Box_2) | \Box_1 \in \lambda^{(\mathrm{rk}(T))}, \Box_2 \in \lambda^{(n)}/\lambda^{(\mathrm{rk}(T))}\}.$$

Therefore,

$$\begin{aligned}
&\prod_{(\Box_1,\Box_2)\in \mathrm{I}(T|_{\lambda^{(n)}})} \left(\frac{1-q^{T(\Box_2)-T(\Box_1)}t^{c(\Box_2)-c(\Box_1)+1}}{1-q^{T(\Box_2)-T(\Box_1)}t^{c(\Box_2)-c(\Box_1)}} \right) \\
&= \prod_{(\Box_1,\Box_2)\in \mathrm{I}(T|_{\lambda^{(\mathrm{rk}(T))}})} \left(\frac{1-q^{T(\Box_2)-T(\Box_1)}t^{c(\Box_2)-c(\Box_1)+1}}{1-q^{T(\Box_2)-T(\Box_1)}t^{c(\Box_2)-c(\Box_1)}} \right) \\
&\times \prod_{\Box_1\in\lambda^{(\mathrm{rk}(T))}} \prod_{\Box_2\in\lambda^{(n)}/\lambda^{(\mathrm{rk}(T))}} \left(\frac{1-q^{T(\Box_2)-T(\Box_1)}t^{c(\Box_2)-c(\Box_1)+1}}{1-q^{T(\Box_2)-T(\Box_1)}t^{c(\Box_2)-c(\Box_1)}} \right) \\
&= \prod_{(\Box_1,\Box_2)\in \mathrm{I}(T|_{\lambda^{(\mathrm{rk}(T))}})} \left(\frac{1-q^{T(\Box_2)-T(\Box_1)}t^{c(\Box_2)-c(\Box_1)+1}}{1-q^{T(\Box_2)-T(\Box_1)}t^{c(\Box_2)-c(\Box_1)}} \right) \prod_{\Box\in\lambda^{(\mathrm{rk}(T))}} \prod_{i=\mathrm{rk}(T)-|\lambda|}^{n-|\lambda|-1} \left(\frac{1-q^{-T(\Box)}t^{i-c(\Box)+1}}{1-q^{-T(\Box)}t^{i-c(\Box)}} \right).
\end{aligned}$$

Note that the following product telescopes:

$$\begin{aligned}
&\prod_{i=\mathrm{rk}(T)-|\lambda|}^{n-|\lambda|-1} \left(\frac{1-q^{-T(\Box)}t^{i-c(\Box)+1}}{1-q^{-T(\Box)}t^{i-c(\Box)}} \right) \\
&= \left(\frac{1-q^{-T(\Box)}t^{\mathrm{rk}(T)-|\lambda|-c(\Box)+1}}{1-q^{-T(\Box)}t^{\mathrm{rk}(T)-|\lambda|-c(\Box)}} \right) \left(\frac{1-q^{-T(\Box)}t^{\mathrm{rk}(T)-|\lambda|-c(\Box)+2}}{1-q^{-T(\Box)}t^{\mathrm{rk}(T)-|\lambda|-c(\Box)+1}} \right) \cdots \left(\frac{1-q^{-T(\Box)}t^{(n-|\lambda|-1)-c(\Box)+1}}{1-q^{-T(\Box)}t^{(n-|\lambda|-1)-c(\Box)}} \right) \\
&= \left(\frac{1-q^{-T(\Box)}t^{n-|\lambda|-c(\Box)}}{1-q^{-T(\Box)}t^{\mathrm{rk}(T)-|\lambda|-c(\Box)}} \right).
\end{aligned}$$

Thus

$$\prod_{(\Box_1,\Box_2)\in \mathrm{I}(T|_{\lambda^{(n)}})} \left(\frac{1-q^{T(\Box_2)-T(\Box_1)}t^{c(\Box_2)-c(\Box_1)+1}}{1-q^{T(\Box_2)-T(\Box_1)}t^{c(\Box_2)-c(\Box_1)}} \right)$$

$$= \prod_{(\Box_1,\Box_2)\in \mathrm{I}(T|_{\lambda^{(\mathrm{rk}(T))}})} \left(\frac{1-q^{T(\Box_2)-T(\Box_1)}t^{c(\Box_2)-c(\Box_1)+1}}{1-q^{T(\Box_2)-T(\Box_1)}t^{c(\Box_2)-c(\Box_1)}} \right) \prod_{\Box\in\lambda^{(\mathrm{rk}(T))}} \left(\frac{1-q^{-T(\Box)}t^{n-|\lambda|-c(\Box)}}{1-q^{-T(\Box)}t^{\mathrm{rk}(T)-|\lambda|-c(\Box)}} \right).$$

Now $\mu(T_{\lambda^{(n)}}) = \mu(T_{\lambda^{(\mathrm{rk}(T))}}) * (n-\mathrm{rk}(T))$ so

$$[\mu(T_{\lambda^{(n)}})]_t! = [\mu(T_{\lambda^{(\mathrm{rk}(T))}})]_t! \cdot [n-\mathrm{rk}(T)]_t!.$$

Putting this together gives

$$\begin{aligned}
&K_{T|_{\lambda^{(n)}}}(q,t) \\
&= \frac{[\mu(T|_{\lambda^{(n)}})]_t!}{[n]_t!} \prod_{(\Box_1,\Box_2)\in \mathrm{I}(T|_{\lambda^{(n)}})} \left(\frac{1-q^{T(\Box_2)-T(\Box_1)}t^{c(\Box_2)-c(\Box_1)+1}}{1-q^{T(\Box_2)-T(\Box_1)}t^{c(\Box_2)-c(\Box_1)}} \right) \\
&= [\mu(T_{\lambda^{(\mathrm{rk}(T))}})]_t! \frac{[n-\mathrm{rk}(T)]_t!}{[n]_t!} \prod_{(\Box_1,\Box_2)\in \mathrm{I}(T|_{\lambda^{(\mathrm{rk}(T))}})} \left(\frac{1-q^{T(\Box_2)-T(\Box_1)}t^{c(\Box_2)-c(\Box_1)+1}}{1-q^{T(\Box_2)-T(\Box_1)}t^{c(\Box_2)-c(\Box_1)}} \right) \\
&\times \prod_{\Box\in\lambda^{(\mathrm{rk}(T))}} \left(\frac{1-q^{-T(\Box)}t^{n-|\lambda|-c(\Box)}}{1-q^{-T(\Box)}t^{\mathrm{rk}(T)-|\lambda|-c(\Box)}} \right).
\end{aligned}$$

From here it is simple to see

$$
\begin{aligned}
&\lim_{n\to\infty} K_{T|_{\lambda^{(n)}}}(q,t)\\
&= \lim_{n\to\infty} [\mu(T_{\lambda^{(\mathrm{rk}(T))}})]_t! \frac{[n-\mathrm{rk}(T)]_t!}{[n]_t!} \prod_{(\square_1,\square_2)\in \mathrm{I}(T|_{\lambda^{(\mathrm{rk}(T))}})} \left(\frac{1-q^{T(\square_2)-T(\square_1)}t^{c(\square_2)-c(\square_1)+1}}{1-q^{T(\square_2)-T(\square_1)}t^{c(\square_2)-c(\square_1)}} \right)\\
&\times \prod_{\square\in\lambda^{(\mathrm{rk}(T))}} \left(\frac{1-q^{-T(\square)}t^{n-|\lambda|-c(\square)}}{1-q^{-T(\square)}t^{\mathrm{rk}(T)-|\lambda|-c(\square)}} \right)\\
&= [\mu(T_{\lambda^{(\mathrm{rk}(T))}})]_t! \prod_{(\square_1,\square_2)\in \mathrm{I}(T|_{\lambda^{(\mathrm{rk}(T))}})} \left(\frac{1-q^{T(\square_2)-T(\square_1)}t^{c(\square_2)-c(\square_1)+1}}{1-q^{T(\square_2)-T(\square_1)}t^{c(\square_2)-c(\square_1)}} \right)\\
&\times \lim_{n\to\infty} \frac{[n-\mathrm{rk}(T)]_t!}{[n]_t!} \prod_{\square\in\lambda^{(\mathrm{rk}(T))}} \left(\frac{1-q^{-T(\square)}t^{n-|\lambda|-c(\square)}}{1-q^{-T(\square)}t^{\mathrm{rk}(T)-|\lambda|-c(\square)}} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= [\mu(T_{\lambda^{(\mathrm{rk}(T))}})]_t! \prod_{(\square_1,\square_2)\in \mathrm{I}(T|_{\lambda^{(\mathrm{rk}(T))}})} \left(\frac{1-q^{T(\square_2)-T(\square_1)}t^{c(\square_2)-c(\square_1)+1}}{1-q^{T(\square_2)-T(\square_1)}t^{c(\square_2)-c(\square_1)}} \right) (1-t)^{\mathrm{rk}(T)}\\
&\times \prod_{\square\in\lambda^{(\mathrm{rk}(T))}} \left(1-q^{-T(\square)}t^{\mathrm{rk}(T)-|\lambda|-c(\square)}\right)^{-1}\\
&= \frac{(1-t)^{\mathrm{rk}(T)}[\mu(T|_{\lambda^{(\mathrm{rk}(T))}})]_t!}{\prod_{\square\in\lambda^{(\mathrm{rk}(T))}}\left(1-q^{-T(\square)}t^{\mathrm{rk}(T)-|\lambda|-c(\square)}\right)} \prod_{(\square_1,\square_2)\in \mathrm{I}(T|_{\lambda^{(\mathrm{rk}(T))}})} \left(\frac{1-q^{T(\square_2)-T(\square_1)}t^{c(\square_2)-c(\square_1)+1}}{1-q^{T(\square_2)-T(\square_1)}t^{c(\square_2)-c(\square_1)}} \right).
\end{aligned}
$$

$\square$

We will now deal with the series side. For this we need the following lemmas. Here we write $|f(q,t)|$ for the t-adic norm of $f(q,t) \in \mathbb{Q}(q)((t))$ normalized so that $|t^n| = 2^{-n}$.

