Geometry of Selberg's Bisectors in the Symmetric Space $SL(n,\mathbb{R})/SO(n,\mathbb{R})$

By

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*Even if you are lost, keep moving forward!*

– Tomori Takamatsu from *BanG Dream!*

# Contents

## Abstract

We study several problems about $\mathcal{P}(n)$, the symmetric space associated with the real Lie group $SL(n,\mathbb{R})$. We endow the symmetric space $\mathcal{P}(n)$ with an $SL(n,\mathbb{R})$-invariant premetric proposed by Selberg as a substitute for the Riemannian distance. The problems addressed in this study are linked to an algorithm designed to determine generalized geometric finiteness for subgroups of $SL(n,\mathbb{R})$, similar to the algorithm proposed by Riley in hyperbolic spaces based on Poincaré's fundamental polyhedron theorem.

The main results of this dissertation are twofold. The first part consists of Chapters 3-4, focusing on the ridge-cycle condition in Poincaré's fundamental polyhedron theorem. This condition requires us to determine whether given hyperplanes in $\mathcal{P}(n)$ are disjoint. We establish several criteria for the disjointness of hyperplanes in $\mathcal{P}(n)$ and construct an angle-like function between hyperplanes. The second part, spanning Chapters 5 to 7, concerns the proposed Poincaré's algorithm for $SL(n,\mathbb{R})$. We describe and implement an algorithm that computes the face-poset structure of Dirichlet-Selberg domains for finite subsets of $SL(n,\mathbb{R})$. This constitutes a crucial aspect of the proposed Poincaré's algorithm. Notably, Poincaré's algorithm for a given subgroup will not terminate if the subgroup lacks a finitely-sided Dirichlet-Selberg domain. This observation motivates us to categorize the Abelian subgroups of $SL(3,\mathbb{R})$ based on whether their Dirichlet-Selberg domains are finitely-sided or not.

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CHAPTER 1

# Introduction

## 1.1. Overview

This dissertation is motivated by the following question that concerns discrete subgroups of the Lie group $SL(n,\mathbb{R})$, [Kap23]:

QUESTION 1.1.1. (1) *Given elements $\gamma_1,\ldots,\gamma_k \in SL(n,\mathbb{R})$, determine if the subgroup $\Gamma < SL(n,\mathbb{R})$ generated by these elements is free of rank $k$ or finitely presented and if this subgroup is discrete.*

(2) *Given a presentation of a word hyperbolic group $\Gamma$ and a homomorphism $\rho : \Gamma \to SL(n,\mathbb{R})$, determine if this homomorphism is faithful or has a finite kernel and if the image of the homomorphism is discrete.*

Past research studied similar questions for specific classes of groups, including surface groups [LRT11] and knot groups [LR11]. One way to understand this question is by studying the action of $SL(n,\mathbb{R})$ as a Lie group on the corresponding symmetric space, namely $SL(n,\mathbb{R})/SO(n)$.
This action is analogous to the well-known $SO^+(n,1)$-action on the hyperbolic $n$-space $\mathbf{H}^n$. There is a specific type of discrete subgroups, namely **convex cocompact subgroups** of $SO^+(n,1)$:

DEFINITION 1.1.2. *A discrete subgroup $\Gamma < SO^+(n,1)$ is called a convex cocompact subgroup if $\Gamma$ preserves a non-empty convex subset $C \subset \mathbf{H}^n$, such that the quotient space $C/\Gamma$ is compact.*

It is known that convex cocompact subgroups are always finitely-presented, [Bow95].
More generally, one considers a semisimple Lie group $G$ of non-compact type acting on the corresponding symmetric space $X = G/K$, the quotient space of $G$ by its maximal compact subgroup $K$. The **Morse property** is a suitable generalization of the convex cocompactness for subgroups of $G$. Similarly to convex cocompact subgroups of $SO^+(n,1)$, Morse subgroups of $G$ are discrete and finitely presented.

Kapovich, Leeb, and Porti [KLP14] described an algorithm, known as the KLP algorithm, which determines if a homomorphism $\rho$ from a given hyperbolic group $\Gamma$ to $G$ is Morse. The KLP algorithm checks if the orbit map $g \mapsto g.x$ for a point $x \in X$ sends geodesic paths of the Cayley graph of $\Gamma$ to paths in $X$ with good geometric properties (i.e., Morse properties).

In the context of hyperbolic spaces, an alternative algorithm called the **Poincaré's Algorithm** [Ril83], determines if a subgroup of $SO^+(n,1)$ is **geometrically finite**:

DEFINITION 1.1.3. *Let $\Gamma$ be a discrete subgroup of $SO^+(n,1)$. A **fundamental polyhedron** for $\Gamma$ is a convex polyhedron $P$ in $\mathbf{H}^n$ such that:*

- *For any $g \neq e \in \Gamma$, $int(P) \cap int(g.P) = \varnothing$.*
- *The union $\cup_{g\in\Gamma} g.P = \mathbf{H}^n$.*
- *The family $\{g.P | g \in \Gamma\}$ of subsets of $\mathbf{H}^n$ is locally finite, i.e., for each point $x \in \mathbf{H}^n$, there is a neighborhood $U \ni x$ in $\mathbf{H}^n$ that meets only finitely many sets $g.P$ in the family $\{g.P | g \in \Gamma\}$.*

See Chapter 2 for a rigorous definition of hyperbolic convex polyhedra.

DEFINITION 1.1.4. *A discrete subgroup $\Gamma < SO^+(n,1)$ is called **geometrically finite** if $\Gamma$ admits a fundamental polyhedron $P$ that satisfies the following:*

- *The fundamental polyhedron $P$ is exact: every facet $F$ of $P$ is an intersection $P \cap g.P$ for an element $g \in \Gamma$.*
- *The fundamental polyhedron $P$ is geometrically finite: for each point $a \in \overline{P} \cap \partial\mathbf{H}^n$, there is a neighborhood $U \subset \overline{\mathbf{H}^n}$ of $a$, such that for each face $F$ of $P$ that meets $U$, the closure $\overline{F}$ contains $a$.*

Convex cocompactness and geometric finiteness have alternative definitions in terms of the limit set of $\Gamma$ [Bow93]. One views the convex cocompactness through such equivalent definitions as a stronger version of the geometric finiteness. Therefore, one can modify Poincaré's Algorithm to check if a subgroup of $SO^+(n,1)$ is convex cocompact.

Poincaré's Algorithm relies on a significant result in geometric group theory, namely Poincaré's Fundamental Polyhedron Theorem. In the algorithm, one constructs convex polyhedra in $\mathbf{H}^n$ called **Dirichlet domains** for finite subsets of a given subgroup $\Gamma$; these Dirichlet domains are finitely-sided. Then, one examines if such convex polyhedra satisfy the two conditions of Poincaré's theorem. We will describe the details of Poincaré's theorem and Poincaré's Algorithm in Chapter 2.

We expect to resolve Question 1.1.1 by performing an analog of Poincaré's Algorithm for the $SL(n,\mathbb{R})$-action on $SL(n,\mathbb{R})/SO(n)$. However, in the symmetric space $SL(n,\mathbb{R})/SO(n)$, the direct analog of the Dirichlet domain is non-convex and impractical to study. Selberg proposes a two-point invariant under the $SL(n,\mathbb{R})$-action on $SL(n,\mathbb{R})/SO(n)$ as a substitute of the Riemannian distance [Sel62]. One defines **Dirichlet-Selberg domains** [Sel62, Kap23] by replacing the Riemannian distance in the definition of Dirichlet domains with Selberg's invariant. Dirichlet-Selberg domains are convex polyhedra in $SL(n,\mathbb{R})/SO(n)$. We adopt Selberg's invariant for our proposed analog of Poincaré's Algorithm for $SL(n,\mathbb{R})$.

The symmetric space $SL(n,\mathbb{R})/SO(n)$ has more intricate properties compared with the hyperbolic space $\mathbf{H}^n$. Consequently, a few questions emerge when generalizing Poincaré's algorithm to the symmetric space $SL(n,\mathbb{R})/SO(n)$ equipped with Selberg's invariant. In the next section, we will introduce the questions we encountered and our results for answering and resolving such questions.

## 1.2. Organization of the Dissertation

In Chapter 2, we review the background material. In Section 2.1, we recall the definition of symmetric spaces of non-compact type. Afterward, we derive models of the symmetric spaces $\mathbf{H}^n$ and $SL(n,\mathbb{R})/SO(n)$ directly from the corresponding Lie groups. In Section 2.2, we recall the Poincaré's Fundamental Polyhedron Theorem and the Poincaré's Algorithm in the context of $\mathbf{H}^n$. We introduce Selberg's invariant and describe a generalized Poincaré's Algorithm for the symmetric space $SL(n,\mathbb{R})/SO(n)$.

In Chapter 3, we address a problem related to the ridge cycle condition in the symmetric space $SL(n,\mathbb{R})/SO(n)$. The original ridge cycle condition involves the Riemannian angle for hyperplane pairs; however, such an angle in $SL(n,\mathbb{R})/SO(n)$ varies as the choice of the base point changes. One wishes to formulate the ridge cycle condition using an angle-like function for pairs of hyperplanes as a substitute, while the angle-like function satisfies particular properties. In the generic case, we explicitly construct such a function; for particular pairs of hyperplanes, we show that they are not in the domain of any invariant angle function.

In Chapter 4, we describe a criterion that determines if two hyperplanes of $SL(n, \mathbb{R})/SO(n)$ are disjoint. We describe an algorithm to examine this practically. Based on such a criterion, we obtain a sufficient condition for the disjointness of Selberg bisectors $Bis(x, y)$ and $Bis(y, z)$ in terms of the distances and angles between points $x, y, z \in SL(n, \mathbb{R})/SO(n)$. This condition is analogous to a sufficient condition in the context of the hyperbolic $n$-space [KL19]; the latter is related to the quasi-geodesic property of piecewise geodesic paths in $\mathbf{H}^n$.

In Chapter 5, we describe and implement an algorithm to compute the partially ordered set (poset) structure of finitely-sided polyhedra in $SL(n, \mathbb{R})/SO(n)$ from the equations of the hyperplanes that bound the polyhedron. Our algorithm uses the BSS computation model [BSS89], which accomplishes computations with polynomials in one step.

Our algorithm has some similarities and differences with the algorithm proposed by Epstein and Petronio for computing the face posets of polyhedra in hyperbolic spaces [EP94]. The difference between them comes from the following fact: a facet of an $\mathbf{H}^n$-polyhedron is an $\mathbf{H}^{n-1}$-polyhedron, which has no analogy for facets of an $SL(n, \mathbb{R})/SO(n)$-polyhedron. This fact leads us to introduce a sub-algorithm determining if the intersection of a collection of hyperplanes in $SL(n, \mathbb{R})/SO(n)$ is non-empty.

The algorithm described in Chapter 5 is an essential step in Poincaré's algorithm. The algorithm allows us to compute the ridge cycles of the Dirichlet domain for a given finite set of words in the generators $g_1, \ldots, g_n \in SL(n, \mathbb{R})$. Therefore, we can check if the ridge cycle condition in Poincaré's Fundamental Polyhedron Theorem is satisfied.

One knows that the isometry group $SO^+(4, 1)$ of the hyperbolic 4-space contains a geometric-finite subgroup that does not admit a finitely-sided Dirichlet domain [Bow93]. In such a case, Poincaré's algorithm does not terminate. In Chapter 6, we construct similar subgroups of $SL(n, \mathbb{R})$. In particular, we classify Abelian subgroups of $SL(3, \mathbb{R})$ with positive eigenvalues depending on whether their Dirichlet-Selberg domains are finitely-sided.

Despite such drawbacks, we apply Poincaré's Theorem to confirm that particular subgroups of $SL(2n, \mathbb{R})$ are free over their generating sets. Indeed, we show that such subgroups have Dirichlet-Selberg domains with pairwise disjoint facets. The freeness of such subgroups was initially proved by Tits [Tit72] via a different approach. Our research reveals a connection between such subgroups with Schottky groups [Mas67] in $SO^+(n, 1)$.

CHAPTER 2

# Background

## 2.1. The hyperbolic space and the symmetric space $SL(n,\mathbb{R})/SO(n)$

Both the hyperbolic $n$-space $\mathbf{H}^n$ and the space $SL(n,\mathbb{R})/SO(n)$ are Riemannian symmetric spaces of non-compact type. We refer to [Hel79] for the concepts and statements in this section.

**2.1.1. Riemannian Symmetric spaces of non-compact type.** A Riemannian symmetric space is a Riemannian manifold with some special geometric properties:

Definition 2.1.1 ( [Hel79], Section IV.3). *For a Riemannian manifold $X$, a **geodesic symmetry** at a point $x \in X$ is an isometry $s_x \in Isom(X)$, such that $s_x(\gamma(t)) = \gamma(-t)$ for all geodesics $\gamma$ in $X$ with $\gamma(0) = x$, and for all $t$ in the domain of $\gamma$.*
*A (globally) **Riemannian symmetric space** is a connected Riemannian manifold $X$, such that a geodesic symmetry $s_x$ exists for all points $x \in X$.*

Riemannian symmetric spaces are homogeneous; that is, the action of $Isom(X)$ on a Riemannian symmetric space $X$ is transitive. The existence of geodesic symmetries on $X$ implies that $X$ is geodesically complete. Therefore, for any two points $x, y \in X$, a geodesic $c$ of length $d(x,y)$ connects them, thus the geodesic symmetry $s_m \in Isom(X)$ at the midpoint $m$ of $c$ swaps the points $x$ and $y$. This implies that the isometry group $Isom(X)$ acts on $X$ transitively.
The products of Riemannian symmetric spaces are also Riemannian symmetric spaces. We say that a Riemannian symmetric space $X$ is **irreducible** if it is not the product of two or more (non-trivial) Riemannian symmetric spaces.
Both the spherical $n$-space $\mathbf{S}^n$ and the hyperbolic $n$-space $\mathbf{H}^n$ are symmetric spaces. Indeed, they represent two types of symmetric spaces characterized by the sectional curvature:

Definition 2.1.2 (cf. [Hel79], Chapter V, Theorem 3.1). *An irreducible simply connected symmetric space $X$ is of **compact type** (**non-compact type**, respectively) if the sectional curvature on $X$ is not identically zero and is non-negative (non-positive, respectively).*

Simply connected symmetric spaces decompose as products of Euclidean spaces and symmetric spaces of compact and non-compact types:

Proposition 2.1.1 ( [Hel79], Chapter V, Proposition 4.2). *Any simply connected Riemannian symmetric space $X$ is isometric to a product:*

$$X = \mathbf{E}^d \times X_+ \times X_-$$

*for certain $d \geq 0$, where $\mathbf{E}^d$ is the $d$-dimensional Euclidean space; $X_+$ and $X_-$ are products of irreducible Riemannian symmetric spaces of compact- and non-compact types, respectively.*

In this thesis, we study irreducible Riemannian symmetric spaces of non-compact type. Such spaces relate to simple Lie groups with non-compact Lie algebras:

Proposition 2.1.2 ( [Hel79], Chapter IV, Theorem 3.3). *If $X$ is a Riemannian symmetric space of non-compact type, then the identity component $Isom_0(X)$ is a connected real Lie group with a non-compact Lie algebra.*
*On the other hand, if $G$ is a connected semisimple Lie group with a non-compact Lie algebra, let $K$ be a maximal compact subgroup of $G$. Then the quotient space $X = G/K$ equipped with the quotient metric is a Riemannian symmetric space of non-compact type. Moreover, $Isom_0(X) = G$.*
*In addition, if the symmetric space $X$ is irreducible, then the corresponding Lie group $G$ is simple, and vice versa.*

An important tool to study symmetric spaces is the **Cartan decomposition** of Lie algebras:

Proposition 2.1.3 ( [Hel79], Section III.7). *Let $\mathfrak{g}_0$ be a semisimple Lie algebra over $\mathbb{R}$ with the Killing form $\langle -,- \rangle_{Kil} : \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathbb{R}$. Then $\mathfrak{g}_0$ admits a Lie algebra automorphism $\theta : \mathfrak{g}_0 \to \mathfrak{g}_0$, called a* ***Cartan involution****, such that $\theta^2 = Id$ and $\langle A, B \rangle_\theta := -\langle A, \theta B \rangle_{Kil}$ is positive definite on $\mathfrak{g}_0$.*
*The Cartan involution is unique up to an inner automorphism of $Aut(\mathfrak{g}_0)$, i.e., $\theta \to \varphi\theta\varphi^{-1}$, where $\varphi \in \mathrm{Aut}(\mathfrak{g}_0)$.*

Definition 2.1.3 ( [Hel79], Section III.7). *Let $\mathfrak{g}_0$ be a semisimple Lie algebra over $\mathbb{R}$. The* ***Cartan decomposition*** *of $\mathfrak{g}_0$ is the eigendecomposition of $\mathfrak{g}_0$ for the Cartan involution $\theta \in Aut(\mathfrak{g}_0)$:*

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0,$$

*such that* $\theta|_{\mathfrak{k}_0} = Id$ *and* $\theta|_{\mathfrak{p}_0} = -Id$. *The Cartan decomposition is unique up to* $Int(\mathfrak{g}_0)$*, the adjoint group of* $\mathfrak{g}_0$.

COROLLARY 2.1.1. *With the notions above, it follows that*

$$[\mathfrak{k}_0, \mathfrak{k}_0] = \mathfrak{k}_0, \quad [\mathfrak{k}_0, \mathfrak{p}_0] = \mathfrak{p}_0, \quad [\mathfrak{p}_0, \mathfrak{p}_0] = \mathfrak{k}_0,$$

*which implies that* $\mathfrak{k}_0$ *is a Lie subalgebra of* $\mathfrak{g}_0$.

The Cartan decomposition of a Lie algebra induces a decomposition of the corresponding Lie group:

THEOREM 2.1.4 ( [**Hel79**], Chapter V, Theorem 6.7; Chapter VI, Theorem 1.1). *Let* $G$ *be a connected Lie group for* $\mathfrak{g}_0$ *and* $\sigma : G \to G$ *be the group automorphism induced by the Cartan involution* $\theta$ *of* $\mathfrak{g}_0$. *That is to say,* $\sigma$ *is the automorphism such that its differential map* $d\sigma_e : T_eG \to T_eG$ *equals* $\theta : \mathfrak{g}_0 \to \mathfrak{g}_0$.
*Then* $K = \exp(\mathfrak{k}_0)$ *is a Lie subgroup of* $G$. *More precisely, it is the fixed-point subgroup for* $\sigma \in Aut(G)$. *Moreover,* $G$ *admits a decomposition*

$$G = KP = PK,$$

*where* $P = \exp(\mathfrak{p}_0)$. *That is, each element* $g \in G$ *is uniquely expressed as a product of an element in* $K$ *and an element in* $P$ *in either order.*

In the non-compact case, the Cartan decomposition of a Lie group $G$ relates to the maximal compact subgroup of $G$:

PROPOSITION 2.1.4 ( [**Hel79**], Chapter VI, Theorem 1.1). *Let* $G$ *be a connected semisimple Lie group with non-compact Lie algebra* $\mathfrak{g}_0$ *over* $\mathbb{R}$*, and* $G = KP$ *is the Cartan decomposition of* $G$ *as in Theorem 2.1.4. Then* $K$ *is a maximal compact subgroup of* $G$.

Consequently, one identifies the Riemannian symmetric space $X = G/K$ of non-compact type with the subset $P$ of $G$ in the Cartan decomposition.

COROLLARY 2.1.2 ( [**Hel79**], Chapter VI, Theorem 1.1). *The map* $P \to G/K$, $p \mapsto p \cdot K$ *is a diffeomorphism.*

*Furthermore, one defines on $X = G/K$ a $G$-invariant Riemannian metric tensor $\langle -,-\rangle_x$: $T_xX \times T_xX \to \mathbb{R}$ for $x \in X$, such that*

$$\langle A + \mathfrak{k}_0, B + \mathfrak{k}_0\rangle_{e\cdot K} = \langle A, B\rangle_{Kil},\ \forall A, B \in \mathfrak{p}_0,$$

*and $(X, \langle -,-\rangle)$ is the symmetric space of non-compact type corresponding to the Lie group $G$.*

Let us describe the action of $G$ on its corresponding symmetric space. The action of $G$ on $G/K$ is given by left multiplications, i.e. $x.(gK) = (xg)K$ for any $x \in G$. However, this does not yield an explicit form of the $G$-action on $P$. Instead of $P$, one may consider the subset

$$S = \{s = p \cdot \sigma(p^{-1}) | p \in P\}$$

as a model of the symmetric space $X$, where $\sigma : G \to G$ is the automorphism induced by the Cartan involution.

Proposition 2.1.5 ( [Hel79], Chapter VI, Exercise A.5). *The subset $S$ of $G$ is diffeomorphic to $P$ by the map*

$$P \to S,\ p \mapsto p \cdot \sigma(p^{-1}).$$

*Under this mapping, the action $G \curvearrowright S$ is given as*

$$g.s = g \cdot s \cdot \sigma(g^{-1}), \quad \forall g \in G,\ s \in S.$$

Proof. Since the group $G$ acts transitively on both $P$ and $S$, it suffices to show that the map $P \to S$ is a diffeomorphism at the identity $e$. This is true since the differential

$$T_eP \to T_eS,\ X \to 2X$$

is an isomorphism.

For the second claim, we take any $s = p \cdot \sigma(p^{-1}) \in S$. Then for any $g \in G$, there exists an element $k \in K$, such that

$$g.p = g \cdot p \cdot k.$$

Therefore,

$$g.s = (g.p)\cdot\sigma((g.p)^{-1}) = g\cdot p\cdot k\cdot\sigma((g\cdot p\cdot k)^{-1})$$
$$= g\cdot p\cdot k\cdot\sigma(k^{-1})\sigma(p^{-1})\sigma(g^{-1}) = g\cdot p\cdot\sigma(p^{-1})\sigma(g^{-1}) = g\cdot s\cdot\sigma(g^{-1}).$$

□

**2.1.2. Example: The hyperbolic $n$-space as a symmetric space.** It is worth reviewing the hyperbolic $n$-space and recognizing it as a symmetric space of non-compact type. Throughout this subsection, let $G = SO^+(n,1)$, the identity component of the signature $(n,1)$ orthogonal group:

$$SO^+(n,1) = \left\{g = (g_{i,j}) \in SL(n+1,\mathbb{R}) \,\middle|\, g^{\mathrm{T}} I_{n,1} g = I_{n,1},\ g_{n+1,n+1} > 0\right\},$$

where

$$I_{n,1} = \begin{pmatrix} I_n & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & -1 \end{pmatrix}.$$

The corresponding Lie algebra, $\mathfrak{g}_0 = \mathfrak{so}(n,1)$, is given as

$$\mathfrak{so}(n,1) = \left\{A \in Mat_{n+1}(\mathbb{R}) \,\middle|\, A^{\mathrm{T}} I_{n,1} + I_{n,1}A = 0\right\}.$$

The Killing form on $\mathfrak{so}(n,1)$ is

$$\langle A, B\rangle_{Kil} = (n-1)\mathrm{tr}(A\cdot B),$$

while the Cartan involution on $\mathfrak{so}(n,1)$ is $\theta(A) = -A^{\mathrm{T}}$. Indeed, the bilinear form $\langle A, B\rangle_\theta = (n-1)\mathrm{tr}(A\cdot B^{\mathrm{T}})$ is positive definite. Consequently, $\mathfrak{g}_0$ has the following Cartan decomposition:

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0, \quad \mathfrak{k}_0 = \left\{\begin{pmatrix} A_1 & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 0 \end{pmatrix} \,\middle|\, A_1 \in \mathfrak{so}(n)\right\}, \quad \mathfrak{p}_0 = \left\{\begin{pmatrix} O & \mathbf{a} \\ \mathbf{a}^{\mathrm{T}} & 0 \end{pmatrix} \,\middle|\, \mathbf{a} \in \mathbb{R}^n\right\}.$$

Therefore, the maximal compact subgroup $K$ of $G$ is isomorphic to $SO(n)$. By computing matrix exponentials, one shows that the other factor $P$ in the Cartan decomposition $G = KP$ is

$$P = \left\{\begin{pmatrix} \delta_{ij} + \frac{x_i x_j}{1+x_0} & x_i \\ x_j & x_0 \end{pmatrix}_{1\le i,j\le n} \,\middle|\, x_0 \in \mathbb{R}^+, x_i \in \mathbb{R},\ x_0^2 - \sum_{i=1}^n x_i^2 = 1\right\}.$$

In the case $G = SO^+(n,1)$, the right multiplication of $K = SO(n)$ leaves the $(n+1)$-th columns of elements in $G$ unchanged, thus all elements in the same coset $gK = g \cdot SO(n) \in G/K$ share the same $(n+1)$-th column vector. This fact implies that we can assign an $(n+1)$-dimensional coordinate system to the symmetric space $X = G/K$, such that the coordinate of an element $gK \in G/K$ consists of the entries of the $(n+1)$-th column vector of any of its representatives in $gK$. That is,

$$X = \{\mathbf{x} = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} | x_0 > 0,\ x_0^2 - \sum_{i=1}^{n} x_i^2 = 1\}.$$

Furthermore, the group $SO^+(n,1)$ acts on $X$ via the left-multiplication of $(n+1)$-dimensional vectors.

The tangent space of $X$ at the identity element $\mathbf{e} = (1, 0, \ldots, 0)$ coincides with $\mathfrak{p}_0$. Again, let the entries of the $(n+1)$-th column of an element $\mathbf{a} \in \mathfrak{p}_0$ be the coordinates of $\mathbf{a} \in T_{\mathbf{e}}X$:

$$T_{\mathbf{e}}X = \{\mathbf{a} = (a_0, \ldots, a_n) \in \mathbb{R}^{n+1} | a_0 = 0\}.$$

The metric tensor at $\mathbf{e}$ is

$$\langle \mathbf{a}, \mathbf{b} \rangle = 2(n-1) \sum_{i=1}^{n} a_i b_i,\ \forall \mathbf{a}, \mathbf{b} \in T_{\mathbf{e}}X.$$

For a general point $\mathbf{x} = (x_0, x_1, \ldots, x_n) \in X$, the left multiplication of the element

$$\begin{pmatrix} \delta_{ij} + \frac{x_i x_j}{1+x_0} & -x_i \\ -x_j & x_0 \end{pmatrix}_{1 \le i,j \le n} \in P \subset SO^+(n,1)$$

takes $\mathbf{x}$ and the tangent space $T_{\mathbf{x}}X$ to $\mathbf{e}$ and $T_{\mathbf{e}}X$, respectively. Thus computations show that the tangent space is

$$T_{\mathbf{x}}X = \{\mathbf{a} \in \mathbb{R}^{n+1} | x_0 a_0 - \sum_{i=1}^{n} x_i a_i = 0\},$$

and the metric tensor at $\mathbf{x}$ is

$$\langle \mathbf{a}, \mathbf{b} \rangle = 2(n-1) \sum_{i=1}^{n} a_i b_i - 2(n-1) a_0 b_0,\ \forall \mathbf{a}, \mathbf{b} \in T_{\mathbf{x}}X.$$

The space $X$ equipped with the metric tensor $\langle -, - \rangle$ described above is the **hyperboloid model** for the hyperbolic $n$-space $\mathbf{H}^n$.