Lemma 3.6.9. *For $a \neq 0$ and $b \in \mathbb{Z}$*

$$
\left| \frac{1-q^a t^{b-1}}{1-q^a t^{b+1}} \right| = \begin{cases} 1 & b \geq 1 \\ 2 & b = 0 \\ 4 & b \leq -1. \end{cases}
$$

Proof. We proceed in cases. If $b \geq 1$ then

$$\left| \frac{1 - q^a t^{b-1}}{1 - q^a t^{b+1}} \right| = \frac{|1 - q^a t^{b-1}|}{|1 - q^a t^{b+1}|} = 1.$$

If $b = 0$,

$$\left| \frac{1 - q^a t^{-1}}{1 - q^a t} \right| = \left| t^{-1} \frac{-q^a + t}{1 - q^a t} \right| = |t^{-1}| \frac{|-q^a + t|}{|1 - q^a t|} = 2.$$

Lastly, if $b \leq -1$ then

$$\left| \frac{1 - q^a t^{b-1}}{1 - q^a t^{b+1}} \right| = \frac{|1 - q^a t^{b-1}|}{|1 - q^a t^{b+1}|} = \frac{2^{-b+1}}{2^{-b-1}} = 4.$$

$\square$

Lemma 3.6.10. *Let $\tau \in \mathrm{APSYT}_{\geq 0}(\lambda; T)$. If $(\square_1, \square_2) \in \mathrm{Inv}(\tau)$ with $c(\square_2) - c(\square_1) \leq 0$ then $\square_1, \square_2 \in \lambda^{(\mathrm{rk}(T))}$.*

Proof. Suppose $(\square_1, \square_2) \in \mathrm{Inv}(\tau)$ with either $\square_1 \in \lambda^{(\infty)}/\lambda^{(\mathrm{rk}(T))}$ or $\square_2 \in \lambda^{(\infty)}/\lambda^{(\mathrm{rk}(T))}$. Then, since $\lambda^{(\infty)}/\lambda^{(\mathrm{rk}(T))}$ is a horizontal strip, necessarily $\square_2 \in \lambda^{(\infty)}/\lambda^{(\mathrm{rk}(T))}$ and $\square_1 \in \lambda^{(\mathrm{rk}(T))}$. Thus $c(\square_2) \geq c(\square_1) + 1$. $\square$

Using these lemmas gives the following:

Proposition 3.6.11. *Let $T \in \Omega(\lambda)$. The sequence of sums*

$$\left(\sum_{\tau \in \mathrm{PSYT}_{\geq 0}(\lambda^{(n)}; T|_{\lambda^{(n)}})} t^{\mathrm{inv}(\tau)} \prod_{(\square_1, \square_2) \in \mathrm{Inv}(\tau)} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) - 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}} \right) \right)_{n \geq n_\lambda}$$

converges with respect to the t-adic topology on $\mathbb{Q}(q)((t))$ to the series

$$\sum_{\tau \in \mathrm{APSYT}_{\geq 0}(\lambda; T)} t^{\mathrm{inv}(\tau)} \prod_{(\square_1, \square_2) \in \mathrm{Inv}(\tau)} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) - 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}} \right) \in \mathbb{Q}(q)((t)).$$

Proof. Our method will be to first verify that the above infinite series over $\tau \in \mathrm{APSYT}_{\geq 0}(\lambda; T)$ is convergent in $\mathbb{Q}(q)((t))$ and then argue that the above sums over $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda^{(n)}; T|_{\lambda^{(n)}})$ converge to the same element of $\mathbb{Q}(q)((t))$.

We begin by noting that from Lemma 3.6.10 we have the (sufficient but egregiously unoptimal) upper bound

$$\#\{(\square_1, \square_2) \in \mathrm{Inv}(\tau) | c(\square_1) - c(\square_2) \leq -1\} \leq \binom{\mathrm{rk}(T)}{2}.$$

Recall that if $T(\Box_1) = T(\Box_2)$ then by Lemma 3.5.11 $c(\Box_2) - c(\Box_1) \geq 2$. Thus using Lemma 3.6.9 we find

$$\left| \prod_{(\Box_1,\Box_2)\in \mathrm{Inv}(\tau)} \left(\frac{1 - q^{T(\Box_2)-T(\Box_1)} t^{c(\Box_2)-c(\Box_1)-1}}{1 - q^{T(\Box_2)-T(\Box_1)} t^{c(\Box_2)-c(\Box_1)+1}} \right) \right| \leq 4^{\binom{\mathrm{rk}(T)}{2}}$$

and hence

$$\left| t^{\mathrm{inv}(\tau)} \prod_{(\Box_1,\Box_2)\in \mathrm{Inv}(\tau)} \left(\frac{1 - q^{T(\Box_2)-T(\Box_1)} t^{c(\Box_2)-c(\Box_1)-1}}{1 - q^{T(\Box_2)-T(\Box_1)} t^{c(\Box_2)-c(\Box_1)+1}} \right) \right| \leq 2^{-\mathrm{inv}(\tau)} 4^{\binom{\mathrm{rk}(T)}{2}}.$$

Recall that (from the strong triangle inequality) if $(f_m(q,t))_{m\geq 1}$ is any sequence in $\mathbb{Q}(q)((t))$ then the series $\sum_{m=0}^{\infty} f_m(q,t)$ is convergent in $\mathbb{Q}(q)((t))$ if and only if $\lim_{m\to\infty} |f_m(q,t)| = 0$. In turn, this is equivalent to the property that for every $r \geq 0$ there are only finitely many $m \geq 1$ with $|f_m(q,t)| \geq 2^{-r}$. From Corollary 3.6.6 we find that for any $r \geq 0$ there are only finitely many $\tau \in \mathrm{APSYT}_{\geq 0}(\lambda; T)$ with

$$\mathrm{inv}(\tau) \leq 2\binom{\mathrm{rk}(T)}{2} + r \Longrightarrow 2^{-\mathrm{inv}(\tau)} 4^{\binom{\mathrm{rk}(T)}{2}} \geq 2^{-r}.$$

Thus there are only finitely many $\tau \in \mathrm{APSYT}_{\geq 0}(\lambda; T)$ with

$$\left| t^{\mathrm{inv}(\tau)} \prod_{(\Box_1,\Box_2)\in \mathrm{Inv}(\tau)} \left(\frac{1 - q^{T(\Box_2)-T(\Box_1)} t^{c(\Box_2)-c(\Box_1)-1}}{1 - q^{T(\Box_2)-T(\Box_1)} t^{c(\Box_2)-c(\Box_1)+1}} \right) \right| \geq 2^{-r}.$$

We conclude that the series

$$S := \sum_{\tau\in \mathrm{APSYT}_{\geq 0}(\lambda;T)} t^{\mathrm{inv}(\tau)} \prod_{(\Box_1,\Box_2)\in \mathrm{Inv}(\tau)} \left(\frac{1 - q^{T(\Box_2)-T(\Box_1)} t^{c(\Box_2)-c(\Box_1)-1}}{1 - q^{T(\Box_2)-T(\Box_1)} t^{c(\Box_2)-c(\Box_1)+1}} \right)$$

is convergent in $\mathbb{Q}(q)((t))$.

Now let $n \geq \mathrm{rk}(T)$.

$$
\begin{aligned}
&\left| S - \sum_{\tau \in \mathrm{PSYT}_{\geq 0}(\lambda^{(n)}; T|_{\lambda^{(n)}})} t^{\mathrm{inv}(\tau)} \prod_{(\square_1, \square_2) \in \mathrm{Inv}(\tau)} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) - 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}} \right) \right| \\
&= \left| \sum_{\substack{\tau \in \mathrm{APSYT}_{\geq 0}(\lambda; T) \\ \mathrm{rk}(\tau) > n}} t^{\mathrm{inv}(\tau)} \prod_{(\square_1, \square_2) \in \mathrm{Inv}(\tau)} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) - 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}} \right) \right| \\
&\leq \max_{\substack{\tau \in \mathrm{APSYT}_{\geq 0}(\lambda; T) \\ \mathrm{rk}(\tau) > n}} \left| t^{\mathrm{inv}(\tau)} \prod_{(\square_1, \square_2) \in \mathrm{Inv}(\tau)} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) - 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}} \right) \right| \\
&\leq 2^{-(n+1-\mathrm{rk}(T))} 4^{\binom{\mathrm{rk}(\tau)}{2}}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\lim_{n \to \infty} \left| S - \sum_{\tau \in \mathrm{PSYT}_{\geq 0}(\lambda^{(n)}; T|_{\lambda^{(n)}})} t^{\mathrm{inv}(\tau)} \prod_{(\square_1, \square_2) \in \mathrm{Inv}(\tau)} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) - 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}} \right) \right| \\
&\leq \lim_{n \to \infty} 2^{-(n+1-\mathrm{rk}(T))} 4^{\binom{\mathrm{rk}(\tau)}{2}} \\
&= 0.
\end{aligned}
$$

$\square$

We immediately arrive at the following product-series formula:

THEOREM 3.6.12. *For $T \in \Omega(\lambda)$ we have the following equality in $\mathbb{Q}(q)((t))$:*

$$
\begin{aligned}
&\frac{\prod_{\square \in \lambda^{(\mathrm{rk}(T))}} \left(1 - q^{-T(\square)} t^{\mathrm{rk}(T) - |\lambda| - c(\square)}\right)}{(1-t)^{\mathrm{rk}(T)} [\mu(T|_{\lambda^{(\mathrm{rk}(T))}})]_t!} \prod_{(\square_1, \square_2) \in \mathrm{I}(\lambda^{(\mathrm{rk}(T))})} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1)}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}} \right) \\
&= \sum_{\tau \in \mathrm{APSYT}_{\geq 0}(\lambda; T)} t^{\mathrm{inv}(\tau)} \prod_{(\square_1, \square_2) \in \mathrm{Inv}(\tau)} \left(\frac{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) - 1}}{1 - q^{T(\square_2) - T(\square_1)} t^{c(\square_2) - c(\square_1) + 1}} \right).
\end{aligned}
$$

REMARK 35. *Note that the powers of q appearing in the Theorem 3.6.12 are all non-positive i.e. the sum and product are elements of $\mathbb{Q}[q^{-1}]((t))$. In particular, we may limit $q \to \infty$ to obtain the prod-sum equality in $\mathbb{Q}((t))$:*

$$\frac{\prod_{\substack{\square\in\lambda^{(\mathrm{rk}(T))}\\ T(\square)=0}}\left(1-t^{\mathrm{rk}(T)-|\lambda|-c(\square)}\right)}{(1-t)^{\mathrm{rk}(T)}[\mu(T|_{\lambda^{(\mathrm{rk}(T))}})]_t!}\prod_{\substack{(\square_1,\square_2)\in \mathrm{I}(\lambda^{(\mathrm{rk}(T))})\\ T(\square_1)=T(\square_2)}}\left(\frac{1-t^{c(\square_2)-c(\square_1)}}{1-t^{c(\square_2)-c(\square_1)+1}}\right)$$

$$=\sum_{\tau\in \mathrm{APSYT}_{\geq 0}(\lambda;T)} t^{\mathrm{inv}(\tau)}\prod_{\substack{(\square_1,\square_2)\in \mathrm{Inv}(\tau)\\ T(\square_1)=T(\square_2)}}\left(\frac{1-t^{c(\square_2)-c(\square_1)-1}}{1-t^{c(\square_2)-c(\square_1)+1}}\right).$$

By noting that the product term in Theorem 3.6.12 is a ***finite*** product of rational terms we observe the following:

COROLLARY 3.6.13. *For $T\in\Omega(\lambda)$,*

$$\sum_{\tau\in \mathrm{APSYT}_{\geq 0}(\lambda;T)} t^{\mathrm{inv}(\tau)}\prod_{(\square_1,\square_2)\in \mathrm{Inv}(\tau)}\left(\frac{1-q^{T(\square_2)-T(\square_1)}t^{c(\square_2)-c(\square_1)-1}}{1-q^{T(\square_2)-T(\square_1)}t^{c(\square_2)-c(\square_1)+1}}\right)\in\mathbb{Q}(q,t).$$

EXAMPLE. *Here we give a few simple examples of this (q,t) identity. Consider $\lambda=\emptyset$ and $T=$ $\begin{array}{|c|c|c|}\hline 1 & 0 & 0\\ \hline\end{array}\ \ldots$ $\in\Omega(\emptyset)$. Then we get*

$$\frac{1-q^{-1}t}{1-t}=\sum_{k=0}^{\infty}t^k\prod_{j=1}^{k}\left(\frac{1-q^{-1}t^{j-1}}{1-q^{-1}t^{j+1}}\right).$$

Now consider $\lambda=(1)$ and $T=$ $\begin{array}{|c|c|c|}\hline 1 & 0 & 0\\ \hline 0 \\ \cline{1-1}\end{array}\ \ldots$ $\in\Omega(1)$. In this case we get

$$\frac{(1-q^{-1}t)(1-t^2)(1-q^{-1}t^{-1})}{(1-t)^2(1-q^{-1})}$$

$$=\sum_{i,j=1}^{\infty}t^{i+j-3}\prod_{k=1}^{i-2}\left(\frac{1-q^{-1}t^{k-1}}{1-q^{-1}t^{k+1}}\right)\prod_{k=2}^{j-1}\left(\frac{1-t^{k-1}}{1-t^{k+1}}\right)\times$$

$$\left(\mathbb{1}(j\leq i-1)t\left(\frac{1-q^{-1}t^{-2}}{1-q^{-1}}\right)+\mathbb{1}(i+1\leq j)\left(\frac{1-q^{-1}}{1-q^{-1}t^2}\right)\right).$$

Interestingly, in both of these cases we can write the series part of these identities as a finite sum of hypergeometric series.

CHAPTER 4

Double Dyck Path Algebra Representations From DAHA

4.1. Introduction

The algebra $\mathbb{B}_{q,t}$ was introduced by Carlsson-Gorsky-Mellit [7] as an algebra which has a natural geometric action on the equivariant K-theory of the parabolic flag Hilbert schemes of points in $\mathbb{C}^2$. This work built on the prior work of Schiffmann-Vasserot [34] who constructed a geometric action of the elliptic Hall algebra $\mathscr{E}$ on the equivariant K-theory of the Hilbert schemes of points in $\mathbb{C}^2$. These construction are part of a larger story in Macdonald theory of relating geometric properties of the Hilbert schemes of points in $\mathbb{C}^2$ to the algebraic combinatorics underlying the modified Macdonald symmetric functions $\widetilde{H}_\mu$ and of the Macdonald operator Δ (which acts on the space of symmetric functions Λ). Importantly, $\mathbb{B}_{q,t}$ is intimately related to the *double Dyck path algebra* $\mathbb{A}_{q,t}$ introduced by Carlsson-Mellit in their proof of the Shuffle Theorem [8] regarding the Frobenius character of the space of diagonal coinvariants and the combinatorics of Dyck paths.

The quiver path algebra $\mathbb{B}_{q,t}$ has relations very similar to the *positive double affine Hecke algebras* (DAHA) in type GL, $\mathscr{D}_n^+$, introduced by Cherednik [9]. In fact, $\mathbb{B}_{q,t}$ contains many copies of *affine Hecke algebras* of type GL. However, there is no direct algebraic relation (no algebra homomorphisms) between $\mathbb{B}_{q,t}$ (nor $\mathbb{A}_{q,t}$) and DAHAs. Nevertheless, there are approaches to more indirectly relate these algebras. Ion-Wu [26] defined an algebra called the *stable-limit DAHA* along with a polynomial representation on the space of *almost symmetric functions* $\mathscr{P}_{as}^+$ which, in a sense, globalizes the polynomial representation of $\mathbb{A}_{q,t}$ (and as we will see later $\mathbb{B}_{q,t}$). They used a stable-limit procedure to define this representation from the polynomial representations of the finite rank DAHAs $\mathscr{D}_n^+$. This representation of the stable-limit DAHA is much larger than the polynomial representation of $\mathbb{B}_{q,t}$ but the limit Cherednik operators of Ion-Wu, in a sense, behave better on a certain subspace of $\mathscr{P}_{as}^+$ given by the following direct sum:

$$\bigoplus_{k\geq 0} x_1\cdots x_k\mathbb{Q}(q,t)[x_1,\ldots,x_k]\otimes\Lambda.$$

This subspace aligns with the polynomial representation of $\mathbb{B}_{q,t}$.

Motivated by the construction of Ion-Wu we will in this chapter develop a method for constructing modules for $\mathbb{B}_{q,t}$ directly from the representation theory of DAHA in type GL. We will use a stable-limit construction similar to Ion-Wu but will not require any additional non-archimedean topological considerations as they did. First, we will show (Proposition 4.2.4) that given any $\mathscr{D}_n^+$ module V we may construct an action of the subalgebra $\mathbb{B}_{q,t}^{(n)}$ on the space $L_\bullet(V)$ defined by

$$L_\bullet(V) = \bigoplus_{0\leq k\leq n} L_k(V) := \bigoplus_{0\leq k\leq n} X_1 \cdots X_k \epsilon_k(V).$$

Here ϵ_k are the *partial trivial idempotents* of the finite Hecke algebra. Each space may be considered as a module for the partially symmetrized positive DAHA, $\epsilon_k \mathscr{D}_n^+ \epsilon_k$. It will be immediate to show (Theorem 4.2.5) that the map $V \to L_\bullet(V)$ is indeed a functor. We show (Proposition 4.2.7) that in the case of the polynomial representations $V_{\text{pol}}^{(n)}$ of DAHA that $L_\bullet(V_{\text{pol}}^{(n)})$ is a $\mathbb{B}_{q,t}^{(n)}$-module quotient of the restriction of the polynomial representation of $\mathbb{B}_{q,t}$ to $\mathbb{B}_{q,t}^{(n)}$.

Afterwards, we will use stable-limits of the representations $L_\bullet(V)$ of $\mathbb{B}_{q,t}^{(n)}$ to build representations of $\mathbb{B}_{q,t}$. This construction will require the input of an infinite family of representations of DAHAs, $(V^{(n)})_{n\geq n_0}$, along with some connecting maps, $\Pi^{(n)} : V^{(n+1)} \to V^{(n)}$, satisfying some special assumptions. Most interestingly, we require that the following relations holds:

$$\Pi^{(n)}\pi_{n+1}T_n = \pi_n\Pi^{(n)}.$$

This is the same relation used by Ion-Wu in their construction of the limit Cherednik operators and is related to certain natural embeddings of the extended affine symmetric groups $\widetilde{\mathfrak{S}}_n \hookrightarrow \widetilde{\mathfrak{S}}_{n+1}$. We call such families $C = \left((V^{(n)})_{n\geq n_1}, (\Pi^{(n)})_{n\geq n_1}\right)$ *compatible* and construct spaces $\mathfrak{L}_k(C)$ given by

$$\mathfrak{L}_k(C) := \lim_{\leftarrow} L_k(V^{(n)}).$$

These are the stable-limits of the spaces $L_k(V^{(n)})$ with respect to the maps $\Pi^{(n)}$. Finally, we package together these spaces to form $\mathfrak{L}_\bullet(C)$ given as

$$\mathfrak{L}_\bullet(C) := \bigoplus_{k\geq 0} \mathfrak{L}_k(C)$$

which may be also thought of as the stable-limit of the $\mathbb{B}_{q,t}^{(n)}$ modules $L_{\bullet}(V^{(n)})$. We show (Theorem 4.2.12) that there is a natural action of $\mathbb{B}_{q,t}$ on $\mathfrak{L}_{\bullet}(C)$ determined by the $\mathbb{B}_{q,t}^{(n)}$ module structures on $L_{\bullet}(V^{(n)})$. This construction is also functorial.

Lastly, we will use our construction of the functor $C \to \mathfrak{L}_{\bullet}(C)$ to define (Theorem 4.3.5) a large family of $\mathbb{B}_{q,t}$ modules, $\mathfrak{L}_{\bullet}(\mathrm{Ind}(C_{\lambda}))$, indexed by partitions λ. These representations in a sense extend the Murnaghan-type representations of the positive elliptic Hall algebra previously defined by the author [2]. As such we call these the Murnaghan-type representations of $\mathbb{B}_{q,t}$. For $\lambda = \emptyset$, $\mathfrak{L}_{\bullet}(\mathrm{Ind}(C_{\emptyset}))$ recovers the polynomial representation of $\mathbb{B}_{q,t}$.

In a recent paper González-Gorsky-Simental [17] defined an extension $\mathbb{B}_{q,t}^{\mathrm{ext}}$ of $\mathbb{B}_{q,t}$ containing certain additional Δ-operators as well as a class of representations of $\mathbb{B}_{q,t}^{\mathrm{ext}}$ called *calibrated* with special properties. Further, they construct a large class of *calibrated* $\mathbb{B}_{q,t}^{\mathrm{ext}}$ representations built from certain posets with exceptional properties. The author conjectures that the Murnaghan-type representations of $\mathbb{B}_{q,t}$, $\mathfrak{L}_{\bullet}(\mathrm{Ind}(C_{\lambda}))$, have extended actions by $\mathbb{B}_{q,t}^{\mathrm{ext}}$ which are calibrated. More generally, there should be a special set of conditions on a compatible sequence C which guarantees that $\mathfrak{L}_{\bullet}(C)$ has an extended action by $\mathbb{B}_{q,t}^{\mathrm{ext}}$ which is calibrated.

4.1.1. Conventions Change. In an effort to better align conventions with the papers [8], [7], and [17] one minor change are made in this section of the thesis. Namely, we will for the remainder of this section swap the roles played by the in-determinants q and t. This means, for example, that the quadratic relation for the finite Hecke algebra now reads $(T_i - 1)(T_i + q) = 0$.

4.2. Main Construction

4.2.1. Additional Relations. We will often write $\mathscr{A}_n^X$ for the subalgebra of $\mathscr{D}_n^+$ generated by $T_1, \ldots, T_{n-1}, X_1, \ldots, X_n$. We will consider $\mathscr{D}_n^+$ as a graded algebra with

- $\deg(T_i) = \deg(Y_i) = 0$
- $\deg(X_i) = 1$.

It is straightforward to check the following additional relations which are all standard in DAHA theory. We will require all of these relations later in this chapter. Some of these relations appeared in Chapter 1.

Remark 36. *For the element π_n we have:*

- $\pi_n X_i = X_{i+1}\pi_n$ *for* $1 \leq i \leq n-1$
- $\pi_n T_i = T_{i+1}\pi_n$ *for* $1 \leq i \leq n-1$
- $\pi_n^2 T_{n-1} = T_1 \pi_n^2$.

The elements $\epsilon_k^{(n)}$ *are the partial trivial idempotents. They satisfy the relations:*

- $(\epsilon_k^{(n)})^2 = \epsilon_k^{(n)}$
- $\epsilon_k^{(n)} T_i = T_i \epsilon_k^{(n)} = \epsilon_k^{(n)}$ *for* $k+1 \leq i \leq n-1$
- $T_i \epsilon_k^{(n)} = \epsilon_k^{(n)} T_i$ *for* $1 \leq i \leq k-1$
- $\epsilon_k^{(n)} = \frac{1}{[n-k]_q!} \sum_{\sigma \in \mathfrak{S}_{(1^k, n-k)}} q^{\ell(\sigma)} T_\sigma^{-1}$
- $\epsilon_k^{(n)} = \left(\frac{1 + qT_{k+1}^{-1} + \ldots + q^{n-k-1} T_{n-1}^{-1} \cdots T_{k+1}^{-1}}{1 + q + \ldots + q^{n-k-1}} \right) \epsilon_{k+1}^{(n)}$
- $\epsilon_k^{(n)} \epsilon_\ell^{(n)} = \epsilon_{\min(k,\ell)}^{(n)}$.

We have that the important element $\widetilde{\pi}_n := X_1 T_1^{-1} \cdots T_{n-1}^{-1}$ *satisfies the relations:*

- $\widetilde{\pi}_n Y_i = Y_{i+1} \widetilde{\pi}_n$ *for* $1 \leq i \leq n-1$
- $\widetilde{\pi}_n t Y_n = Y_1 \widetilde{\pi}_n$
- $\widetilde{\pi}_n T_i = T_{i+1} \widetilde{\pi}_n$ *for* $1 \leq i \leq n-2$
- $\widetilde{\pi}_n^2 T_{n-1} = T_1 \widetilde{\pi}_n^2$.