**2.1.3. Example: The symmetric space $\mathcal{P}(n) = SL(n,\mathbb{R})/SO(n)$.** Now we proceed to the case of $G = SL(n,\mathbb{R})$. The Lie algebra of $G$ is

$$\mathfrak{g}_0 = \mathfrak{sl}(n,\mathbb{R}) = \{A \in Mat_n(\mathbb{R}) | \mathrm{tr}(A) = 0\},$$

and the Killing form on $\mathfrak{sl}(n,\mathbb{R})$ is

$$\langle A, B\rangle_{Kil} = 2n\mathrm{tr}(A \cdot B).$$

Similarly to the $\mathfrak{so}(n,1)$ case we explained in the previous subsection, the Cartan involution on $\mathfrak{sl}(n,\mathbb{R})$ is $\theta(A) = -A^{\mathrm{T}}$. Therefore, the Cartan decomposition of $\mathfrak{g}_0 = \mathfrak{sl}(n,\mathbb{R})$ reads as

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0, \quad \mathfrak{k}_0 = \mathfrak{so}(n), \quad \mathfrak{p}_0 = \{A \in \mathfrak{sl}(n,\mathbb{R}) | A^{\mathrm{T}} = A\}.$$

The Cartan involution $\theta$ induces the Cartan automorphism $\sigma \in Aut(G)$, $\sigma(g) = {g^{-1}}^{\mathrm{T}}$. By computing the exponential map, it follows that the Cartan decomposition of $G$ is

$$G = KP,\ K = SO(n),\ P = \{x \in Sym_n(\mathbb{R}),\ \det(x) = 1,\ x > 0\},$$

where $Sym_n(\mathbb{R})$ is the vector space of $n \times n$ real symmetric matrices, and $x > 0$ means that $x$ is positive definite.

Let us derive a model for the symmetric space $SL(n,\mathbb{R})/SO(n)$. The sets $S$ and $P$ described in Subsection 2.1.1 are the same subset of $SL(n,\mathbb{R})$, with the diffeomorphism $P \to S$, $x \mapsto x^2$. The group $G = SL(n,\mathbb{R})$ acts on $S$ as $g.x = gxg^{\mathrm{T}}$.

The tangent space at the identity element $e = I_n \in S$ is

$$T_eS = T_eP = \mathfrak{p}_0 = \{A \in Sym_n(\mathbb{R}), \mathrm{tr}(A) = 0\},$$

with the linear map $T_eP \to T_eS,\ A \mapsto 2A$. Since the Killing form on $T_eP$ is $\langle A, B\rangle_{Kil} = 2n\mathrm{tr}(AB)$, the metric tensor on $T_eS$ is

$$\langle A, B\rangle = 2n\mathrm{tr}(A/2 \cdot B/2) = \frac{n}{2}\mathrm{tr}(AB),\ \forall A, B \in T_eS.$$

For a general point $x \in S$, the action of $x^{-1/2} \in SL(n,\mathbb{R})$ takes $x$ to $e$. We see from this that

$$T_xS = \{A \in Sym_n(\mathbb{R}) | \mathrm{tr}(x^{-1/2}Ax^{-1/2}) = 0\} = \{A \in Sym_n(\mathbb{R}) | \mathrm{tr}(x^{-1}A) = 0\},$$

and the metric tensor at $x$ is

$$\langle A, B\rangle = \frac{n}{2}\mathrm{tr}(x^{-1/2}Ax^{-1}Bx^{-1/2}) = \frac{n}{2}\mathrm{tr}(x^{-1}Ax^{-1}B),\ \forall A, B \in T_xS.$$

The space $S$ equipped with the metric tensor above describes a model of the symmetric space for the Lie group $SL(n, \mathbb{R})$. We denote this symmetric space by $\mathcal{P}(n)$. For convenience, we omit the $(n/2)$ factor from the metric tensor above:

DEFINITION 2.1.5. *The (**symmetric matrix model** of) symmetric space for $SL(n, \mathbb{R})$ is the set*

$$\mathcal{P}(n) = \{x \in Sym_n(\mathbb{R}) |\det(x) = 1,\ x > 0\},$$

*equipped with the metric tensor*

$$\langle A, B\rangle = \mathrm{tr}(x^{-1}Ax^{-1}B),\ \forall A, B \in T_x\mathcal{P}(n).$$

Our dissertation also considers another model of this symmetric space:

DEFINITION 2.1.6. *The **projective model** of $\mathcal{P}(n)$ is the following set:*

$$\mathcal{P}(n) = \{[x] \in \mathbf{P}(Sym_n(\mathbb{R}))|x > 0\},$$

*which is identified with the symmetric matrix model by the following diffeomorphism:*

$$\mathcal{P}_{proj}(n) \to \mathcal{P}_{mat}(n), \quad [x] \mapsto (\det(x))^{-1/n}) \cdot x.$$

One defines the **standard Satake compactification** and **Satake boundary** of $\mathcal{P}(n)$ through the projective model.

DEFINITION 2.1.7. *The standard Satake compactification of $\mathcal{P}(n)$ is the set*

$$\overline{\mathcal{P}(n)}_S = \{[x] \in \mathbf{P}(Sym_n(\mathbb{R}))|x \geq 0\},$$

*and the Satake boundary of $\mathcal{P}(n)$ is the set*

$$\partial_S\mathcal{P}(n) = \overline{\mathcal{P}(n)}_S \backslash \mathcal{P}(n).$$

*Here, $x \geq 0$ means that $x$ is positive semi-definite.*

From now on we adopt a different group action: $SL(n,\mathbb{R}) \curvearrowright \mathcal{P}(n)$, $g.x = g^{\mathrm{T}}xg$. This is a right group action isomorphic to the left group action $g.x = gxg^{\mathrm{T}}$ we used previously in this section.

**2.1.4. Rank and vector-valued distances for symmetric spaces.** One way to classify the symmetric spaces is by their **rank**:

Definition 2.1.8. *Let $X$ be a symmetric space. A submanifold $\Sigma \subset X$ is called* ***totally geodesic*** *if every geodesic of $\Sigma$ equipped with the pullback metric from $X$ is also a geodesic of $X$.*
*The* ***rank*** *of $X$ is the maximal dimension of flat (i.e., zero curvature) totally geodesic submanifolds of $X$.*

Totally geodesic submanifolds of $X$ correspond to **Lie triple systems** of the Lie algebra for $X$:

Definition 2.1.9 ( [**Hel79**], Section IV.7). *A* ***Lie triple system*** *of a real Lie algebra $\mathfrak{g}_0$ is a subspace $\mathfrak{s} \subset \mathfrak{g}_0$ that is closed under the ternary operator $[-,[-,-]]$, where $[-,-]$ denotes the Lie bracket on $\mathfrak{g}_0$.*

Proposition 2.1.6 ( [**Hel79**], Chapter IV, Theorem 7.2). *Let $G$ be the Lie group for the symmetric space $X$ of non-compact type, $\mathfrak{g}_0$ be the corresponding Lie algebra, with the Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, and $\exp : \mathfrak{g}_0 \to G$ be the exponential map.*
*Suppose that $\mathfrak{s}$ is a Lie triple system of $\mathfrak{g}_0$ contained in $\mathfrak{p}_0$. Then $\Sigma = \exp(\mathfrak{s})$ is a totally geodesic submanifold of $X$.*
*On the other hand, suppose that $\Sigma$ is a totally geodesic submanifold of $X$ that contains the identity. Then $\Sigma$ is the image of a Lie triple system $\mathfrak{s}$ in $\mathfrak{p}_0$ under the exponential map.*

Proposition 2.1.7 ( [**Hel79**], Chapter V, Proposition 6.1). *The totally geodesic submanifold $\Sigma$ of $X$ is flat if and only if the corresponding Lie triple system $\mathfrak{s}$ is Abelian, i.e., the Lie bracket $[-,-]$ vanishes on $\mathfrak{s}$.*

Using Propositions 2.1.6 and 2.1.7, one obtains the rank of a symmetric space directly from the structure of its corresponding Lie algebra.

Example 2.1.10. *Let $\mathfrak{g}_0$ be the Lie algebra $\mathfrak{so}(n,1)$. Recall that the subspace*

$$\mathfrak{p}_0 = \left\{ \begin{pmatrix} O & \mathbf{x} \\ \mathbf{x}^{\mathrm{T}} & 0 \end{pmatrix} \middle| \mathbf{x} \in \mathbb{R}^n \right\},$$

*is identified with* $\mathbb{R}^n$*, with the basis* $\{e_1, \ldots, e_n\}$*,* $e_i = E_{i,n+1} + E_{n+1,i} \in \mathfrak{g}_0$*. For any* $i \neq j$*,*

$$[e_i, e_j] = E_{ij} - E_{ji} \neq 0,$$

*which implies that* $\mathfrak{p}_0$ *does not contain any Abelian subspaces of dimension* $> 1$*. Thus, the corresponding symmetric space* $\mathbf{H}^n$ *is of rank* $1$*.*

EXAMPLE 2.1.11. *Let* $\mathfrak{g}_0$ *be the Lie algebra* $\mathfrak{sl}(n, \mathbb{R})$*. Then*

$$\mathfrak{p}_0 = \{X^{\mathrm{T}} = X, \mathrm{tr}(X) = 0\}$$

*contains an Abelian Lie subalgebra of dimension* $(n-1)$ *that consists of all the diagonal elements in* $\mathfrak{p}_0$*. Furthermore, one verifies that this is a maximal Abelian Lie subalgebra contained in* $\mathfrak{p}_0$*. That is to say, the rank of the symmetric space* $\mathcal{P}(n)$ *is* $(n-1)$*. The corresponding maximal totally geodesic flat submanifold of* $\mathcal{P}(n)$ *is*

$$F = \{diag(x_1, \ldots, x_n) | x_i > 0, \ \prod x_i = 1\}.$$

*By taking logarithm on* $F$ *and letting* $a_i = \log x_i$*, one identifies* $F$ *with a hyperplane in the Euclidean* $n$*-space*

$$\{(a_1, \ldots, a_n) \in \mathbb{R}^n | \sum a_i = 0\}.$$

The Riemannian distance is not the complete two-point invariant in a symmetric space $X$ of rank $\geq 2$. The actual invariant is a vector-valued function whose range dimension equals the rank of $X$. We refer to [**KLP17**] for the concepts below.

Let $X$ be a symmetric space of non-compact type, and suppose that $rank(X) = r \geq 2$. Fix a maximal flat totally geodesic submanifold $F_{mod}$ of $X$ that contains the identity element, we call it a **model flat** of $X$. The dimension of $F_{mod}$ equals $r = rank(X)$.

PROPOSITION 2.1.8. *The isometries of* $F_{mod}$ *induced by elements in* $G = Isom_0(X)$ *form a subgroup* $W_{aff} < G$*. Furthermore,* $W_{aff}$ *is a semi-direct product*

$$W_{aff} = \mathbb{R}^r \rtimes W,$$

*where* $\mathbb{R}^r$ *acts on* $F_{mod}$ *as translations, and* $W$ *is a finite reflection group fixing the identity element of* $X$*. We call* $W$ *the* ***Weyl group*** *of the symmetric space* $X$*.*

Definition 2.1.12. *We call the quotient space $\Delta = F_{mod}/W$ the (Euclidean)* ***model Weyl chamber*** *of $X$.*

Proposition 2.1.9. *One has a natural identification*

$$(X \times X)/G \cong X/K \cong \Delta.$$

*That is, for any pair of points $(x_1, x_2) \in X \times X$, there is a unique element $g \in G$, such that $g.x_1 = e$ and $g.x_2 \in \Delta$.*

Definition 2.1.13. *Define the* ***vector-valued distance*** *$d_\Delta : X \times X \to \Delta$ as the quotient map induced from the identification $(X \times X)/G \cong \Delta$.*

The identification above shows that the vector-valued distance $d_\Delta$ is $G$-invariant. Moreover, it relates to the Riemannian distance on $X$ by $d = ||d_\Delta||$.

Example 2.1.14. *Recall that the model flat of $\mathcal{P}(n)$ is $F_{mod} = \{(a_1, \ldots, a_n) \in \mathbb{R}^n | \sum a_i = 0\}$. The Weyl group for $\mathcal{P}(n)$ is the symmetric group $\mathfrak{S}_n$, realized as the permutation group of the $n$ diagonal entries of $F_{mod}$. Thus the model Weyl chamber of $\mathcal{P}(n)$ is*

$$\Delta = \{(a_1, \ldots, a_n) \in \mathbb{R}^n | a_1 \geq \cdots \geq a_n, \ \sum a_i = 0\}.$$

*To understand the vector-valued distance between points $x, y \in \mathcal{P}(n)$, we note that the element $x^{-1/2} \in G = SL(n, \mathbb{R})$ takes $x$ to the identity and takes $y$ to $x^{-1/2}yx^{-1/2}$. Furthermore, there is an element in $K = SO(n)$ that fixes the identity and diagonalizes the matrix $x^{-1/2}yx^{-1/2}$. The diagonal entries of such a diagonalization are the eigenvalues of $x^{-1/2}yx^{-1/2}$, which coincide with the eigenvalues of $x^{-1}y$.*
*Therefore, the vector-valued distance from $x$ to $y \in \mathcal{P}(n)$ reads as*

$$d_\Delta(x, y) = (\log \lambda_1, \ldots, \log \lambda_n) \in \Delta,$$

*where $\lambda_1 \geq \cdots \geq \lambda_n$ are the eigenvalues of $x^{-1}y$.*

### 2.2. Poincaré's Theorem and Algorithm

The algorithm we study in the dissertation is closely related to Poincaré's Fundamental Polyhedron Theorem. Poincaré's theorem was initially proven for spaces of constant curvature, i.e., spherical, Euclidean, and hyperbolic spaces. Thus, we will first review Poincaré's theorem and algorithm in the context of the hyperbolic $n$-space.