Note that the last two of these relations only depend of the structure of the subalgebra $\mathscr{A}_n^X$ *of* $\mathscr{D}_n^+$ *and thus hold more generally for all* $2 \leq k \leq n$*:*

- $(X_1 T_1^{-1} \cdots T_{k-1}^{-1}) T_i = T_{i+1} (X_1 T_1^{-1} \cdots T_{k-1}^{-1})$ *for* $1 \leq i \leq k-2$
- $(X_1 T_1^{-1} \cdots T_{k-1}^{-1})^2 T_{k-1} = T_1 (X_1 T_1^{-1} \cdots T_{k-1}^{-1})^2$.

Lastly, we have the following expansion of the Hecke algebra analogues of the Jucys-Murphy elements the standard T_σ *basis:*

$$q^{n-k} T_k^{-1} \cdots T_{n-1}^{-1} T_{n-1}^{-1} \cdots T_k^{-1} = 1 + (q-1)(T_k^{-1} + q T_{k+1}^{-1} T_k^{-1} + \ldots + q^{n-k-1} T_{n-1}^{-1} \cdots T_k^{-1}).$$

4.2.2. $\mathbb{B}_{q,t}^{(n)}$ Modules From $\mathscr{D}_n^+$. In this section we will take any graded $\mathscr{D}_n^+$ module V and construct a corresponding graded $\mathbb{B}_{q,t}^{(n)}$ module $L_\bullet(V)$. To do this we will first define the spaces which constitute $L_\bullet(V)$.

Definition 4.2.1. *For any graded* $\mathscr{D}_n^+$ *module* V *and* $0 \leq k \leq n$ *define the space* $L_k = L_k(V)$ *as* $L_k := X_1 \cdots X_k \epsilon_k(V)$*. We let* $L_\bullet = L_\bullet(V)$ *denote the space* $L_\bullet := \bigoplus_{0 \leq k \leq n} L_k$.

For the remainder of this section let V be a graded module for $\mathscr{D}_n^+$. We are going to now construct operators T_i, z_i, d_+, d_- on $L_\bullet$ which we will show generate a representation of $\mathbb{B}_{q,t}^{(n)}$.

Definition 4.2.2. *Define the operators*

- $T_i : L_k \to L_k$ *for* $1 \leq i \leq k-1$
- $z_i : L_k \to L_k$ *for* $1 \leq i \leq k$
- $d_+ : L_k \to L_{k+1}$ *for* $0 \leq k \leq n-1$
- $d_- : L_k \to L_{k-1}$ *for* $1 \leq k \leq n$

as follows:

- $T_i(v)$ *is defined by the action of* T_i *on* V
- $z_i(v) := Y_i(v)$ *as defined by the action of* Y_i *on* V
- $d_+(v) := -q^k X_1 T_1^{-1} \cdots T_k^{-1} v$
- $d_-(v) := (1-q)(1 + qT_k^{-1} + \ldots + q^{n-k} T_{n-1}^{-1} \cdots T_k^{-1})(v)$.

It is not immediately obvious that these operators are well defined i.e. that their ranges are correctly specified above. We show this now.

Lemma 4.2.3. *If* $v \in L_k$ *then* $T_i(v), z_j(v) \in L_k$ *for all* $1 \leq i \leq k-1$ *and* $1 \leq j \leq k$. *If* $k \leq n-1$ *then* $d_+(v) \in L_{k+1}$ *and if* $1 \leq k$ *then* $d_-(v) \in L_{k-1}$.

Proof. Let $v \in L_k$ say, $v = X_1 \cdots X_k \epsilon_k(w)$ for $w \in V$. First we have,

$$\begin{aligned} T_i(v) &= T_i X_1 \cdots X_k \epsilon_k(w) \\ &= X_1 \cdots X_k \epsilon_k(T_i w) \in L_k. \end{aligned}$$

Next we have

$$
\begin{aligned}
z_1(v) &= Y_1 X_1 \cdots X_k \epsilon_k(w) \\
&= q^n \pi T_{n-1}^{-1} \cdots T_1^{-1} X_1 \cdots X_k \epsilon_k(w) \\
&= q^n \pi T_{n-1}^{-1} \cdots T_k^{-1} X_1 \cdots X_k T_{k-1}^{-1} \cdots T_1^{-1} \epsilon_k(w) \\
&= q^n \pi X_1 \cdots X_{k-1} T_{n-1}^{-1} \cdots T_k^{-1} X_k T_{k-1}^{-1} \cdots T_1^{-1} \epsilon_k(w) \\
&= q^n X_2 \cdots X_k \pi T_{n-1}^{-1} \cdots T_k^{-1} X_k T_{k-1}^{-1} \cdots T_1^{-1} \epsilon_k(w) \\
&= q^n X_2 \cdots X_k t q^{-(n-k)} X_1 \pi T_{n-1} \cdots T_k T_{k-1}^{-1} \cdots T_1^{-1} \epsilon_k(w) \\
&= t q^k X_1 \cdots X_k \pi T_{n-1} \cdots T_k T_{k-1}^{-1} \cdots T_1^{-1} \epsilon_k(w) \\
&= X_1 \cdots X_k \pi T_{n-1} \cdots T_k \epsilon_k(t q^k T_{k-1}^{-1} \cdots T_1^{-1} w).
\end{aligned}
$$

Now for all $k < i \leq n-1$ we have

$$
\begin{aligned}
T_i \pi T_{n-1} \cdots T_k &= \pi T_{i-1} T_{n-1} \cdots T_k \\
&= \pi T_{n-1} \cdots T_{i+1} T_{i-1} T_i T_{i-1} T_{i-2} \cdots T_k \\
&= \pi T_{n-1} \cdots T_{i+1} T_i T_{i-1} T_i T_{i-2} \cdots T_k \\
&= \pi T_{n-1} \cdots T_k T_i.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
& X_1 \cdots X_k \pi T_{n-1} \cdots T_k \epsilon_k(t q^k T_{k-1}^{-1} \cdots T_1^{-1} w) \\
&= X_1 \cdots X_k \epsilon_k(t q^k \pi T_{n-1} \cdots T_k T_{k-1}^{-1} \cdots T_1^{-1} w)
\end{aligned}
$$

which is clearly in L_k.

Now for any $1 < i \leq k$ since $Y_i = q^{-1} T_{i-1} Y_{i-1} T_{i-1}$ we see that

$$
Y_i = q^{-i+1} T_{i-1} \cdots T_1 Y_1 T_1 \cdots T_{i-1}
$$

and so

$$z_i = q^{-i+1} T_{i-1} \cdots T_1 Y_1 T_1 \cdots T_{i-1}.$$

Since $T_1 \cdots T_{i-1} v \in L_k$ we see that $Y_1(T_1 \cdots T_{i-1} v) \in L_k$ as well and so

$$z_i(v) = q^{-i+1} T_{i-1} \cdots T_1 Y_1 T_1 \cdots T_{i-1}(v) = q^{-i+1} T_{i-1} \cdots T_1 Y_1 (T_1 \cdots T_{i-1} v) \in L_k.$$

We now look at d_+. We find that

$$\begin{aligned}
& d_+(v) = \\
& = -q^k X_1 T_1^{-1} \cdots T_k^{-1}(v) \\
& = -q^k X_1 T_1^{-1} \cdots T_k^{-1}(X_1 \cdots X_k \epsilon_k(w)) \\
& = -T_1 \cdots T_k X_{k+1}(X_1 \cdots X_k \epsilon_k(w)) \\
& = -T_1 \cdots T_k X_1 \cdots X_{k+1} \epsilon_k(w) \\
& = -X_1 \cdots X_{k+1}(T_1 \cdots T_k \epsilon_k(w)) \\
& = -X_1 \cdots X_{k+1}(T_1 \cdots T_k \epsilon_{k+1} \epsilon_k(w)) \\
& = X_1 \cdots X_{k+1} \epsilon_{k+1}(-T_1 \cdots T_k \epsilon_k(w)) \in L_{k+1}.
\end{aligned}$$

Lastly, we look at d_-. We suppose $v \in L_{k+1}$ say, $v = X_1 \cdots X_{k+1} \epsilon_{k+1}(w)$ for $w \in V$. We get that

$$\begin{aligned}
& d_-(v) \\
& = (1-q)(1 + qT_{k+1}^{-1} + \ldots + q^{n-k-1} T_{n-1}^{-1} \cdots T_{k+1}^{-1})(v) \\
& = (1-q)(1 + qT_{k+1}^{-1} + \ldots + q^{n-k-1} T_{n-1}^{-1} \cdots T_{k+1}^{-1})(X_1 \cdots X_{k+1} \epsilon_{k+1}(w)) \\
& = (1-q)(1 + qT_{k+1}^{-1} + \ldots + q^{n-k-1} T_{n-1}^{-1} \cdots T_{k+1}^{-1})(\epsilon_{k+1}(X_1 \cdots X_{k+1} w)) \\
& = (1-q)(1 + q + \ldots + q^{n-k-1}) \epsilon_k(X_1 \cdots X_{k+1} w) \\
& = X_1 \cdots X_k \epsilon_k((1-q^{n-k}) X_{k+1} w) \in L_k.
\end{aligned}$$

$\square$

Now we will show that the collection of operators T_i, z_j, d_-, d_+ acting on the space $L_\bullet$ generates an action of $\mathbb{B}_{q,t}$.

PROPOSITION 4.2.4. *$L_\bullet$ is a $\mathbb{B}_{q,t}^{(n)}$-module.*

PROOF. We will show that the operators T_i, z_j, d_-, d_+ on $L_\bullet$ defined in Definition 4.2.2 satisfy the relations in Definition 1.6.1. Note first that the relations involving only T_i's and z_j's follow immediately from their definition and the fact that V is a $\mathscr{D}_n^+$-module.

We will start by verifying the relations between d_+ and the T_i. We will for the remainder of this proof let $v \in L_k$ and specify various conditions on k as needed. Suppose $0 \leq k \leq n-1$. Then for $1 \leq i \leq k-1$ using the braid relations we see directly that $d_+T_i(v) = T_{i+1}d_+(v)$.

Now if $0 \leq k \leq n-2$ we see from the braid relations and the fact that $T_{k+1}(v) = v$

$$
\begin{aligned}
&T_1^{-1}d_+^2(v)\\
&= T_1^{-1}d_+(-q^kX_1T_1^{-1}\cdots T_k^{-1}(v))\\
&= T_1^{-1}(-q^{k+1}X_1T_1^{-1}\cdots T_{k+1}^{-1})(-q^kX_1T_1^{-1}\cdots T_k^{-1}(v))\\
&= q^{2k+1}T_1^{-1}X_1T_1^{-1}T_2^{-1}\cdots T_{k+1}^{-1}X_1T_1^{-1}\cdots T_k^{-1}(v)\\
&= q^{2k}X_2T_2^{-1}\cdots T_{k+1}^{-1}X_1T_1^{-1}\cdots T_k^{-1}(v)\\
&= q^{2k}X_1X_2T_2^{-1}\cdots T_{k+1}^{-1}T_1^{-1}\cdots T_k^{-1}(v)\\
&= q^{2k}X_1X_2T_2^{-1}\cdots T_{k+1}^{-1}T_1^{-1}\cdots T_k^{-1}T_{k+1}(v)\\
&= q^{2k}X_1X_2T_1T_2^{-1}\cdots T_{k+1}^{-1}T_1^{-1}\cdots T_k^{-1}(v)\\
&= q^{2k+1}X_1T_1^{-1}X_1T_2^{-1}\cdots T_{k+1}^{-1}T_1^{-1}\cdots T_k^{-1}(v)\\
&= q^{2k+1}X_1T_1^{-1}T_2^{-1}\cdots T_{k+1}^{-1}X_1T_1^{-1}\cdots T_k^{-1}(v)\\
&= (-q^{k+1}X_1T_1^{-1}\cdots T_{k+1}^{-1})(-q^kX_1T_1^{-1}\cdots T_k^{-1}(v))\\
&= d_+^2(v).
\end{aligned}
$$

We will now show that $d_+ z_i = z_{i+1} d_+$. Suppose $1 \leq i \leq k \leq n-1$. Then we have by using Remark 36

$$
\begin{aligned}
& z_{i+1} d_+(v) \\
& = -q^k Y_{i+1} X_1 T_1^{-1} \cdots T_k^{-1}(v) \\
& = -q^k Y_{i+1} X_1 T_1^{-1} \cdots T_k^{-1} T_{k+1}^{-1} \cdots T_{n-1}^{-1}(v) \\
& = -q^k Y_{i+1} (X_1 T_1^{-1} \cdots T_{n-1}^{-1})(v) \\
& = -q^k (X_1 T_1^{-1} \cdots T_{n-1}^{-1}) Y_i(v) \\
& = -q^k X_1 T_1^{-1} \cdots T_k^{-1} Y_i T_{k+1}^{-1} \cdots T_{n-1}^{-1}(v) \\
& = -q^k X_1 T_1^{-1} \cdots T_k^{-1} Y_i(v) \\
& = d_+(z_i(v)).
\end{aligned}
$$

Next we note that the relations between just d_- and the T_i follow trivially from the fact that $d_- : L_{k+1} \to L_k$ is a scalar multiple of $\epsilon_k|_{L_{k+1}}$ which follows from the relations (see Remark 36). Further, the relation $z_i d_- = d_- z_i$ also follows easily from the fact that $Y_i T_j = T_j Y_i$ for $i \notin \{j, j+1\}$.

Now we are left to show that the relations involving $\varphi := \frac{1}{q-1}[d_+, d_-]$ hold. Notice that φ may be computed for $1 \leq k \leq n-1$ as

$$
\begin{aligned}
& (q-1)\varphi(v) \\
& = [d_+, d_-](v) \\
& = (d_+ d_- - d_- d_+)(v) \\
& = d_+((1-q)(1 + qT_k^{-1} + \ldots + q^{n-k} T_{n-1}^{-1} \cdots T_k^{-1})(v)) - d_-(-q^k X_1 T_1^{-1} \cdots T_k^{-1} v)
\end{aligned}
$$

$$= (1-q)(-q^{k-1}X_1T_1^{-1}\cdots T_{k-1}^{-1})(1+qT_k^{-1}+\ldots+q^{n-k}T_{n-1}^{-1}\cdots T_k^{-1})(v)$$
$$-(1-q)(1+qT_{k+1}^{-1}+\ldots+q^{n-k-1}T_{n-1}^{-1}\cdots T_{k+1}^{-1})(-q^kX_1T_1^{-1}\cdots T_k^{-1}v)$$
$$= (q-1)q^{k-1}X_1T_1^{-1}\cdots T_{k-1}^{-1}$$
$$\times\Big((1+qT_k^{-1}+\ldots+q^{n-k}T_{n-1}^{-1}\cdots T_k^{-1})-q(1+qT_{k+1}^{-1}+\ldots+q^{n-k-1}T_{n-1}^{-1}\cdots T_{k+1}^{-1})T_k^{-1}\Big)$$
$$= (q-1)q^{k-1}X_1T_1^{-1}\cdots T_{k-1}^{-1}$$

so that

$$\varphi(v) = q^{k-1}X_1T_1^{-1}\cdots T_{k-1}^{-1}(v).$$

Let $2 \le k \le n$. Then

$$q\varphi d_-(v)$$
$$= q\varphi(1-q)(1+qT_k^{-1}+\ldots+q^{n-k}T_{n-1}^{-1}\cdots T_k^{-1})v$$
$$= q(1-q)q^{k-2}X_1T_1^{-1}\cdots T_{k-2}^{-1}(1+qT_k^{-1}+\ldots+q^{n-k}T_{n-1}^{-1}\cdots T_k^{-1})v$$
$$= (1-q)(1+qT_k^{-1}+\ldots+q^{n-k}T_{n-1}^{-1}\cdots T_k^{-1})q^{k-1}X_1T_1^{-1}\cdots T_{k-2}^{-1}(v)$$
$$= d_-\varphi T_{k-1}(v).$$

Let us now show that $T_1\varphi d_+ = qd_+\varphi$. Suppose $1 \le k \le n-2$. Then

$$T_1\varphi d_+(v)$$
$$T_1\varphi(-q^kX_1T_1^{-1}\cdots T_k^{-1})(v)$$
$$= T_1(q^kX_1T_1^{-1}\cdots T_k^{-1})(-q^kX_1T_1^{-1}\cdots T_k^{-1})(v)$$
$$= -q^{2k}T_1X_1T_1^{-1}\cdots T_k^{-1}X_1T_1^{-1}\cdots T_k^{-1}(v)$$
$$= -q^{2k}T_1(XT_1^{-1}\cdots T_k^{-1})^2(v)$$

$$= -q^{2k}(X_1T_1^{-1}\cdots T_k^{-1})^2T_k(v)$$

$$= -q^{2k}X_1T_1^{-1}\cdots T_k^{-1}X_1T_1^{-1}\cdots T_{k-1}^{-1}(v)$$

$$= q(-q^kX_1T_1^{-1}\cdots T_k^{-1})(q^{k-1}X_1T_1^{-1}\cdots T_{k-1}^{-1})(v)$$

$$= qd_+\varphi(v).$$

Lastly, we show that $z_1(qd_+d_- - d_-d_+) = qt(d_+d_- - d_-d_+)z_k$. Take $1 \le k \le n-1$. Then we find

$$z_1(qd_+d_- - d_-d_+)$$

$$= z_1(qd_+d_-(v) - d_-d_+(v))$$

$$= z_1\left(qd_+(1-q)(1+qT_k^{-1}+\ldots+q^{n-k}T_{n-1}^{-1}\cdots T_k^{-1})(v) - d_-(-q^kX_1T_1^{-1}\cdots T_k^{-1})(v)\right)$$

$$= Y_1q(1-q)(-q^{k-1}X_1T_1^{-1}\cdots T_{k-1}^{-1})(1+qT_k^{-1}+\ldots+q^{n-k}T_{n-1}^{-1}\cdots T_k^{-1})(v)$$

$$- Y_1(1-q)(1+qT_{k+1}^{-1}+\ldots+q^{n-k-1}T_{n-1}^{-1}\cdots T_{k+1}^{-1})(-q^kX_1T_1^{-1}\cdots T_k^{-1})(v)$$

$$= (q-1)q^kY_1X_1T_1^{-1}\cdots T_{k-1}^{-1}$$

$$\times\left(1+qT_{k+1}^{-1}+\ldots+q^{n-k-1}T_{n-1}^{-1}\cdots T_{k+1}^{-1} - (1+qT_{k+1}^{-1}+\ldots+q^{n-k-1}T_{n-1}^{-1}\cdots T_{k+1}^{-1}T_k^{-1})\right)(v)$$

$$= (q-1)q^kY_1X_1T_1^{-1}\cdots T_{k-1}^{-1}\left(1+(q-1)(T_k^{-1}+qT_{k+1}^{-1}T_k^{-1}+\ldots+q^{n-k-1}T_{n-1}^{-1}\cdots T_k^{-1})\right)(v)$$

$$= (q-1)q^kY_1X_1T_1^{-1}\cdots T_{k-1}^{-1}\left(q^{n-k}T_k^{-1}\cdots T_{n-1}^{-1}T_{n-1}^{-1}\cdots T_k^{-1}\right)(v)$$

$$= q^n(q-1)Y_1\widetilde{\pi}T_{n-1}^{-1}\cdots T_k^{-1}(v)$$

$$= q^n(q-1)\widetilde{\pi}(tY_n)T_{n-1}^{-1}\cdots T_k^{-1}(v)$$

Continuing we find:

$$
\begin{aligned}
&= tq^n(q-1)X_1T_1^{-1}\cdots T_{k-1}^{-1}(T_k^{-1}\cdots T_{n-1}^{-1}Y_nT_{n-1}^{-1}\cdots T_k^{-1})(v)\\
&= tq^n(q-1)X_1T_1^{-1}\cdots T_{k-1}^{-1}(q^{-(n-k)}Y_k)(v)\\
&= tq^k(q-1)X_1T_1^{-1}\cdots T_{k-1}^{-1}Y_k(v)\\
&= qt(q-1)\left(q^{k-1}X_1T_1^{-1}\cdots T_{k-1}^{-1}\right)Y_k(v)\\
&= qt(q-1)\varphi z_k(v)\\
&= qt[d_+,d_-]z_k(v).
\end{aligned}
$$

□

COROLLARY 4.2.5. *The map $W \to L_\bullet(W)$ is a covariant functor $\mathscr{D}_n^+ - \mathrm{Mod} \to \mathbb{B}_{q,t}^{(n)} - \mathrm{Mod}$.*

PROOF. Suppose $\phi : U \to W$ is a homogeneous $\mathscr{D}_n^+$-module map. Now for any $0 \le k \le n$ we see that if $v = X_1\cdots X_k\epsilon_k(u) \in L_k(U)$ then

$$\phi(v) = \phi(X_1\cdots X_k\epsilon_k(u)) = X_1\cdots X_k\epsilon_k(\phi(u)) \in L_k(W).$$

Thus ϕ yields a map $\phi_\bullet : L_\bullet(U) \to L_\bullet(W)$ given by restricting ϕ to each of the subspaces $L_k(U) \subset U$. From Definition 4.2.2 we see that each of the operators T_i, z_i, d_-, d_+ is expressed *entirely* in terms of the action of $\mathscr{D}_n^+$ on U and as such we conclude that $\phi_\bullet$ is a $\mathbb{B}_{q,t}^{(n)}$ module map. □

4.2.3. The Polynomial Case. The goal of this section is to relate the $\mathbb{B}_{q,t}^{(n)}$ modules $L_\bullet(W)$ constructed above to the polynomial representation $V_\bullet^{\mathrm{pol}}$ of $\mathbb{B}_{q,t}$ in the case when $W = V_{\mathrm{pol}}^{(n)}$. We will show that there are natural maps

$$x_1\cdots x_k\mathbb{Q}(q,t)[x_1,\ldots,x_k]\otimes\Lambda \to x_1\cdots x_k\mathbb{Q}(q,t)[x_1,\ldots,x_n]^{\mathfrak{S}_{(1^k,n-k)}}$$

which are $\mathbb{B}_{q,t}^{(n)}$ module projections. This is nontrivial since the definitions of z_i and d_- are quite different in both modules. We will use the work of Ion-Wu to bridge this gap.

DEFINITION 4.2.6. *[26] Recall Definition 1.3.12 from Chapter 1. Consider the following operators given on $f \in \mathbb{Q}(q,t)[x_1,\ldots,x_k]\otimes\Lambda$ as follows:*

- $T_i(f) := s_i(f) + (1-q)x_i \frac{f - s_i(f)}{x_i - x_{i+1}}$
- $d_+(f) := -q^k X_1 T_1^{-1} \cdots T_k^{-1}(f)$
- $d_-(x_1^{a_1} \cdots x_{k+1}^{a_{k+1}} F[\mathfrak{X}_{k+1}]) := x_1^{a_1} \cdots x_k^{a_k} F[\mathfrak{X}_k - u^{-1}] \operatorname{Exp}[(1-q)u\mathfrak{X}_k] \mid_{u^{a_{k+1}}}$
- $z_i(f) = \mathscr{Y}_i(f) := \lim_m \widetilde{Y}_i^{(m)} \Xi^{(m)}(f)$

where $\mathscr{Y}_i$ are the limit Cherednik operators, $\lim_m$ is the limit as defined by Ion-Wu (with q and t swapped) and $\widetilde{Y}_i^{(m)}$ are the deformed Cherednik operators.