Here we realize $\mathbf{H}^n$ via the hyperboloid model, which is a hypersurface in $\mathbb{R}^{n+1}$. We refer to [**Rat94**] for the contents in this section.

**2.2.1. Poincaré's Fundamental Polyhedron Theorem for hyperbolic spaces.** We begin with **convex polyhedra** in the hyperbolic $n$-space:

DEFINITION 2.2.1. *A $d$-**plane** of $\mathbf{H}^n$ is the non-empty intersection of $\mathbf{H}^n$ with a $(d+1)$-dimensional linear subspace of $\mathbb{R}^{n+1}$. An $(n-1)$-plane of $\mathbf{H}^n$ is called a **hyperplane**.*

*The complement of a hyperplane in $\mathbf{H}^n$ consists of two connected components; each of them is called an **open half-space** of $\mathbf{H}^n$. A **closed half-space** is the closure of an open half-space.*

PROPOSITION 2.2.1. *For any $x \neq y \in \mathbf{H}^n$, the **bisector** between them:*

$$Bis(x, y) = \{z \in \mathbf{H}^n | d(x, z) = d(y, z)\},$$

*is a hyperplane of $\mathbf{H}^n$. Here and in what follows, $d(-,-)$ is the Riemannian distance on $\mathbf{H}^n$.*

PROOF. Suppose that $x = (x_0, x_1, \ldots, x_n)$ and $y = (y_0, y_1, \ldots, y_n)$, then

$$Bis(x, y) = \{z \in \mathbf{H}^n | \langle x, z\rangle = \langle y, z\rangle\} = \{z \in \mathbf{H}^n | (x_0 - y_0)z_0 - \sum_{i=1}^{n} (x_i - y_i)z_i = 0\}.$$

Here, $\langle -,-\rangle$ is the bilinear form of signature $(n, 1)$. The bisector $Bis(x, y)$ is non-empty as it contains the midpoint between $x$ and $y$. This equation is linear and thus satisfies our definition of a hyperplane of $\mathbf{H}^n$. $\square$

DEFINITION 2.2.2. *A (closed) **convex polyhedron** in $\mathbf{H}^n$ is the intersection of a **locally finite** collection of closed half-spaces in $\mathbf{H}^n$. That is to say, it is an intersection $\bigcap_{i\in\mathcal{I}} H_i$, where $H_i$'s are closed half-spaces in $\mathbf{H}^n$; furthermore, for each point $x \in \mathbf{H}^n$, there is a neighborhood $U \ni x$ in $\mathbf{H}^n$, such that the subset*

$$\{i \in \mathcal{I} | \partial H_i \cap U \neq \varnothing\}$$

*is finite.*

We note that the definition of convex polyhedra in $\mathbf{H}^n$ allows the situation that $\partial H_i = \partial H_j = H_i \cap H_j$ for specific $i, j \in \mathcal{I}$. This case yields a polyhedron that lies in the hyperplane $\partial H_i \subset \mathbf{H}^n$.

Definition 2.2.3. *The* ***dimension*** *of a convex polyhedron $P$ is the dimension of the minimal plane $\Sigma \subset \mathbf{H}^n$ containing $P$. Such a plane is unique and denoted span$(P)$.*

*A* ***facet*** *of a convex polyhedron $P$ is a maximal subset $F \subset \partial P$ such that $F$ itself is a convex polyhedron in $\mathbf{H}^n$. We denote by $\mathcal{S}(P)$ the set of facets of $P$.*

*A* ***face*** *of a convex polyhedron $P$ is defined inductively as follows:*

- *The convex polyhedron $P$ is a face of itself.*
- *A facet of a face of $P$ is yet also a face of $P$.*

*We denote by $\mathcal{F}(P)$ the set of proper faces of $P$.*

*A* ***ridge*** *of $P$ is a codimension 2 face of $P$. We denote by $\mathcal{R}(P)$ the set of ridges of $P$.*

An element $g \in SO^+(n,1)$ takes hyperplanes, half-spaces, and convex polyhedra to the same types of geometric objects, respectively. Thus, one considers the action of elements in $SO^+(n,1)$ on convex polyhedra in $\mathbf{H}^n$. In particular, one asks if a discrete subgroup of $SO^+(n,1)$ admits a fundamental polyhedron in $\mathbf{H}^n$ (see Definition 1.1.3). The answer to this question is positive:

Definition 2.2.4. *For a point $x \in \mathbf{H}^n$ and a discrete subgroup $\Gamma < SO^+(n,1)$, define the* ***Dirichlet domain*** *for $\Gamma$ centered at $x$ as*

$$D(x,\Gamma) = \{y \in \mathbf{H}^n | d(x,y) \leq d(g.x,y),\ \forall g \in \Gamma\}.$$

Proposition 2.2.2. *The Dirichlet domain $D(x,\Gamma)$ is a convex polyhedron bounded by the bisectors $Bis(x,g.x)$ for some elements $g \in \Gamma$. Moreover, if the stabilizer subgroup $Stab_\Gamma(x)$ is trivial (which holds for a generic choice of $x$), then $D(x,\Gamma)$ is a fundamental polyhedron for $\Gamma$.*

For computational purposes (see Subsection 2.2.2), we also define the Dirichlet domain $D(x,\Gamma)$ in the same way if $\Gamma$ is a discrete closed subset of $SO^+(n,1)$.

The converse question is more difficult: one asks if a given convex polyhedron $P$ is the fundamental polyhedron of any discrete subgroup $\Gamma < SO^+(n,1)$. One answers this question by **Poincaré's fundamental polyhedron theorem**. We will state this theorem after a couple of definitions.

Definition 2.2.5. *A convex polyhedron is* ***exact****, if for any $F \in \mathcal{S}(P)$, there exists an element $g_F \in SO^+(n,1)$, such that*

$$F = P \cap g_F.P,$$

*and such that $F' := g_F^{-1}.F$ is also a facet of $P$. Such an isometry $g_F$ is called a* ***facet pairing transformation*** *for the facet $F$ of $P$.*

Definition 2.2.6. *For an exact convex polyhedron $P$, a* ***facet pairing*** *is a set*

$$\Phi = \{g_F \in SO^+(n,1) | F \in \mathcal{S}(P)\},$$

*such that:*

- *For any facet $F \in \mathcal{S}(P)$, $g_F$ is a facet pairing transformation for $F$.*
- *For the facet $F' = g_F^{-1}.F$, its corresponding facet pairing transformation $g_{F'}$ coincides with $g_F^{-1}$.*

If $P$ is a polyhedron equipped with a facet pairing, one sees that $g_{F'}^{-1}.F' = F$ for any $F \in \mathcal{S}(P)$. That is, facets of $P$ occur in pairs $\{F, F'\}$. One considers the space obtained by **gluing** the paired facets of an exact convex polyhedron equipped with a facet pairing.

Definition 2.2.7. *Two points $x, x'$ in an exact convex polyhedron $P$ are said to be* ***paired*** *by the facet pairing $\Phi$, written $x \cong x'$, if there is a pair of facets $F$, $F' \in \mathcal{S}(P)$ such that $x \in F$, $x' \in F'$, and $x = g_F.x'$. The pairing for points in $P$ defines an equivalence relation, namely, two points $x \sim x' \in P$ if and only if $x = x_1 \cong x_2 \cong \ldots \cong x_m = x'$ for a finite number $m$.*
*This equivalence relation defines a quotient space $M := P/\sim$, and the metric on $P$ descends to a path-metric on $M$. This metric space $M$ is called the* ***quotient space*** *of $P$ obtained by* ***gluing*** *the facets of $P$ together.*

If $P$ is a fundamental domain of a discrete subgroup $\Gamma < SO^+(n,1)$, one shows that the resulting quotient space of $P$ is isometric to the hyperbolic manifold or orbifold $\mathbf{H}^n/\Gamma$ [**Rat94**]. Moreover, the metric space $\mathbf{H}^n/\Gamma$ is complete. Therefore, if an exact convex polyhedron $P$ is a fundamental polyhedron for a certain discrete subgroup $\Gamma$, two conditions should be satisfied:

- The quotient space $M = P/\sim$ is either a hyperbolic manifold or orbifold.
- The quotient space $M$ is complete.

The second condition is usually formulated as a **cusp link condition**, which is omitted here (in Poincaré's Algorithm, one does not need to test this condition; see Proposition 2.2.4 below).

Regarding the first condition, one knows that $M$ is an orbifold (or manifold) whenever the ridges of $P$ satisfy certain conditions:

DEFINITION 2.2.8. *A **ridge cycle** of the facet pairing $\Phi$ for $P$ is an equivalence class $[x]$ under the equivalence condition $\sim$, where $x$ is an interior point of a ridge $r \in \mathcal{R}(P)$.*

PROPOSITION 2.2.3. *Any finite ridge cycle of $\Phi$ is a set $[x] = \{x_1, \ldots, x_m\}$, where the point $x_i$ is contained in the ridge $r_i$, such that:*

- *The points $x_i \cong x_{i+1}$ for $i = 1, \ldots, m$, $i$ is taken modulo $m$.*
- *For any $i > 1$, $r_i = F'_{i-1} \cap F_i$, and $g_{F_{i-1}}.r_i = r_{i-1}$, $i$ is taken modulo $m$.*

*The cycle $[x]$ is called **cyclic** if all $x_i$'s are distinct. It is called **dihedral** if $m$ is even, $x_i = x_{m+1-i}$, and $F_i = F'_{m-i}$, $i$ is taken modulo $m$.*

DEFINITION 2.2.9. *For a finite ridge-cycle $[x] = \{x_1, \ldots, x_m\}$, define its **dihedral angle sum** as*

$$\theta[x] = \sum_{i=1}^{m} \theta(x_i),$$

*where $\theta(x_i)$ is the dihedral angle of $P$ along the ridge $r_i$ containing $x_i$.*

THEOREM 2.2.10 ( [**Rat94**], Theorem 13.4.2). *Let $P$ be an exact convex polyhedron in $\mathbf{H}^n$ equipped with facet pairing $\Phi$, and $M$ be the quotient space obtained by gluing the facets of $P$ together. Suppose that all ridge cycles $[x]$ of $\Phi$ in $P$ satisfy the **ridge cycle condition**, i.e.:*

- *The ridge cycle $[x]$ is finite, and*
- *The dihedral angle sum $\theta[x] = 2\pi/k$, where $k \in \mathbb{N}$.*

*Then, $M$ is a hyperbolic orbifold or manifold. Specifically, if all ridge cycles are cyclic with $\theta[x] = 2\pi$, then $M$ is a hyperbolic manifold.*

Poincaré's Fundamental Polyhedron Theorem claims that the two conditions above suffice for $P$ to be a fundamental polyhedron:

THEOREM 2.2.11 (Poincaré). *Let $P$ be an exact convex polyhedron in $\mathbf{H}^n$ equipped with a facet pairing $\Phi$, and let $M$ be the quotient space resulting from gluing $P$ by $\Phi$. Assume that*

- *The facet pairing $\Phi$ for $P$ satisfies the ridge cycle condition.*
- *The quotient space $M$ is complete.*

*Then the following holds for the convex polyhedron $P$:*

- *The group $\Gamma = \langle \Phi \rangle$ generated by the facet pairing $\Phi$ is a discrete subgroup of $SO^+(n, 1)$.*
- *The convex polyhedron $P$ is a fundamental polyhedron for $\Gamma$.*
- *The group $\Gamma$ has a presentation with the generating set $\Phi$. The relators for this group presentation consist of the words $g_F g_{F'}$ corresponding to the facet pairing transformations, and the words $(g_{F_1} \dots g_{F_m})^k$ corresponding to the ridge cycles.*

*Here, the notions of the facets $F_1, \dots, F_m$ agree with those in Proposition 2.2.3, and $k \in \mathbb{N}$ is the divisor appeared in the ridge cycle condition in Theorem 2.2.10.*

**2.2.2. Poincaré's algorithm for hyperbolic spaces.** Let us introduce Poincaré's Algorithm based on Poincaré's Fundamental Polyhedron Theorem. The algorithm checks if a subgroup of $SO^+(n, 1)$ is geometrically finite. A similar algorithm was initially suggested by Riley [**Ril83**] for the case $n = 3$ and by Epstein and Petronio [**EP94**] for the general case.

Recall that in Definition 2.2.4, we defined Dirichlet domains for discrete subsets of $SO^+(n, 1)$. In particular, we consider Dirichlet domains for finite subsets of $SO^+(n, 1)$; such domains are finitely-sided. A benefit of having finitely-sided Dirichlet domains is the completeness property:

Proposition 2.2.4 ( [**Kap23**]). *Suppose that $\Gamma$ is a finite subset of $SO^+(n, 1)$, $x \in \mathbf{H}^n$, and the Dirichlet domain $D(x, \Gamma)$ satisfies the ridge cycle condition. Then the quotient space for $D(x, \Gamma)$ is complete, hence the polyhedron $D(x, \Gamma)$ satisfies the assumptions for Poincaré's Fundamental Polyhedron Theorem.*

Below we describe Poincaré's Algorithm, starting with a finite set of generators $\{g_1, \dots, g_m\}$ of a subgroup $\Gamma < SO^+(n, 1)$, as well as a center $x \in \mathbf{H}^n$.

**Poincaré's Algorithm.**

(1) Starting with $l = 1$, compute the subset $\Gamma_l \subset \Gamma$ of elements represented by words of length $\leq l$ in the letters of $g_i$ and $g_i^{-1}$, $i = 1, \dots, m$. The result is a finite subset of $SO^+(n, 1)$.