We can use the work of Ion-Wu to relate the above $\mathbb{B}_{q,t}$ module $\mathscr{P}_\bullet$ to the $\mathbb{B}_{q,t}$ module $W_\bullet^{\text{pol}}$ defined by Carlsson-Gorsky-Mellit as follows.

THEOREM. *[26] The maps T_i, d_+, d_-, z_i on $\mathscr{P}_\bullet$ define a representation of $\mathbb{B}_{q,t}$. This representation is isomorphic to the $\mathbb{B}_{q,t}$ representation on $W_\bullet^{\text{pol}} := \bigoplus_{k \geq 0} (y_1 \cdots y_k)^{-1} V_k^{\text{pol}}$ defined by Carlsson-Gorsky-Mellit via the map $\Phi_\bullet = \bigoplus_{k \geq 0} \Phi_k$ defined by*

$$\Phi_k(x_1^{a_1} \cdots x_k^{a_k} F[\mathfrak{X}_k]) := y_1^{a_1 - 1} \cdots y_k^{a_k - 1} F\left[\frac{X}{q-1}\right].$$

REMARK 37. *Ion-Wu in their paper also construct the additional operator d_+^* on $\mathscr{P}_\bullet$ from which they obtain an action of $\mathbb{A}_{q,t}$ on $\mathscr{P}_\bullet$. Further, they show that this $\mathbb{A}_{q,t}$ module is isomorphic to the standard $\mathbb{A}_{q,t}$ representation as defined by Mellit [31] which is the same as the Carlsson-Gorsky-Mellit action of $\mathbb{A}_{q,t}$ on $W_\bullet^{\text{pol}}$. The result as stated above is thus a strictly weaker result than the main theorem of Ion-Wu but as we are only interested in the subalgebra $\mathbb{B}_{q,t}$ of $\mathbb{A}_{q,t}$, we will only require the above result as stated.*

By the above theorem of Ion-Wu we find that each of the spaces $y_1 \cdots y_k W_k^{\text{pol}} = V_k^{\text{pol}}$ gets mapped by Φ_k^{-1} to the space $x_1 \cdots x_k \mathbb{Q}(q,t)[x_1, \ldots, x_k] \otimes \Lambda$ which we will call L_k^{pol}. Thus we see that $\mathbb{B}_{q,t}$ acts on the space $L_\bullet^{\text{pol}} := \bigoplus_{k \geq 0} L_k^{\text{pol}} \subset \mathscr{P}_\bullet$. For all $n \geq 0$ can relate $L_\bullet^{\text{pol}}$ to $L_\bullet(V_{\text{pol}}^{(n)})$ in the following way:

PROPOSITION 4.2.7. *The map $\Xi_\bullet^{(n)} : L_\bullet^{\text{pol}} \to L_\bullet(V_{\text{pol}}^{(n)})$ defined component-wise by $\Xi^{(n)}$ is a $\mathbb{B}_{q,t}^{(n)}$ module map.*

PROOF. We need to show that $\Xi^{(n)}$ commutes with the operators T_i, z_i, d_+, d_- as defined on both of the spaces $L_\bullet^{\text{pol}}$ and $L_\bullet(V_{\text{pol}}^{(n)})$ respectively. For T_i and d_+ this is immediate. For z_i we note that from the construction of the deformed Cherednik operators $\widetilde{Y}_i^{(n)}$ [26] we have that if $1 \leq i \leq k$ then

$$\widetilde{Y}_i^{(n)} X_i = Y_i^{(n)} X_i.$$

Further, from Ion-Wu we also know that (using this chapter's conventions) for all $n \geq i$

$$Y_i^{(n)} X_i \; \Xi^{(n)} = \Xi^{(n)} Y_i^{(n+1)} X_i.$$

Thus for $f \in L_k^{\mathrm{pol}}$ and $1 \leq i \leq k$ we find

$$Y_i^{(n)} \Xi^{(n)}(f) = \Xi^{(n)} \mathscr{Y}_i(f)$$

and so

$$z_i \Xi^{(n)} = \Xi^{(n)} z_i.$$

Let $0 \leq k \leq n-1$ and $f = x_1 \cdots x_{k+1} g$ for $g \in \mathbb{Q}(q,t)[x_1, \ldots, x_k]$. From Chapter 2, we know that

$$\begin{aligned}
d_-(f) &= \lim_m \left(\frac{1 + qT_{k+1}^{-1} + \ldots + q^{m-k-1} T_{m-1}^{-1} \cdots T_{k+1}^{-1}}{1 + q + \ldots + q^{m-k-1}} \right) \Xi^{(m)}(f) \\
&= \left(\frac{1}{1 + q + q^2 + \ldots} \right) \lim_m \left(1 + qT_{k+1}^{-1} + \ldots + q^{m-k-1} T_{m-1}^{-1} \cdots T_{k+1}^{-1} \right) \Xi^{(m)}(f) \\
&= (1-q) \lim_m \left(1 + qT_{k+1}^{-1} + \ldots + q^{m-k-1} T_{m-1}^{-1} \cdots T_{k+1}^{-1} \right) \Xi^{(m)}(f) \\
&= (1-q) \lim_m \left(1 + qT_{k+1}^{-1} + \ldots + q^{m-k-1} T_{m-1}^{-1} \cdots T_{k+1}^{-1} \right) X_1 \cdots X_{k+1} \Xi^{(m)}(g).
\end{aligned}$$

Now if $m \geq k$ then

$$\begin{aligned}
&\Xi^{(m)} \left(1 + qT_{k+1}^{-1} + \ldots + q^{m-k} T_m^{-1} \cdots T_{k+1}^{-1} \right) X_1 \cdots X_{k+1} \\
&\Xi^{(m)} \left(1 + qT_{k+1}^{-1} + \ldots + q^{m-k-1} T_{m-1}^{-1} \cdots T_{k+1}^{-1} \right) X_1 \cdots X_{k+1} + \Xi^{(m)} q^{m-k} T_m^{-1} \cdots T_{k+1}^{-1} X_1 \cdots X_{k+1} \\
&= \left(1 + qT_{k+1}^{-1} + \ldots + q^{m-k-1} T_{m-1}^{-1} \cdots T_{k+1}^{-1} \right) X_1 \cdots X_{k+1} \Xi^{(m)} + \Xi^{(m)} X_{m+1} T_m \cdots T_{k+1} X_1 \cdots X_k \\
&= \left(1 + qT_{k+1}^{-1} + \ldots + q^{m-k-1} T_{m-1}^{-1} \cdots T_{k+1}^{-1} \right) X_1 \cdots X_{k+1} \Xi^{(m)}.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\Xi^{(n)}(d_-(f)) \\
&= (1-q)\left(1+qT_{k+1}^{-1}+\ldots+q^{n-k-1}T_{n-1}^{-1}\cdots T_{k+1}^{-1}\right)X_1\cdots X_{k+1}\Xi^{(n)}(g) \\
&= (1-q)\left(1+qT_{k+1}^{-1}+\ldots+q^{n-k-1}T_{n-1}^{-1}\cdots T_{k+1}^{-1}\right)\Xi^{(n)}(X_1\cdots X_{k+1}g) \\
&= (1-q)\left(1+qT_{k+1}^{-1}+\ldots+q^{n-k-1}T_{n-1}^{-1}\cdots T_{k+1}^{-1}\right)\Xi^{(n)}(f) \\
&= d_-\Xi^{(n)}(f).
\end{aligned}
$$

Thus $\Xi^{(n)}d_- = d_-\Xi^{(n)}$ and so $\Xi_\bullet^{(n)}$ is a $\mathbb{B}_{q,t}^{(n)}$ module map. □

REMARK 38. *Since $L_\bullet^{\mathrm{pol}}$ is isomorphic as a $\mathbb{B}_{q,t}$ module to $V_\bullet^{\mathrm{pol}}$ via the map $\Phi_\bullet$ and from Proposition 4.2.7 we know that $\Xi^{(n)} : L_\bullet^{\mathrm{pol}} \to L_\bullet(V_{\mathrm{pol}}^{(n)})$ is a $\mathbb{B}_{q,t}^{(n)}$ module quotient, it follows that $L_\bullet(V_{\mathrm{pol}}^{(n)})$ is a $\mathbb{B}_{q,t}^{(n)}$ module quotient of $\mathrm{Res}_{\mathbb{B}_{q,t}^{(n)}}^{\mathbb{B}_{q,t}} V_\bullet^{\mathrm{pol}}$.*

4.2.4. $\mathbb{B}_{q,t}$ Modules From Compatible Sequences. We will now build representations for the full $\mathbb{B}_{q,t}$ algebra given certain special families of DAHA representations.

DEFINITION 4.2.8. *Let $C = \left((V^{(n)})_{n\geq n_1}, (\Pi^{(n)})_{n\geq n_1}\right)$ be a collection of $\mathbb{Q}(q,t)$-vector spaces and maps $\Pi^{(n)} : V^{(n+1)} \to V^{(n)}$ with $n_1 \geq 1$. We call C a* ***compatible sequence*** *if the following conditions hold:*

- *Each $V^{(n)}$ is a graded $\mathscr{D}_n^+$-module*
- *The maps $\Pi^{(n)} : V^{(n+1)} \to V^{(n)}$ are degree-preserving.*
- *Each map $\Pi^{(n)}$ is a $\mathscr{A}_n^X$ module map.*
- $\Pi^{(n)}X_{n+1} = 0$
- $\Pi^{(n)}\pi_{n+1}T_n = \pi_n\Pi^{(n)}$.

Given compatible sequences $C = \left((V^{(n)})_{n\geq n_1}, (\Pi^{(n)})_{n\geq n_1}\right)$ and $D = \left((W^{(n)})_{n\geq n_2}, (\Psi^{(n)})_{n\geq n_2}\right)$ a homomorphism $\phi : C \to D$ is a collection of maps $\phi = (\phi^{(n)})_{n\geq \max(n_1,n_2)}$ with $\phi^{(n)} : V^{(n)} \to W^{(n)}$ such that

- *$\phi^{(n)}$ are degree-preserving $\mathscr{D}_n^+$ module maps.*
- $\phi^{(n)}\Pi^{(n)} = \Psi^{(n)}\phi^{(n+1)}$.

We will write $\mathfrak{C}$ for the category of compatible sequences.

Remark 39. *The importance of the relation $\Pi^{(n)}\pi_{n+1}T_n = \pi_n\Pi^{(n)}$ can be traced back to at least the work of Ion-Wu [26] on their stable-limit DAHA. This relation allowed Ion-Wu to construct the limit Cherednik operators on the space of almost symmetric functions utilizing a remarkable stability relation for the classical Cherednik operators. We will be following a similar idea in a different setting in this section of the chapter.*

This relation may be interpreted as relating to the natural inclusion map on extended affine symmetric groups $\widehat{\mathfrak{S}}_n \to \widehat{\mathfrak{S}}_{n+1}$ given by $s_i \to s_i$ for $1 \leq i \leq n-1$ and $\pi \to \pi s_n$. Diagrammatically, this map sends the crossing diagram for some $\sigma \in \widehat{\mathfrak{S}}_n$ on n-strands to the corresponding crossing diagram on $(n+1)$-strands where we send $n+1$ to itself.

For the remainder of this section we fix a compatible sequence $C = \left((V^{(n)})_{n\geq n_0}, (\Pi^{(n)})_{n\geq n_0}\right)$. It is easy to check that for $0 \leq k \leq n$, $\Pi^{(n)}(L_k(V^{(n+1)})) \subset L_k(V^{(n)})$ so that the following definition makes sense.

Definition 4.2.9. *For $k \geq 0$ define $\mathfrak{L}_k = \mathfrak{L}_k(C)$ to be the stable-limit $\mathfrak{L}_k := \lim_{\leftarrow} L_k(V^{(n)})$ with respect to the maps $\Pi^{(n)}$. We define $\mathfrak{L}_\bullet = \mathfrak{L}_\bullet(C)$ as $\mathfrak{L}_\bullet = \bigoplus_{k\geq 0}\mathfrak{L}_k$. We will write $\Pi^{(n)}_\bullet : L_\bullet(V^{(n+1)}) \to L_\bullet(V^{(n)})$ for the map obtained by restricting $\Pi^{(n)}$ to each component $L_k(V^{(n+1)})$.*

If we let $\widetilde{V}$ denote the stable-limit of the spaces $V^{(n)}$ with respect to the maps $\Pi^{(n)}$ then we can reinterpret the spaces $\mathfrak{L}_k$ as

$$\mathfrak{L}_k = \{v \in X_1 \cdots X_k\widetilde{V} |\ T_i(v) = v \text{ for } i > k\}.$$

Lemma 4.2.10. *For $n \geq n_0$ the map $\Pi^{(n)}_\bullet : L_\bullet(V^{(n+1)}) \to L_\bullet(V^{(n)})$ is a $\mathbb{B}^{(n)}_{q,t}$-module map.*

Proof. By definition $\Pi^{(n)}$ is a $\mathscr{A}^X_n$-module map so for $1 \leq i \leq k-1$, $\Pi^{(n)}T_i = T_i\Pi^{(n)}$. Further, we also know that if $k \leq n-1$ then $\Pi^{(n)}d_+ = d_+\Pi^{(n)}$ since on L_k, $d_+ = -q^kX_1T_1^{-1}\cdots T_k^{-1}$.

Now let $1 \leq k \leq n$ and $v \in L_k$ say, $v = X_1 \cdots X_k \epsilon_k(w)$. We see that from a nearly identical calculation to one seen in the proof of Proposition 4.2.7

$$
\begin{aligned}
&\Pi^{(n)} d_-(v) \\
&= \Pi^{(n)}(1-q)(1 + qT_k^{-1} + \ldots + q^{n+1-k}T_n^{-1} \cdots T_k^{-1})(v) \\
&= \Pi^{(n)}(1-q)(1 + qT_k^{-1} + \ldots + q^{n+1-k}T_n^{-1} \cdots T_k^{-1})(X_1 \cdots X_k \epsilon_k(w)) \\
&= \Pi^{(n)}(1-q)(1 + qT_k^{-1} + \ldots + q^{n-k}T_{n-1}^{-1} \cdots T_k^{-1})(X_1 \cdots X_k \epsilon_k(w)) \\
&\ + \Pi^{(n)}(1-q)q^{n+1-k}T_n^{-1} \cdots T_k^{-1} X_1 \cdots X_k \epsilon_k(w) \\
&= (1-q)(1 + qT_k^{-1} + \ldots + q^{n-k}T_{n-1}^{-1} \cdots T_k^{-1})\Pi^{(n)}(X_1 \cdots X_k \epsilon_k(w)) \\
&+ \Pi^{(n)}(1-q)X_{n+1}T_n \cdots T_k X_1 \cdots X_{k-1} \epsilon_k(w) \\
&= (1-q)(1 + qT_k^{-1} + \ldots + q^{n-k}T_{n-1}^{-1} \cdots T_k^{-1})\Pi^{(n)}(X_1 \cdots X_k \epsilon_k(w)) \\
&= (1-q)(1 + qT_k^{-1} + \ldots + q^{n-k}T_{n-1}^{-1} \cdots T_k^{-1})\Pi^{(n)}(v) \\
&= d_- \Pi^{(n)}(v).
\end{aligned}
$$

Lastly, let $1 \leq i \leq k$. Using the relation $\Pi^{(n)} \pi_{n+1} T_n = \pi_n \Pi^{(n)}$ we find

$$
\begin{aligned}
&\Pi^{(n)} z_i(v) \\
&= \Pi^{(n)} Y_i(v) \\
&= \Pi^{(n)} q^{n-i+2} T_{i-1} \cdots T_1 \pi_{n+1} T_n^{-1} \cdots T_i^{-1}(v) \\
&= \Pi^{(n)} q^{n-i+2} T_{i-1} \cdots T_1 \pi_{n+1} T_n^{-1} \cdots T_i^{-1}(X_1 \cdots X_k \epsilon_k(w)) \\
&= \Pi^{(n)} q^{n-i+2} T_{i-1} \cdots T_1 \pi_{n+1} T_n^{-1} \cdots T_i^{-1} X_i (X_1 \cdots X_{i-1} X_{i+1} \cdots X_k \epsilon_k(w)) \\
&= \Pi^{(n)} q^2 T_{i-1} \cdots T_1 \pi_{n+1} X_{n+1} T_n \cdots T_i (X_1 \cdots X_{i-1} X_{i+1} \cdots X_k \epsilon_k(w))
\end{aligned}
$$

$$= \Pi^{(n)} q^2 t T_{i-1} \cdots T_1 X_1 \pi_{n+1} T_n \cdots T_i (X_1 \cdots X_{i-1} X_{i+1} \cdots X_k \epsilon_k(w))$$
$$= q^2 t T_{i-1} \cdots T_1 X_1 \Pi^{(n)} \pi_{n+1} T_n \cdots T_i (X_1 \cdots X_{i-1} X_{i+1} \cdots X_k \epsilon_k(w))$$
$$= q^2 t T_{i-1} \cdots T_1 X_1 \pi_n \Pi^{(n)} T_{n-1} \cdots T_i (X_1 \cdots X_{i-1} X_{i+1} \cdots X_k \epsilon_k(w))$$
$$= q^2 T_{i-1} \cdots T_1 \pi_n X_n T_{n-1} \cdots T_i (X_1 \cdots X_{i-1} X_{i+1} \cdots X_k \Pi^{(n)}(\epsilon_k(w))$$
$$= q^{n-i+1} T_{i-1} \cdots T_1 \pi_n T_{n-1}^{-1} \cdots T_i^{-1} X_i (X_1 \cdots X_{i-1} X_{i+1} \cdots X_k \Pi^{(n)}(\epsilon_k(w))$$
$$= q^{n-i+1} T_{i-1} \cdots T_1 \pi_n T_{n-1}^{-1} \cdots T_i^{-1} \Pi^{(n)} (X_1 \cdots X_k \epsilon_k(w))$$
$$= z_i \Pi^{(n)}(v).$$

□

As an immediate consequence of Lemma 4.2.10 and Lemma 1.7.2 we may make the following definition.

DEFINITION 4.2.11. *We define the graded $\mathbb{B}_{q,t}$ module structure on $\mathfrak{L}_\bullet$ given by the stable-limit of the graded $\mathbb{B}_{q,t}^{(n)}$ modules $\mathfrak{L}_\bullet^{(n)}$ with respect to the maps $\Pi_\bullet^{(n)} : \mathfrak{L}_\bullet^{(n+1)} \to \mathfrak{L}_\bullet^{(n)}$.*

EXAMPLE. *In the case of the polynomial representations of $\mathscr{D}_n^+$, $V_{\text{pol}}^{(n)}$, we see using Proposition 4.2.7 that $\mathfrak{L}_\bullet(C_{\text{pol}}) \cong V_\bullet^{\text{pol}}$ where*

$$C_{\text{pol}} := \left((V_{\text{pol}}^{(n)})_{n \geq 1}, (\Xi_\bullet^{(n)})_{n \geq 1} \right).$$

The construction in Definition 4.2.11 associates to any compatible sequence C a graded module $\mathfrak{L}_\bullet(C)$ of $\mathbb{B}_{q,t}$. We can easily see that this construction is functorial.

THEOREM 4.2.12. *(Main Theorem) The map $C \to \mathfrak{L}_\bullet(C)$ is a covariant functor $\mathfrak{C} \to \mathbb{B}_{q,t} - \mathrm{Mod}$.*

PROOF. This follows immediately using the functoriality described in Remark 11 and from the fact that the operators on $\mathfrak{L}_\bullet(C)$ are described *entirely* in terms of the action of each $\mathscr{D}_n^+$ on $V^{(n)}$. □

REMARK 40. *Recently, González-Gorsky-Simental [17] introduced the extended algebra $\mathbb{B}_{q,t}^{\text{ext}}$ and the notion of calibrated $\mathbb{B}_{q,t}^{\text{ext}}$ modules. The extended algebra $\mathbb{B}_{q,t}^{\text{ext}}$ contains additional Δ-operators* *with specific relations motivated by the Δ-operators in Macdonald theory. Calibrated $\mathbb{B}_{q,t}^{\mathrm{ext}}$ modules are those modules with a basis of joint eigenvectors for the z_i's and the additional operators Δ_{p_m} with simple nonzero spectrum.*

In the case of the polynomial representations of DAHAs, the $\mathbb{B}_{q,t}$ representation $\mathfrak{L}_\bullet(C_{\mathrm{pol}})$ has an extended action by $\mathbb{B}_{q,t}^{\mathrm{ext}}$ using Δ-operators and this representation is calibrated. It is an interesting question to figure out exactly which properties of the family of DAHA modules C_{pol} allow for this extended calibrated action by $\mathbb{B}_{q,t}^{\mathrm{ext}}$.

4.3. Compatible Sequences From AHA

In this section we give a method for defining compatible sequences. We will consider families of representations for the affine Hecke algebras $\mathscr{A}_n$ in type GL with special properties which we call *pre-compatible*. These families of representations for $\mathscr{A}_n$ can then be induced to give representations of the corresponding $\mathscr{D}_n^+$ which can be shown to be compatible after carefully defining the correct connecting maps.

Definition 4.3.1. *Let $C = \left((U^{(n)})_{n\geq n_0}, (\kappa^{(n)})_{n\geq n_0}\right)$ be a collection of $\mathbb{Q}(q,t)$-vector spaces and maps $\kappa^{(n)} : U^{(n+1)} \to U^{(n)}$ with $n_1 \geq 1$. We call C a **pre-compatible** sequence if the following hold:*

- *Each $U^{(n)}$ is a graded $\mathscr{A}_n$ module (grading is arbitrary)*
- *The maps $\kappa^{(n)} : U^{(n+1)} \to U^{(n)}$ are degree preserving $\mathscr{H}_n$ module maps*
- *$\kappa^{(n)}\pi_{n+1}T_n = \pi_n\kappa^{(n)}$.*

Given any pre-compatible sequence C we define the spaces $(V_C^{(n)})_{n\geq n_0}$ by

$$V_C^{(n)} := \mathrm{Ind}_{\mathscr{A}_n}^{\mathscr{D}_n^+} U^{(n)}$$

which we endow with the grading inherited by $\mathbb{Q}(q,t)[X_1,\ldots,X_n]\otimes U^{(n)}$ (which is isomorphic as a vector space). Define the maps $(\Pi_C^{(n)} : V_C^{(n+1)} \to V_C^{(n)})_{n\geq n_0}$ by

$$\Pi_C^{(n)}(X_1^{\alpha_1}\cdots X_{n+1}^{\alpha_{n+1}}\otimes v) := \mathbb{1}(\alpha_{n+1}=0)\otimes \kappa^{(n)}(v).$$

We will write $\mathrm{Ind}(C)$ for the family

$$\mathrm{Ind}(C) := \left((V_C^{(n)})_{n\geq n_0}, (\Pi_C^{(n)})_{n\geq n_0}\right).$$