(2) Compute the Dirichlet domain $D(x, \Gamma_l)$ centered at $x$ for the finite set $\Gamma_l$. Namely, we compute the equations for all ridges of $D(x, \Gamma_l)$. Epstein and Petronio [**EP94**] provide an algorithm for this task.

(3) Having the data for all ridges of $D(x, \Gamma_l)$, we check if this convex polyhedron is exact. That is, for any $g \in \Gamma_l$ and facets $F_g$, $F_{g^{-1}}$ of $D(x, \Gamma_l)$ contained in $Bis(x, g.x)$ and

$Bis(x, g^{-1}.x)$ respectively, we check if $g.F_{g^{-1}} = F_g$. Algorithms for this task can be found in, e.g., [EF82].

(4) We check if this convex polyhedron satisfies the ridge cycle condition in Poincaré's Fundamental Polyhedron Theorem.

(5) If the condition is not satisfied, replace $l$ with $(l+1)$ and repeat the steps above.

(6) If the ridge cycle condition is satisfied, then Proposition 2.2.4 implies that $D = D(x, \Gamma_l)$ satisfies the requirements for Poincaré's theorem. Therefore, $D(x, \Gamma_l)$ is the fundamental domain for the group $\Gamma' = \langle \Gamma_l \rangle$. Check as follows if $g_i \in \Gamma'$ for each generator $g_i$, $i = 1, \dots, m$. First, we connect $x$ and $g_i.x$ with a path avoiding the $\Gamma'$-image of the ridges of $D$. This path determines a word in letters of the facet pairings of $D$. Then we check if this word equals the generator $g_i$. [Ril83]

(7) If there is a generator $g_i \notin \Gamma'$, we replace $l$ with $(l+1)$ and repeat the steps from the beginning.

(8) If all generators $g_i \in \Gamma'$, then $\Gamma = \Gamma'$ is a discrete subgroup of $SO^+(n,1)$. Moreover, $\Gamma$ is finitely presented; we derive the relators for the presentation of $\Gamma$ from the ridge-cycle data of $D(x, \Gamma_l)$.

REMARK 2.2.1. *For $n \geq 4$, Bowditch [**Bow93**] gives examples of discrete subgroups of $SO^+(n,1)$ that admit finitely-sided fundamental domains while all their Dirichlet domains are infinitely-sided. Although Poincaré's algorithm fails for such subgroups, the algorithm works for all convex cocompact subgroups [**KLP14**].*

**2.2.3. Selberg's invariant and convex polyhedra in $\mathcal{P}(n)$.** [1] In Section 2.2.1 we introduced Dirichlet domains in the context of hyperbolic spaces. However, studying Dirichlet domains in the symmetric space $\mathcal{P}(n)$ is impractical due to the nonlinear nature of the Riemannian distance on $\mathcal{P}(n)$. Selberg [Sel62] introduced a function for pairs of points in $\mathcal{P}(n)$, which is invariant under the $SL(n, \mathbb{R})$-action:

DEFINITION 2.2.12. *For $X, Y \in \mathcal{P}(n)$, define the **Selberg's invariant** from $X$ to $Y$ as*

$$s(X, Y) = \operatorname{tr}(X^{-1}Y).$$

[1] From now on, we denote points in $\mathcal{P}(n)$ by capital letters such as $X$ or $Y$. This notation should not be confused with the notation $X$ for a symmetric space; the latter will not appear for the rest of the dissertation.

Proposition 2.2.5. *The function $s(-,-)$ satisfies the following properties:*

- *The function $s(-,-)$ behaves similarly to a distance function: $s(X,Y) \geq n$ for any $X, Y \in \mathcal{P}(n)$, and $s(X,Y) = n$ if and only if $X = Y$.*
- *The function $s(-,-)$ is $SL(n,\mathbb{R})$-invariant: for any $g \in SL(n,\mathbb{R})$ and $X, Y \in \mathcal{P}(n)$, one has $s(X,Y) = s(g.X, g.Y)$.*

Proof. For the first claim, we notice that $s(X,Y) = \mathrm{tr}(X^{-1/2}YX^{-1/2})$, while $X^{-1/2}YX^{-1/2}$ is a positive definite matrix with determinant 1. Thus, the trace of $X^{-1/2}YX^{-1/2}$, which is the sum of the eigenvalues of $X^{-1/2}YX^{-1/2}$, is no less than $n$. If the equality holds, then all the eigenvalues of $X^{-1/2}YX^{-1/2}$ are equal to 1. Therefore, $X^{-1/2}YX^{-1/2} = I_n$, which implies that $X = Y$.
For the second claim:

$$s(g.X, g.Y) = \mathrm{tr}((g.X)^{-1}(g.Y)) = \mathrm{tr}(g^{-1}X^{-1}{g^{\mathrm{T}}}^{-1}g^{\mathrm{T}}Yg) = \mathrm{tr}(g^{-1}X^{-1}Yg) = \mathrm{tr}(X^{-1}Y) = s(X,Y).$$

□

Remark 2.2.2. *Selberg's invariant relates to the vector-valued distance on $\mathcal{P}(n)$ as follows:*

$$s = \sum_{i=1}^{n} \exp(d_i),$$

*where $(d_1, \ldots, d_n) = d_\Delta$. This fact also implies Proposition 2.2.5.*

Bisectors defined via Selberg's invariant are linear:

Proposition 2.2.6. *Define the (Selberg)* ***bisector*** *$Bis(X,Y)$ for $X, Y \in \mathcal{P}(n)$:*

$$Bis(X,Y) = \{Z \in \mathcal{P}(n) | s(X,Z) = s(Y,Z)\}.$$

*Then for any $X, Y \in \mathcal{P}(n)$, $Bis(X,Y)$ is defined by a linear equation over the entries of the symmetric matrix $Z$.*

Proof. One has that $Bis(X,Y) = \{Z \in \mathcal{P}(n) | \mathrm{tr}((X^{-1} - Y^{-1})Z)\} = 0$, which is obviously linear. □

The linear nature of Selberg's invariant on $\mathcal{P}(n)$ allows Selberg to define a polyhedral analog of the Dirichlet domain. We begin by defining **convex polyhedra** in $\mathcal{P}(n)$ with respect to the symmetric matrix model of $\mathcal{P}(n)$:

Definition 2.2.13. *A $d$-**plane** of $\mathcal{P}(n)$ is the non-empty intersection of $\mathcal{P}(n)$ with a $(d+1)$-dimensional linear subspace of the vector space $Sym_n(\mathbb{R}) = \mathbb{R}^{n(n+1)/2}$. An $((n-1)(n+2)/2-1)$-plane of $\mathcal{P}(n)$ is called a **hyperplane** of $\mathcal{P}(n)$.*

*We define (open and closed) **half-spaces** of $\mathcal{P}(n)$ and **convex polyhedra** in $\mathcal{P}(n)$ analogously to Definition 2.2.1 and 2.2.2.*

*We define the **dimension** of convex polyhedra in $\mathcal{P}(n)$, as well as **facets**, faces, and ridges of convex polyhedra in $\mathcal{P}(n)$ analogously to Definition 2.2.3.*

*The group $SL(n,\mathbb{R})$ acts on convex polyhedra in $\mathcal{P}(n)$. We define **fundamental polyhedra** for a discrete subgroup $\Gamma < SL(n,\mathbb{R})$ analogously to Definition 1.1.3.*

Following Selberg, we define Dirichlet domains in $\mathcal{P}(n)$ with respect to Selberg's invariant instead of the Riemannian distance:

Definition 2.2.14. *The **Dirichlet-Selberg domain** for a discrete subset $\Gamma \subset SL(n,\mathbb{R})$ centered at $X \in \mathcal{P}(n)$ is the set*

$$DS(X,\Gamma) = \{Y \in \mathcal{P}(n) | s(X,Y) \leq s(g.X,Y),\ \forall g \in \Gamma\}.$$

Proposition 2.2.7 ( [**Kap23**]). *For a discrete subgroup $\Gamma < SL(n,\mathbb{R})$ and a point $X \in \mathcal{P}(n)$, the Dirichlet-Selberg domain $DS(X,\Gamma)$ is a convex polyhedron in $\mathcal{P}(n)$. Moreover, if $Stab_\Gamma(X)$ is trivial, $DS(X,\Gamma)$ is a fundamental polyhedron for $\Gamma$.*

**2.2.4. Poincaré's algorithm for $\mathcal{P}(n)$.** Poincaré's theorem for $\mathcal{P}(n)$ is similar to that for hyperbolic spaces. However, the dihedral angle in hyperbolic spaces lacks an analog in $\mathcal{P}(n)$. The **tiling condition**, avoiding using dihedral angles, is equivalent to the ridge cycle condition: [**Kap23**]

Definition 2.2.15. *Let $P$ be an exact convex polyhedron in $\mathcal{P}(n)$, $[x] = \{x_1, \ldots, x_m\}$ be a finite ridge cycle, and $x_i$ is contained in the ridge $r_i$ for $i = 1, \ldots, m$. For any $i$, the ridge $r_i$ is the intersection $F_i \cap F'_{i-1}$, where $i$ is taken modulo $m$. The facet $F'_i = g_{F_i}^{-1} F_i$ is paired with $F_i$ by $g_{F_i} \in SL(n,\mathbb{R})$ (cf. Proposition 2.2.3). The ridge cycle $[x]$ satisfies the **tiling condition** if there exists a neighborhood $U$ of $r = r_1$ in $\mathcal{P}(n)$ such that:*

- *The set $U$ is a union*

$$U = \bigcup_{i=1}^{km} U_i,\ U_i = (\prod_{j=1}^{i-1} g_{F_j}).V_i,$$

*for a number $k \in \mathbb{N}$. Here $V_i$ is a neighborhood of $r_i$ in $P$, $i$ is taken modulo $m$.*

- *The intersection*

$$int(U_i) \cap int(U_j) = \varnothing,$$

*for any $i \neq j$.*

- *The intersection*

$$S_i = U_i \cap U_{i+1} = U \cap (\prod_{j=1}^{i-1} g_{F_j}).F_i,$$

*for any $i = 1, \ldots, km$ taken modulo $km$. If $i - j \not\equiv \pm 1$ modulo $km$, the intersection*

$$U_i \cap U_j = r.$$

*We say the convex polyhedron $P$ satisfies the tiling condition, if every ridge cycle of $P$ is finite and satisfies the tiling condition.*

Intuitively, for a ridge cycle of a polyhedron $D$, the tiling condition implies that the facet pairing transformations glue the neighborhoods of the ridges in $D$ together. This is the condition one needs for Poincaré's Fundamental Polyhedron Theorem:

THEOREM 2.2.16 (Poincaré's theorem for $\mathcal{P}(n)$ [Kap23]). *Let $P$ be an exact convex polyhedron in $\mathcal{P}(n)$ with facet pairing $\Phi$. Let $M$ be the quotient space resulting from gluing the facets by $\Phi$, equipped with the quotient path-metric from the convex polyhedron $P$ in $\mathcal{P}(n)$. Suppose that*

- *The facet pairing $\Phi$ satisfies the tiling condition.*
- *The quotient space $M$ is complete.*

*Then the following holds:*

- *The group $\Gamma = \langle \Phi \rangle$ is a discrete subgroup of $SL(n, \mathbb{R})$.*
- *The convex polyhedron $P$ is a fundamental polyhedron for $\Gamma$.*
- *The group $\Gamma$ is finitely presented, with the generating set $\Phi$. The set of relators for the presentation consists of the following: the words $g_F g_{F'}$ corresponding to the facet pairings, and the words $g_{F_1} \ldots g_{F_m}$ corresponding to the ridge-cycles, as we defined in Definition 2.2.15.*

We finally arrive at the analog of Poincaré's algorithm for Dirichlet-Selberg domains in the symmetric space $\mathcal{P}(n)$. Suppose we have a finite set of generators $\{g_1, \ldots, g_n\} \subset SL(n, \mathbb{R})$ and a point $X \in \mathcal{P}(n)$ as the center of Dirichlet-Selberg domains. The algorithm below determines if the subgroup $\Gamma < SL(n,\mathbb{R})$ generated by $\{g_1,\ldots,g_n\}$ is discrete and finitely presented.

**Poincaré's Algorithm (tentative).**

(1) Starting with $l=1$, compute the subset $\Gamma_l \subset \Gamma$ of elements represented by words of length $\leq l$ in the letters of $g_i$ and $g_i^{-1}$. This is a finite subset of $SL(n,\mathbb{R})$.

(2) Compute the Dirichlet-Selberg domain $DS(X,\Gamma_l)$ for the finite set $\Gamma_l$.

(3) Having the data of ridges of $DS(X,\Gamma_l)$, we check if this convex polyhedron is exact.

(4) We check if this convex polyhedron satisfies the tiling condition. (To implement this step, we consider a "ridge cycle condition" in Chapter 3, which is equivalent to the tiling condition.)

(5) If the condition is not satisfied, replace $l$ with $(l+1)$, and repeat the steps above.

(6) If the ridge cycle condition is satisfied, we check if the quotient space is complete. If so, then $DS(X,\Gamma_l)$ satisfies the conditions in Poincaré's theorem for $\mathcal{P}(n)$. Therefore, $DS(X,\Gamma_l)$ is the fundamental domain for the group $\Gamma'$ generated by $\Gamma_l$. For each generator $g_i$ in the generating set, check if $g_i \in \Gamma'$, similarly to the algorithm for hyperbolic spaces.

(7) If any generator $g_i \notin \Gamma'$, replace $l$ with $(l+1)$ and repeat the steps from the beginning.

(8) If all generators $g_i \in \Gamma'$, then $\Gamma = \Gamma'$ is a discrete subgroup of $SL(n,\mathbb{R})$. Moreover, $\Gamma$ is finitely presented. We obtain the relators of the presentation for $\Gamma$ from the ridge-cycle data of $DS(X,\Gamma_l)$.

Regarding step (6), Kapovich conjectures that finitely-sided Dirichlet-Selberg domains $DS(X,\Gamma_l)$ satisfy the completeness property (similarly to Dirichlet domains in the hyperbolic spaces):

Conjecture 2.2.17 ( [Kap23]). *Let $D = DS(X,\Gamma_l)$ be a finitely-sided Dirichlet-Selberg domain in $\mathcal{P}(n)$ that satisfies the tiling condition. Then, the quotient space $M = D/\sim$ is complete.*

## 2.3. Preliminaries

Below we review the necessary preliminaries before presenting the main results of this dissertation.

**2.3.1. Matrix pencils and generalized eigenvalues.** Some of our main results use matrix pencils. We briefly review the concepts related to our research.

Definition 2.3.1. *A real (or complex)* ***matrix pencil*** *is a set* $\{A - \lambda B | \lambda \in \mathbb{R}\}$ *(or* $\lambda \in \mathbb{C}$*, respectively), where* $A$ *and* $B$ *are real* $n \times n$ *matrices. We denote this matrix pencil by* $(A, B)$*.*
*We say a matrix pencil* $(A, B)$ *is* ***regular*** *if* $\det(A - \lambda B) \neq 0$ *for at least one value* $\lambda \in \mathbb{C}$ *(equivalently, for almost every* $\lambda$*). We say* $(A, B)$ *is* ***singular*** *if both* $A$ *and* $B$ *are singular and* $A - \lambda B$ *is singular for all* $\lambda \in \mathbb{C}$*.*

We define the **generalized eigenvalues** of a matrix pencil:

Definition 2.3.2. *A* ***generalized eigenvalue*** *of a matrix pencil* $(A, B)$ *is a number* $\lambda_0 \in \mathbb{C}$ *such that* $A - \lambda_0 B$ *is singular.*
*For a regular pencil* $(A, B)$*, the* ***multiplicity*** *of a generalized eigenvalue* $\lambda_0$ *is the multiplicity of the root* $\lambda = \lambda_0$ *for the polynomial* $\det(A - \lambda B)$ *over* $\lambda$*.*
*If* $B$ *is singular, we adopt the convention that* $\infty$ *is a generalized eigenvalue of the pencil* $(A, B)$*. The multiplicity of* $\infty$ *is* $n - \deg(\det(A - \lambda B))$*.*
*In particular, every* $\lambda \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ *is a generalized eigenvalue of a singular matrix pencil.*

A matrix pencil $(A, B)$ is **symmetric** if both $A$ and $B$ are symmetric matrices. We define **definiteness** for symmetric matrix pencils:

Definition 2.3.3. *We say that a symmetric matrix pencil* $(A, B)$ *is (semi-) definite, if either* $A$ *or* $B$ *is (semi-) definite, or if* $A - \lambda B$ *is (semi-) definite for at least one number* $\lambda \in \mathbb{R}$*.*

We define congruence transformations of symmetric matrix pencils as

$$(A, B) \to (Q^{\mathrm{T}} A Q, Q^{\mathrm{T}} B Q),$$

where $Q \in GL(n, \mathbb{R})$, and $A, B \in Sym_n(\mathbb{R})$. Generalized eigenvalues are invariant under these transformations:

Proposition 2.3.1. *For any* $Q \in GL(n, \mathbb{R})$*, the matrix pencils* $(A, B)$ *and* $(Q^{\mathrm{T}} A Q, Q^{\mathrm{T}} B Q)$ *have the same generalized eigenvalues as well as the same multiplicities of them.*

Proof. Notice that $\det(P^{\mathrm{T}} A P - \lambda P^{\mathrm{T}} B P) = \det(P)^2 \det(A - \lambda B)$, while $\det(P) \neq 0$. Thus, these polynomials have the same roots as well as the same multiplicities of roots. $\square$

If $A'$ and $B'$ are linearly independent linear combinations of $A$ and $B$, the generalized eigenvalues of $(A', B')$ relate to those of $(A, B)$ by a Möbius transformation:

LEMMA 2.3.3.1. *Suppose that $\lambda_1, \ldots, \lambda_n$ are the generalized eigenvalues of the matrix pencil $(A, B)$. Then for any $p, q, r, s \in \mathbb{R}$ with $ps - qr \neq 0$, the generalized eigenvalues of $(pA + qB, rA + sB)$ are $\lambda_i' := \frac{p\lambda_i + q}{r\lambda_i + s}$, $i = 1, \ldots, n$.*

PROOF. Notice that

$$(pA + qB) - \frac{p\lambda + q}{r\lambda + s}(rA + sB) = \frac{(ps - qr)(A - \lambda B)}{r\lambda + s}.$$

Since $ps - qr \neq 0$, one has that $\frac{p\lambda + q}{r\lambda + s}$ is a generalized eigenvalue of $(pA + qB, rA + sB)$ if and only if $\lambda$ is a generalized eigenvalue of $(A, B)$.

If $\infty$ is a generalized eigenvalue of $(A, B)$, then $B$ is singular. Therefore,

$$A' - \frac{p}{r}B' = \frac{rA' - pB'}{r} = \frac{(qr - ps)B}{r}$$

is singular. That is, $\frac{p}{r}$ is a generalized eigenvalue of $(A', B')$. This agrees with the statement of the lemma if one interprets the formal expression $\frac{p\infty + q}{r\infty + s}$ as $\frac{p}{r}$.

In both cases, the corresponding eigenvalues are related by a Möbius transformation: $\overline{\mathbb{C}} \to \overline{\mathbb{C}}, \lambda \mapsto \frac{p\lambda + q}{r\lambda + s}$. □

Our work uses a normal form of matrix pencils under congruence transformation. For this reason, we introduce block-diagonal matrix pencils:

DEFINITION 2.3.4. *A **block-diagonal matrix pencil** is a matrix pencil $(A, B)$, where $A = diag(A_1, \ldots, A_m)$ and $B = diag(B_1, \ldots, B_m)$; for $i = 1, \ldots, m$, $A_i$ and $B_i$ are square matrices of the same dimension $d_i$.*

*The blocks of an $n \times n$ block-diagonal matrix pencil $(A, B)$ define a partition of the set $\{1, \ldots, n\}$. We say the matrix pencil $(A', B')$ is (strictly) **finer** than the matrix pencil $(A, B)$ if the partition corresponding to the pencil $(A', B')$ is (strictly) finer than the one corresponding to $(A, B)$, up to a permutation of the $n$ numbers.*

Uhlig characterizes the "finest" block-diagonalization of regular symmetric matrix pencils:

THEOREM 2.3.5 ( [Uhl73]). *Let $(A, B)$ be a symmetric matrix pencil with $A$ invertible. Suppose that the Jordan canonical form of $B^{-1}A$ is $Q^{-1}B^{-1}AQ = J = diag(J_1, \ldots, J_m)$, where $J_i$ is a Jordan block of dimension $d_i$, $i = 1, \ldots, m$. Then $(A', B') = (Q^{\mathrm{T}}AQ, Q^{\mathrm{T}}BQ)$ is a block-diagonal matrix pencil; the block $(A_i, B_i)$ is of dimension $d_i$ for $i = 1, \ldots, m$. Moreover, $(A', B')$ is finer than any matrix pencil in its congruence equivalence class.*

DEFINITION 2.3.6. *For a regular symmetric matrix pencil $(A, B)$, suppose that there exists $c \in \mathbb{R}$ such that $B + cA$ is invertible, and $Q^{-1}(B + cA)^{-1}AQ$ is the Jordan canonical form of $(B + cA)^{-1}A$. Define the **normal form** of $(A, B)$ under congruence transformations as*

$$(A', B') = (Q^{\mathrm{T}}AQ, Q^{\mathrm{T}}BQ).$$

Blocks of $(A', B')$ satisfy additional properties:

LEMMA 2.3.6.1. *In the notation of Theorem 2.3.5, let $(A_i, B_i)$ be the diagonal blocks of the congruence normal form $(A', B')$ of the matrix pencil $(A, B)$, $i = 1, \ldots, m$. Suppose that $A_i = (a_i^{j,k})_{j,k=1}^{d_i}$ and $B_i = (b_i^{j,k})_{j,k=1}^{d_i}$. Then the entries $a_i^{j,k}$ satisfy:*

*(1) $a_i^{j,k} = a_i^{j',k'}$, for any $j + k = j' + k'$,*

*(2) $a_i^{j,k} = 0$, for any $j + k \leq d_i$.*

*The entries $b_i^{j,k}$ satisfy the same property.*

PROOF. The matrices satisfy the relation $A_i = B_iJ_i$, where $J_i = J_{\lambda_i, d_i}$ is the $d_i \times d_i$ Jordan block matrix with the eigenvalue $\lambda_i$. Thus, for any $j$ and any $k > 1$,

$$a_i^{j,k} = \lambda_i b_i^{j,k} + b_i^{j,k-1}, \tag{2.1}$$

and $a_i^{j,1} = \lambda_i b_i^{j,1}$.

Since both $A_i$ and $B_i$ are symmetric, for any $j, k > 1$,

$$b_i^{j-1,k} = b_i^{k-1,j} = a_i^{k,j} - \lambda_i b_i^{k,j} = a_i^{j,k} - \lambda_i b_i^{j,k} = b_i^{j,k-1},$$

which is the property (1).

For any $k < d_i$,

$$b_i^{1,k} = a_i^{1,k+1} - \lambda_i b_i^{1,k+1} = a_i^{k+1,1} - \lambda_i b_i^{k+1,1} = 0.$$

Therefore, if $j+k \le d_i$, property (1) implies that

$$b_i^{j,k} = \cdots = b_i^{1,j+k-1} = 0,$$

which is the property (2). The entries of $A_i$ satisfy the same property since $A_i = B_i J_i$. $\square$

The normal form in Theorem 2.3.5 does not apply to singular symmetric matrix pencils. Nevertheless, Jiang and Li prove the following result:

LEMMA 2.3.6.2 ( [JL16]). *Let $(A, B)$ be a singular symmetric $n \times n$ matrix pencil. Then $(A, B)$ is congruent to $(A', B')$, where the matrices $A'$ and $B'$ satisfy*

$$A' = \begin{pmatrix} A_1 & O & O \\ O & O & O \\ O & O & O \end{pmatrix}, \quad B' = \begin{pmatrix} B_1 & B_2 & O \\ B_2^{\mathrm{T}} & O & O \\ O & O & B_3 \end{pmatrix},$$

*for $n_1 \times n_1$ matrices $A_1$ and $B_1$, an $n_1 \times n_2$ matrix $B_2$, and an $n_3 \times n_3$ matrix $B_3$, $n_1 + n_2 + n_3 = n$. Moreover, $A_1$ and $B_3$ are invertible.*

**2.3.2. Co-oriented hyperplanes.** We introduce **co-oriented hyperplanes**, which will be utilized to define invariant angle functions in Chapter 3:

DEFINITION 2.3.7. *The normal space of a nonzero matrix $A \in Sym_n(\mathbb{R})$ is defined as*

$$A^{\perp} = \{X \in \mathcal{P}(n) | \mathrm{tr}(X \cdot A) = 0\},$$

*which is a hyperplane in $\mathcal{P}(n)$ whenever it is non-empty. We designate $A$ as a **normal vector** of the hyperplane $A^{\perp}$. A hyperplane associated with a normal vector is called a **co-oriented hyperplane**. The normal vector of a hyperplane is unique up to a nonzero multiple. Identical co-oriented hyperplanes with normal vectors that differ by a positive multiple are regarded as the same co-oriented hyperplanes. Identical co-oriented hyperplanes with normal vectors that differ by a negative multiple from each other are said to be oppositely oriented. If $\sigma$ is a co-oriented hyperplane given by $A^{\perp}$, then the co-oriented hyperplane with the opposite orientation is denoted by $-\sigma$ or $(-A)^{\perp}$.*
*We say that a co-oriented hyperplane $\sigma$ **lies between** two co-oriented hyperplanes $A^{\perp}$ and $B^{\perp}$ if the normal vector associated with $\sigma$ is a positive linear combination of $A$ and $B$.*

The idea of a co-oriented hyperplane lying between two co-oriented hyperplanes is presented in Figure 2.1. We also define the co-orientation for facets of convex polyhedra in $\mathcal{P}(n)$:

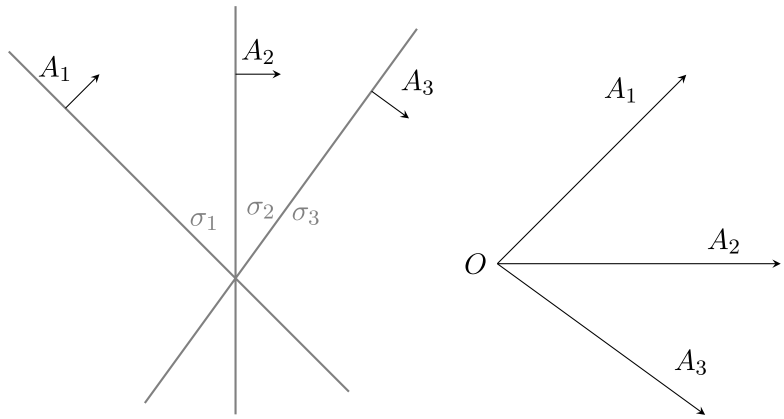


FIGURE 2.1. The hyperplane $\sigma_2$ lies between the hyperplanes $\sigma_1$ and $\sigma_3$.