Proposition 4.3.2. *If C is pre-compatible then* $\mathrm{Ind}(C)$ *is compatible.*

Proof. By construction each space $V_C^{(n)}$ is a graded $\mathscr{D}_n^+$ module and the maps $\Pi_C^{(n)}$ are degree preserving $\mathscr{A}_n^X$ maps with $\Pi_C^{(n)} X_{n+1} = 0$. Thus we only need to show that $\Pi_C^{(n)}\pi^{n+1}T_n = \pi_n \Pi_C^{(n)}$. To see this we have the following:

$$\begin{aligned}
&\Pi_C^{(n)}\pi_{n+1}T_n(X_1^{\alpha_1}\cdots X_{n+1}^{\alpha_{n+1}}\otimes v)\\
&= X_2^{\alpha_1}\cdots X_n^{\alpha_{n-1}}\Pi_C^{(n)}\pi_{n+1}T_n(X_n^{\alpha_n}X_{n+1}^{\alpha_{n+1}}\otimes v)\\
&= X_2^{\alpha_1}\cdots X_n^{\alpha_{n-1}}\Pi_C^{(n)}\pi_{n+1}\left(\left(X_n^{\alpha_{n+1}}X_{n+1}^{\alpha_n}T_n + (1-q)X_n\frac{X_n^{\alpha_n}X_{n+1}^{\alpha_{n+1}} - X_n^{\alpha_{n+1}}X_{n+1}^{\alpha_n}}{X_n - X_{n+1}}\right)\otimes v\right)\\
&= X_2^{\alpha_1}\cdots X_n^{\alpha_{n-1}}\Pi_C^{(n)}\pi_{n+1}\left(X_n^{\alpha_{n+1}}X_{n+1}^{\alpha_n}\otimes T_n v\right)\\
&\;\; + (1-q)X_2^{\alpha_1}\cdots X_n^{\alpha_{n-1}}\Pi_C^{(n)}\pi_{n+1}X_n\frac{X_n^{\alpha_n}X_{n+1}^{\alpha_{n+1}} - X_n^{\alpha_{n+1}}X_{n+1}^{\alpha_n}}{X_n - X_{n+1}}(1\otimes v)\\
&= X_2^{\alpha_1}\cdots X_n^{\alpha_{n-1}}\Pi_C^{(n)}X_{n+1}^{\alpha_{n+1}}(tX_1)^{\alpha_n}\pi_{n+1}\left(1\otimes T_n v\right)\\
&\;\; + (1-q)X_2^{\alpha_1}\cdots X_n^{\alpha_{n-1}}\Pi_C^{(n)}X_{n+1}\pi_{n+1}\frac{X_n^{\alpha_n}X_{n+1}^{\alpha_{n+1}} - X_n^{\alpha_{n+1}}X_{n+1}^{\alpha_n}}{X_n - X_{n+1}}(1\otimes v)\\
&= X_2^{\alpha_1}\cdots X_n^{\alpha_{n-1}}\Pi_C^{(n)}X_{n+1}^{\alpha_{n+1}}(tX_1)^{\alpha_n}\pi_{n+1}\left(1\otimes T_n v\right)\\
&= \mathbb{1}(\alpha_{n+1}=0)(tX_1)^{\alpha_n}X_2^{\alpha_1}\cdots X_n^{\alpha_{n-1}}\Pi_C^{(n)}(1\otimes \pi_{n+1}T_n(v))\\
&= \mathbb{1}(\alpha_{n+1}=0)(tX_1)^{\alpha_n}X_2^{\alpha_1}\cdots X_n^{\alpha_{n-1}}\otimes \kappa^{(n)}(\pi_{n+1}T_n(v))\\
&= \mathbb{1}(\alpha_{n+1}=0)(tX_1)^{\alpha_n}X_2^{\alpha_1}\cdots X_n^{\alpha_{n-1}}\otimes \pi_n\kappa^{(n)}(v)\\
&= \mathbb{1}(\alpha_{n+1}=0)(tX_1)^{\alpha_n}X_2^{\alpha_1}\cdots X_n^{\alpha_{n-1}}\pi_n\otimes \kappa^{(n)}(v)\\
&= \pi_n\left(\mathbb{1}(\alpha_{n+1}=0)X_1^{\alpha_1}\cdots X_n^{\alpha_n}\otimes\kappa^{(n)}(v)\right)\\
&= \pi_n\Pi_C^{(n)}(X_1^{\alpha_1}\cdots X_{n+1}^{\alpha_{n+1}}\otimes v).
\end{aligned}$$

Thus $\Pi_C^{(n)}\pi_{n+1}T_n = \pi_n\Pi_C^{(n)}$ and so $\mathrm{Ind}(C)$ is compatible. □

We will now give a large family of pre-compatible sequences built from Young diagrams. The modules in these sequences are the same (up to changing conventions) as the modules in [12] and Chapter 3.

Definition 4.3.3. *Define the* $\mathbb{Q}(q,t)$*-algebra homomorphism* $\rho_n : \mathscr{A}_n \to \mathscr{H}_n$ *by*

- $\rho_n(T_i) = T_i$ *for* $1 \leq i \leq n-1$
- $\rho_n(\pi_n) = T_1^{-1} \cdots T_{n-1}^{-1}$.

For a $\mathscr{H}_n$*-module* V *we will denote by* $\rho_n^*(V)$ *the* $\mathscr{A}_n$*-module with action defined for* $v \in V$ *and* $X \in \mathscr{A}_n$ *by* $X(v) := \rho_n(X)(v)$.

Definition 4.3.4. *Recall from Definition 3.3.1 the irreducible* $\mathscr{H}_n$*-modules* S_λ *corresponding to* $\lambda \in \mathbb{Y}$. *Note that the roles of* q *and* t *have been reversed in this chapter. For* $n \geq n_\lambda$ *define the* $\mathscr{A}_n$ *modules* $U_\lambda^{(n)} := \rho_n^*(S_{\lambda^{(n)}})$ *and maps* $\kappa_\lambda^{(n)} : U_\lambda^{(n+1)} \to U_\lambda^{(n)}$ *given for* $\tau \in \mathrm{SYT}(\lambda^{(n+1)})$ *as*

$$\kappa_\lambda^{(n)}(e_\tau) := \begin{cases} e_{\tau|_{\lambda^{(n)}}} & \tau(\square_0) = n+1 \\ 0 & \tau(\square_0) \neq n+1. \end{cases}$$

where $\square_0$ *is the unique square in* $\lambda^{(n+1)}/\lambda^{(n)}$.

We consider the $\mathscr{A}_n$ *modules* $U_\lambda^{(n)}$ *as graded with the trivial grading i.e.* $U_\lambda^{(n)} = U_\lambda^{(n)}(0)$. *We will write* C_λ *for the family*

$$C_\lambda := \left((U_\lambda^{(n)})_{n \geq n_\lambda}, (\kappa_\lambda^{(n)})_{n \geq n_\lambda} \right).$$

Remark 41. *As constructed, the elements* e_τ *of the* $\mathscr{A}_n$ *module* $U_\lambda^{(n)}$ *are* ***not*** *weight vectors for the Cherednik elements* Y_i *but rather for the reversed orientation Cherednik elements* θ_i *given by* $\theta_i = q^{i-1} T_{i-1}^{-1} \cdots T_1^{-1} \pi T_{n-1} \cdots T_i$. *Explicitly, we have that for* $\tau \in \mathrm{SYT}(\lambda)$ *and* $1 \leq i \leq n$, $\theta_i(e_\tau) = q^{c_\tau(i)} e_\tau$.

Theorem 4.3.5. *For any* λ, $\mathrm{Ind}(C_\lambda)$ *is a compatible sequence with* $\mathfrak{L}_\bullet(\mathrm{Ind}(C_\lambda))$ *a nonzero graded* $\mathbb{B}_{q,t}$ *module.*

PROOF. It is easy from the explicit T_i relations given in Definition 4.3.4 to verify that for every $n \geq n_\lambda$ the map $\kappa_\lambda^{(n)} : U_\lambda^{(n+1)} \to U_\lambda^{(n)}$ is a $\mathscr{H}_n$ module map. We therefore also have that

$$\begin{aligned}
&\kappa_\lambda^{(n)} \pi_{n+1} T_n \\
&= \kappa_\lambda^{(n)} \rho_{n+1}(\pi_{n+1}) T_n \\
&= \kappa_\lambda^{(n)} T_1^{-1} \cdots T_n^{-1} T_n \\
&= \kappa_\lambda^{(n)} T_1^{-1} \cdots T_{n-1}^{-1} \\
&= T_1^{-1} \cdots T_{n-1}^{-1} \kappa_\lambda^{(n)} \\
&= \pi_n \kappa_\lambda^{(n)}.
\end{aligned}$$

Hence, C_λ is a pre-compatible sequence and so by Proposition 4.3.2 it follows that $\mathrm{Ind}(C_\lambda)$ is a compatible sequence. Thus we may consider the graded $\mathbb{B}_{q,t}$ module $\mathfrak{L}_\bullet(\mathrm{Ind}(C_\lambda))$.

To show that $\mathfrak{L}_\bullet(\mathrm{Ind}(C_\lambda))$ is nonzero it suffices to show that $\mathfrak{L}_0(\mathrm{Ind}(C_\lambda))$ is nonzero. This is space is the stable-limit of the symmetrized spaces $\epsilon_0^{(n)}(\mathrm{Ind}_{\mathscr{A}_n}^{\mathscr{D}_n^+} U_\lambda^{(n)})$ with respect to the maps $\kappa_\lambda^{(n)}$. However, this space is the *Murnaghan-type representation* $\widetilde{W}_\lambda$ of the positive elliptic Hall algebra of shape λ from the Chapter 3. This space is infinite dimensional for any λ and so clearly $\mathfrak{L}_0(\mathrm{Ind}(C_\lambda))$ is nonzero. □

We can show further that for all $k \geq 0$, $\mathfrak{L}_k(\mathrm{Ind}(C_\lambda))$ is infinite dimensional. To see this note that $d_+^k : \mathfrak{L}_0(\mathrm{Ind}(C_\lambda)) \to \mathfrak{L}_k(\mathrm{Ind}(C_\lambda))$ is given by

$$(-q^{k-1} X_1 T_1^{-1} \cdots T_{k-1}^{-1}) \cdots (-q^2 X_1 T_1^{-1} T_2^{-1})(-q X_1 T_1^{-1})(-X_1)$$

which is clearly injective. Thus since $\mathfrak{L}_0(\mathrm{Ind}(C_\lambda)) = \widetilde{W}_\lambda$ is infinite-dimensional the same is true for $\mathfrak{L}_k(\mathrm{Ind}(C_\lambda))$.

As the $\mathbb{B}_{q,t}$ modules $\mathfrak{L}_\bullet(\mathrm{Ind}(C_\lambda))$ contain the Murnaghan-type representation $\widetilde{W}_\lambda$ of EHA we will refer to these modules as the $\mathbb{B}_{q,t}$ modules of Murnaghan-type.

REMARK 42. *The author conjectures that each of the Murnaghan-type $\mathbb{B}_{q,t}$ modules, $\mathfrak{L}_\bullet(\mathrm{Ind}(C_\lambda))$, has an extended action by $\mathbb{B}_{q,t}^{\mathrm{ext}}$ and that these extended modules are calibrated. Evidence for this conjecture comes from chapter 3 where we constructed Δ-operators on the space $\widetilde{W}_\lambda = \mathfrak{L}_0(\mathrm{Ind}(C_\lambda))$*

which have distinct nonzero spectrum. Extending these Δ*-operators to the whole space* $\mathfrak{L}_\bullet(\mathrm{Ind}(C_\lambda))$ *is nontrivial.*

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