DEFINITION 2.3.8. *Let $P$ be a convex polyhedron in $\mathcal{P}(n)$. A facet $F \in \mathcal{S}(P)$ associated with a normal vector $A$ of span$(F)$ is called a* ***co-oriented facet*** *of $P$.*

*For any facet $F \in \mathcal{S}(P)$, the convex polyhedron $P$ lies within one of the two closed half-spaces bounded by span$(F)$. When there are no special instructions, we make a convention that the normal vector $A$ associated with $F$ is selected so that*

$$P \subset \{X \in \mathcal{P}(n) | \mathrm{tr}(X \cdot A) \leq 0\},$$

*and say that $A$ is outward-pointing (and $-A$ is inward-pointing). We also make a convention that span$(F)$ is associated with the same normal vector as $F$.*

The lemma below is self-evident.

LEMMA 2.3.8.1. *Let $P$ be a convex polyhedron, with $r \in \mathcal{R}(P)$ being a ridge of $P$ such that $r = S_1 \cap S_2$, $S_1, S_2$ being facets of $P$.*

*Suppose that a hyperplane $\sigma$ contains $r$ and divides $P$ into two convex polyhedra $P_1$ and $P_2$, where $S_1 \subset P_1$ and $S_2 \subset P_2$. Denote $S = \sigma \cap P$, which is a facet of $P_1$ associated with an outward-pointing normal vector (thus a facet of $P_2$ associated with an inward-pointing normal vector).*

*Under these assumptions, span$(S)$ lies between $-$span$(S_1)$ and span$(S_2)$.*

CHAPTER 3

# Angle-like Functions between Hyperplanes

Our first goal is to formulate an analog of the angle sum condition that is equivalent to the tiling condition in Definition 2.2.15. This analog requires us to introduce an angle-like function for pairs of co-oriented hyperplanes in $\mathcal{P}(n)$. Such an angle-like function must satisfy specific properties; for instance, it is natural to assume that this function is additive and invariant under the $SL(n,\mathbb{R})$-action, akin to the dihedral angle in hyperbolic spaces.

## 3.1. Main Result

Below we define an **invariant angle function** for $\mathcal{P}(n)$:

DEFINITION 3.1.1. *An invariant angle function $\theta(-,-)$ is a function for pairs of co-oriented hyperplanes $(\sigma_1,\sigma_2)$ in $\mathcal{P}(n)$ with the following properties:*

1. *For any co-oriented hyperplanes $\sigma_1$ and $\sigma_2$, $0 \leq \theta(\sigma_1,\sigma_2) \leq \pi$. Furthermore, $\theta(\sigma_1,\sigma_2) = 0$ if and only if $\sigma_1 = \sigma_2$, while $\theta(\sigma_1,\sigma_2) = \pi$ if and only if $\sigma_1 = -\sigma_2$.*
2. *For any co-oriented hyperplanes $\sigma_1$ and $\sigma_2$ and any $g \in SL(n,\mathbb{R})$, $\theta(g.\sigma_1, g.\sigma_2) = \theta(\sigma_1,\sigma_2)$.*
3. *For any co-oriented hyperplanes $\sigma_1$ and $\sigma_2$, $\theta(\sigma_2,\sigma_1) = \theta(\sigma_1,\sigma_2)$, $\theta(-\sigma_1,\sigma_2) = \pi - \theta(\sigma_1,\sigma_2)$.*
4. *For any co-oriented hyperplane $\sigma_2$ lying between $\sigma_1$ and $\sigma_3$, $\theta(\sigma_1,\sigma_2)+\theta(\sigma_2,\sigma_3) = \theta(\sigma_1,\sigma_3)$.*

PROPOSITION 3.1.1. *Let $\theta$ be an invariant angle function. For any exact convex polyhedron $P$ in $\mathcal{P}(n)$, the tiling condition for $P$ as defined in Definition 2.2.15 is equivalent to the following* ***angle sum condition*** *for $P$:*

- *Any ridge cycle $[x]$ is a finite set, $[x] = \{x_1,\ldots,x_m\}$.*
- *Furthermore, $\theta[x] = \sum_{i=1}^{m} \theta(x_i) = 2\pi/k$ for $k \in \mathbb{N}$. Here, $\theta(x_i) = \theta(F_i, F'_{i-1})$ represents the invariant angle $\theta$ for the two co-oriented hyperplanes spanned by the two facets $F_i$, $F'_{i-1}$ of $P$ containing $x_i$.*

Proof. A finite ridge cycle for the polyhedron $P$ is a set of points $\{x_1, \dots, x_m\}$ in $P$, where the ridges $r_i = F_i \cap F'_{i-1} \supset x_i$, $F_i$ and $F'_i$ are facets of $P$, $i = 1, \dots, m$. We suppose that all facets of $P$ are co-oriented via outward-pointing normal vectors.

On one hand, suppose that a ridge cycle $[x]$ in $P$ satisfies the tiling condition. That is, the ridge $r = r_1$ containing $x = x_1$ has a neighborhood $U$ in $\mathcal{P}(n)$, such that

$$U = \bigcup_{i=1}^{km} U_i,$$

and the sets $U_i$, $i = 1, \dots, km$ satisfy the conditions in Definition 2.2.15; specifically, $U_i = \left(\prod_{j=1}^{i-1} g_{F_j}\right).V_i$, $V_i \subset P$. Denote

$$P_i = \left(\prod_{j=1}^{i-1} g_{F_j}\right).P, \quad S_i = \left(\prod_{j=1}^{i-1} g_{F_j}\right).F_i,$$

then $S_i$ is a facet of both $P_i$ and $P_{i+1}$, and $U_i \cap U_{i+1} \subset S_i$. Let $S_i$ be co-oriented by associating it with an outward-pointing normal vector for $P_i$ (thus an inward-pointing normal vector for $P_{i+1}$). Let $\sigma_i = span(S_i)$ denote the co-oriented hyperplane with the same co-orientation by $S_i$, $i = 1, \dots, km$. There exists a certain $1 < j < km$ such that the hyperplane $\sigma_0$ intersects with $U_j$ and then divides $P_j$ into two convex polyhedra. Denoted these polyhedra by $P'_j$ and $P''_j$, where $S_{j-1} \subset P'_j$ and $S_j \subset P''_j$. The polyhedra $P_1, \dots, P_{j-1}$ and $P'_j$ lie within the same connected component of $\sigma_0^{\mathsf{c}}$, and $S_0$ is inward-pointing as a facet of $P_0$. Consequently, the normal vector associated with $-\sigma_0$ is outward-pointing with respect to $P'_j$. According to Lemma 2.3.8.1, $-\sigma_0$ lies between $\sigma_{j-1}$ and $\sigma_j$. For any $0 < i < j-1$, $\sigma_i$ lies between $\sigma_0$ and $\sigma_{j-1}$. Therefore, by properties (3) and (4) of invariant angle functions,

$$\sum_{i=1}^{j-1} \theta(\sigma_i, \sigma_{i-1}) + \theta(\sigma_{j-1}, -\sigma_0) = \theta(\sigma_0, \sigma_{j-1}) + \theta(\sigma_{j-1}, -\sigma_0) = \pi.$$

Similarly we have

$$\theta(-\sigma_0, \sigma_j) + \sum_{i=j+1}^{km} \theta(\sigma_i, \sigma_{i-1}) = \theta(-\sigma_0, \sigma_j) + \theta(\sigma_j, \sigma_{km}) = \theta(-\sigma_0, \sigma_j) + \theta(\sigma_j, \sigma_0) = \pi.$$

Since $-\sigma_0$ lies between $\sigma_{j-1}$ and $\sigma_j$,

$$\theta(\sigma_{j-1}, -\sigma_0) + \theta(-\sigma_0, \sigma_j) = \theta(\sigma_{j-1}, \sigma_j),$$

by property (4) of invariant angle functions. Therefore,

$$\sum_{i=1}^{km} \theta(\sigma_i, \sigma_{i-1}) = 2\pi.$$

By property (2) of invariant angle functions,

$$\theta(\sigma_i, \sigma_{i-1}) = \theta(\prod_{j=1}^{i-1} g_{F_j}.F_i, \prod_{j=1}^{i-1} g_{F_j}.F'_{i-1}) = \theta(F_i, F'_{i-1}) = \theta(x_i),$$

where $i$ is taken modulo $m$. Thus,

$$2\pi = \sum_{i=1}^{km} \theta(x_i) = k \cdot \sum_{i=1}^{m} \theta(x_i) = k\theta[x],$$

i.e., the ridge cycle $[x]$ satisfies the angle sum condition.

On the other hand, suppose that a ridge cycle $[x]$ in $P$ satisfies the angle sum condition. The ridge $r$ is contained in the convex polyhedron

$$P_i = \left(\prod_{j=1}^{i-1} g_{F_j}\right).P,$$

for all $i = 1, \ldots, km$. Moreover, $r$ is the intersection of the facets

$$\left(\prod_{j=1}^{i-1} g_{F_j}.F_i\right) \cap \left(\prod_{j=1}^{i-1} g_{F_j}.F'_{i-1}\right) := S_i \cap S_{i-1}$$

of $P_i$. Analogously to the proof of the other direction given above, the angle sum condition implies that $\sum_{i=1}^{km} \theta(S_i, S_{i-1}) = k\theta[x] = 2\pi$. Therefore,

$$S_0 := F'_0 = \left(\prod_{j=1}^{km-1} g_{F_j}\right).F_{km} := S_{km},$$

i.e., the sets $P_1$ and $P_{km}$ meet at $S_0 = S_{km}$. Moreover, since $\sum_{j=1}^{i} \theta(S_i, S_{i-1}) < 2\pi$ for $i < km$, the interiors $int(P_i)$ are pairwise disjoint for $i = 1, \ldots, km$.

Let $U$ be a sufficiently small neighborhood of the ridge $r$ in $\mathcal{P}(n)$, such that for any $i = 1, \ldots, km$, the set $U$ does not intersect with $P_i$ at facets other than $S_i$ and $S_{i-1}$. Let $U_i = U \cap P_i$, then the union $U = \bigcup_{i=1}^{km} U_i$ satisfies the conditions in Definition 2.2.15. In other words, the ridge cycle $[x]$ satisfies the tiling condition. $\square$

Proposition 3.1.1 implies that, to formulate the tiling condition for convex polyhedra in $\mathcal{P}(n)$, it is sufficient to construct an invariant angle function that satisfies properties (1) to (4) as defined in Definition 3.1.1. For generic pairs of co-oriented hyperplanes, we explicitly construct an invariant angle function, which is presented in the main theorem below.

THEOREM 3.1.2. *Let $\sigma_1 = A^\perp$ and $\sigma_2 = B^\perp$ be co-oriented hyperplanes in $\mathcal{P}(n)$ and suppose that the matrix pencil $(A, B)$ is regular.*

*(1) Suppose that the set of generalized eigenvalues of $(A, B)$ contains nonreal numbers, denoted by $\lambda_1, \ldots, \lambda_k$ and $\lambda_1^*, \ldots, \lambda_k^*$. The following serves as an invariant angle function:*

$$\theta(\sigma_1, \sigma_2) = \frac{1}{k} \sum_{i=1}^{k} |\arg(\lambda_i)| \,. \tag{3.1}$$

*(2) Suppose that all distinct generalized eigenvalues of $(A, B)$ are real or infinity, ordered as $\lambda_k > \cdots > \lambda_1$, where $k \geq 3$. The following serves as an invariant angle function (which is the limit as $\lambda_k \to \infty$ if $\infty$ is a generalized eigenvalue):*

$$\theta(\sigma_1, \sigma_2) = \arccos \frac{\sum_{i=1}^{k} \frac{\lambda_{i+1}+\lambda_i}{\lambda_{i+1}-\lambda_i}}{\sqrt{\left(\sum_{i=1}^{k} \frac{1}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^{k} \frac{(\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)}}. \tag{3.2}$$

*(3) If $(A, B)$ has at most 2 distinct generalized eigenvalues and all of these are real, then $(\sigma_1, \sigma_2)$ is not in the domain of any invariant angle function.*

REMARK 3.1.1. *We will describe the motivation of the formula (3.2) in Appendix A.*

We say that a given pair of co-oriented hyperplanes $(\sigma_1, \sigma_2) = (A^\perp, B^\perp)$ in $\mathcal{P}(n)$ is of type (1), (2), or (3) if the set of generalized eigenvalues of $(A, B)$ corresponds to case (1), (2), or (3) in Theorem 3.1.2, respectively. The following fact is a direct consequence of the properties of matrix pencils:

PROPOSITION 3.1.2. *(1) For any $g \in SL(n, \mathbb{R})$ the hyperplane pairs $(\sigma_1, \sigma_2)$ and $(g.\sigma_1, g.\sigma_2)$ share the same type.*

*(2) If $\sigma_3$ lies between $\sigma_1$ and $\sigma_2$, both $(\sigma_1, \sigma_3)$ and $(\sigma_2, \sigma_3)$ belong to the same type as $(\sigma_1, \sigma_2)$.*

PROOF. Claim (1) is clear. For claim (2), suppose that $A$ and $B$ are the normal vectors associated with $\sigma_1$ and $\sigma_2$, respectively. After positive rescalings of $A$ and $B$, we assume that the normal vector associated with $\sigma_3$ is $C = A + B$. Suppose that the generalized eigenvalues of the pencil $(A, B)$ are $\lambda_i$, where $i = 1, \ldots, n$. Lemma 2.3.3.1 implies that the generalized eigenvalues of $(A, C)$ are $(1 + \lambda_i)$ and the generalized eigenvalues of $(C, B)$ are $\frac{\lambda_i}{1+\lambda_i}$, $i = 1, \ldots, n$. Note that $\lambda$ is real if and only if $\frac{\lambda}{1+\lambda}$ is real, which implies claim (2) of the proposition. $\square$

Following Proposition 3.1.2, we will prove the three statements in Theorem 3.1.2 individually in the subsequent sections.

### 3.2. Proof of Theorem 3.1.2, case (1)

We will prove that the function

$$\theta(\sigma_1, \sigma_2) = \frac{1}{k} \sum_{i=1}^{k} |\arg(\lambda_i)|$$

defined for all pairs $(\sigma_1, \sigma_2)$ of type (1) satisfies the properties listed in Definition 3.1.1.

Proof of Theorem 3.1.2, case (1). To begin, we establish the well-definedness of the function for pairs of co-oriented hyperplanes in (3.1). According to Lemma 2.3.3.1, $\frac{c_2}{c_1}\lambda_i$ and $\frac{c_2}{c_1}\lambda_i^*$ will be the nonreal generalized eigenvalues of $(c_1 A, c_2 B)$, for any $c_1, c_2 > 0$, where $i = 1, \ldots, k$. The arguments of these numbers equal the arguments of $\lambda_i$ and $\lambda_i^*$, respectively. Consequently, (3.1) implies that $\theta((c_1 A)^\perp, (c_2 B)^\perp) = \theta(A^\perp, B^\perp)$ for any $c_1, c_2 > 0$, i.e., the expression (3.1) remains the same for $(c_1 A, c_2 B)$.

Furthermore, since $|\arg(\lambda_i^*)| = |-\arg(\lambda_i)| = |\arg(\lambda_i)|$, the outcome of (3.1) remains unchanged when replacing $\lambda_i$ with $\lambda_i^*$.

Next, we will verify properties (1) to (4) in Definition 3.1.1 for the function $\theta$ defined by (3.1). The property (1) is obvious. Regarding the property (2), we notice that $(g^{-1})^{\mathrm{T}}.A$ and $(g^{-1})^{\mathrm{T}}.B$ serve as normal vectors of $g.\sigma_1$ and $g.\sigma_2$, respectively. Since the pencil $((g^{-1})^{\mathrm{T}}.A, (g^{-1})^{\mathrm{T}}.B)$ shares the same generalized eigenvalues as $(A, B)$, the angle $\theta(g.\sigma_1, g.\sigma_2) = \theta(\sigma_1, \sigma_2)$.

To verify the property (3), notice that the pencil $(B, A)$ possesses generalized eigenvalues $\lambda_i^{-1}$ and $\lambda_i^{-1*}$, where $i = 1, \ldots, k$. Since $\arg(\lambda_i^{-1}) = -\arg(\lambda_i)$, it follows $\theta(\sigma_2, \sigma_1) = \theta(\sigma_1, \sigma_2)$. Furthermore, the generalized eigenvalues of the pencil $(-A, B)$ equal $-\lambda_i$ and $-\lambda_i^*$, where $i = 1, \ldots, k$. Since $|\arg(-\lambda_i)| = \pi - |\arg(\lambda_i)|$, we deduce that $\theta(-\sigma_1, \sigma_2) = \pi - \theta(\sigma_1, \sigma_2)$.

Lastly, we verify the property (4). The normal vector $C$ of $\sigma_3$ is a positive linear combination of $A$ and $B$. Since positive rescalings of $A$ and $B$ preserve the angles $\theta(A^\perp, C^\perp)$, $\theta(C^\perp, A^\perp)$ and $\theta(A^\perp, B^\perp)$, we assume that $C = A + B$. Under this condition, Lemma 2.3.3.1 shows that the

nonreal generalized eigenvalues of $(A, C)$ are $(1 + \lambda_i)$ and $(1 + \lambda_i^*)$, while the nonreal generalized eigenvalues of $(C, B)$ are $\frac{\lambda_i}{1+\lambda_i}$ and $\frac{\lambda_i^*}{1+\lambda_i^*}$, where $i = 1, \dots, k$.
We note that $\arg\left(\frac{\lambda}{1+\lambda}\right) > 0$ if and only if $\arg(\lambda) > 0$. Thus,

$$\theta(\sigma_1, \sigma_3) + \theta(\sigma_2, \sigma_3) = \frac{1}{k} \sum \left( |\arg(1 + \lambda_i)| + \left|\arg\left(\frac{\lambda_i}{1 + \lambda_i}\right)\right| \right)$$
$$= \frac{1}{k} \sum \left( \left|\arg(1 + \lambda_i) + \arg\left(\frac{\lambda_i}{1 + \lambda_i}\right)\right| \right) = \frac{1}{k} \sum (|\arg(\lambda_i)|) = \theta(\sigma_1, \sigma_2).$$

This concludes the verification of the property (4) in Definition 3.1.1. In summary, the function $\theta$ defined by (3.1) serves as an invariant angle function. □

### 3.3. Proof of Theorem 3.1.2, case (2)

We will prove that the function

$$\theta(\sigma_1, \sigma_2) = \arccos \frac{\sum_{i=1}^{k} \frac{\lambda_{i+1}+\lambda_i}{\lambda_{i+1}-\lambda_i}}{\sqrt{\left(\sum_{i=1}^{k} \frac{1}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^{k} \frac{(\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)}}.$$

defined for all pairs $(\sigma_1, \sigma_2)$ of type (2) satisfies the properties listed in Definition 3.1.1. For simplicity, we define

$$t(x_1, \dots, x_k) = \frac{\sum_{i=1}^{k} \frac{x_{i+1}+x_i}{x_{i+1}-x_i}}{\sqrt{\left(\sum_{i=1}^{k} \frac{1}{x_{i+1}-x_i}\right)\left(\sum_{i=1}^{k} \frac{(x_{i+1}+x_i)^2}{x_{i+1}-x_i}\right)}},$$

and $\bar{t}(x_1, \dots, x_k) = t(x_{\sigma_k}, \dots, x_{\sigma_1})$, where $\{\sigma_1, \dots, \sigma_k\}$ represents the permutation of $\{1, \dots, k\}$ such that $x_{\sigma_k} \geq \cdots \geq x_{\sigma_1}$. Our first lemma concerns the compositions of $\bar{t}$ and Möbius transformations:

Lemma 3.3.0.1. *Let $\varphi$ be a Möbius transformation on $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, and let $\lambda_k > \cdots > \lambda_1$ represent real numbers. If $\varphi$ is orientation-preserving, then*

$$\bar{t}(\varphi(\lambda_1), \dots, \varphi(\lambda_k)) = t(\varphi(\lambda_1), \dots, \varphi(\lambda_k)). \tag{3.3}$$

*If $\varphi$ is orientation-reversing, then*

$$\bar{t}(\varphi(\lambda_1), \dots, \varphi(\lambda_k)) = -t(\varphi(\lambda_1), \dots, \varphi(\lambda_k)). \tag{3.4}$$

Proof. Let $(\sigma_1, \ldots, \sigma_k)$ denote the permutation of $(1, \ldots, k)$ such that

$$\varphi(\lambda_{\sigma_k}) > \cdots > \varphi(\lambda_{\sigma_1}).$$

If $\varphi$ is orientation-preserving, then $(\sigma_1, \ldots, \sigma_k)$ is a cyclic permutation, satisfying $t(x_1, \ldots, x_k) = t(x_{\sigma_1}, \ldots, x_{\sigma_k})$ for any $x_1, \ldots, x_k$. Hence, equation (3.3) holds.

If $\varphi$ is orientation-reversing, then $(\sigma_1, \ldots, \sigma_k)$ is a cyclic permutation of $(k, \ldots, 1)$. Note that $t(x_k, \ldots, x_1) = -t(x_1, \ldots, x_k)$ for any $x_1, \ldots, x_k$. Therefore, equation (3.4) holds. $\square$

We also need the following lemma:

Lemma 3.3.0.2. *Let $\lambda_k > \cdots > \lambda_1$ be real numbers, then the following inequalities hold:*

$$\sum\nolimits_{1 \le i \ne j \le k} \frac{(\lambda_{i+1} + \lambda_i - \lambda_{j+1} - \lambda_j)^2}{(\lambda_{i+1} - \lambda_i)(\lambda_{j+1} - \lambda_j)} > 0, \tag{3.5a}$$

$$\sum\nolimits_{i=1}^{k} \frac{(2 + \lambda_{i+1} + \lambda_i)^2}{\lambda_{i+1} - \lambda_i} > 0. \tag{3.5b}$$

Proof. Denote

$$s_i = \lambda_{i+1} + \lambda_i, \ d_i = \lambda_{i+1} - \lambda_i,$$

where the index is taken modulo $k$. Then the numbers $s_i$, $i = 1, \ldots, k$ satisfy the following inequalities:

$$s_1 < s_2 < \cdots < s_{k-1}, \quad s_1 < s_k < s_{k-1},$$

and the numbers $d_i$, $i = 1, \ldots, k$ satisfy

$$d_1, \ldots, d_{k-1} > 0, \quad d_k = -\sum_{i=1}^{k-1} d_i < 0.$$

In terms of $s_i$ and $d_i$, inequalities (3.5a) and (3.5b) reduce to

$$\sum_{1 \le i \ne j \le k} \frac{(s_i - s_j)^2}{d_i d_j} > 0, \quad \sum_{i=1}^{k} \frac{(2 + s_i)^2}{d_i} > 0.$$

Assume that $j$ is the number satisfying $s_j \le s_k \le s_{j+1}$, $1 \le j \le k-2$, then

$$\begin{aligned}
&\frac{(s_1 - s_{k-1})^2}{d_1 d_{k-1}} > \frac{(s_1 - s_k)^2 + (s_k - s_{k-1})^2}{d_1 d_{k-1}} > -\frac{(s_1 - s_k)^2}{d_1 d_k} - \frac{(s_{k-1} - s_k)^2}{d_{k-1} d_k},\\
&\frac{(s_i - s_1)^2}{d_1 d_i} > -\frac{(s_i - s_k)^2}{d_k d_i}, \quad \forall\ j+1 \le i < k-1,\\
&\frac{(s_i - s_{k-1})^2}{d_{k-1} d_i} > -\frac{(s_i - s_k)^2}{d_k d_i}, \quad \forall\ 1 < i \le j.
\end{aligned}$$

These inequalities yield (3.5a).

We divide the proof of inequality (3.5b) in two cases. If $s_k + 2 \le 0$, then

$$-\frac{(2+s_k)^2}{d_k} < \frac{(2+s_k)^2}{d_1} < \frac{(2+s_1)^2}{d_1}.$$

If $s_k + 2 \ge 0$, then

$$-\frac{(2+s_k)^2}{d_k} < \frac{(2+s_k)^2}{d_{k-1}} < \frac{(2+s_{k-1})^2}{d_{k-1}}.$$

For both cases, inequality (3.5b) holds. $\square$

We return to the proof of Theorem 3.1.2.

Proof of Theorem 3.1.2, case (2). Firstly, we show that (3.2) always yields real values. That is,

$$-1 \le \frac{\sum_{i=1}^{k} \frac{\lambda_{i+1}+\lambda_i}{\lambda_{i+1}-\lambda_i}}{\sqrt{\left(\sum_{i=1}^{k} \frac{1}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^{k} \frac{(\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)}} \le 1,$$

for any real numbers $\lambda_k > \cdots > \lambda_1$. By the Cauchy-Binet identity,

$$\left(\sum_{i=1}^{k} \frac{1}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^{k} \frac{(\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right) - \left(\sum_{i=1}^{k} \frac{\lambda_{i+1}+\lambda_i}{\lambda_{i+1}-\lambda_i}\right)^2 = \frac{1}{2}\sum_{i\ne j} \frac{(\lambda_{i+1}+\lambda_i-\lambda_{j+1}-\lambda_j)^2}{(\lambda_{i+1}-\lambda_i)(\lambda_{j+1}-\lambda_j)}.$$

Lemma 3.3.0.2 implies that the right-hand side is positive.

Next, we will prove properties (1) to (4) in Definition 3.1.1. Properties (1) and (2) are proved similarly to the corresponding arguments in Section 3.2. To show that the property (3) holds, notice that the generalized eigenvalues of $(B, A)$ are $\lambda_i^{-1}$, which are values of an orientation-reversing

Möbius transformation of $\lambda_i$, $i = 1, \dots, k$. By Lemma 3.3.0.1,

$$\cos\theta(\sigma_2, \sigma_1) = \bar{t}(\lambda_1^{-1}, \dots, \lambda_k^{-1}) = -\frac{\sum_{i=1}^k \frac{\lambda_{i+1}^{-1}+\lambda_i^{-1}}{\lambda_{i+1}^{-1}-\lambda_i^{-1}}}{\sqrt{\left(\sum_{i=1}^k \frac{1}{\lambda_{i+1}^{-1}-\lambda_i^{-1}}\right)\left(\sum_{i=1}^k \frac{(\lambda_{i+1}^{-1}+\lambda_i^{-1})^2}{\lambda_{i+1}^{-1}-\lambda_i^{-1}}\right)}}.$$

Since

$$\sum_{i=1}^k \frac{1}{\lambda_i^{-1}-\lambda_{i+1}^{-1}} = \sum_{i=1}^k \frac{\lambda_i\lambda_{i+1}}{\lambda_{i+1}-\lambda_i} = \sum_{i=1}^k \left(\frac{\lambda_{i+1}-\lambda_i}{4} + \frac{\lambda_i\lambda_{i+1}}{\lambda_{i+1}-\lambda_i}\right) = \sum_{i=1}^k \frac{(\lambda_{i+1}+\lambda_i)^2/4}{\lambda_{i+1}-\lambda_i},$$

$$\sum_{i=1}^k \frac{(\lambda_i^{-1}+\lambda_{i+1}^{-1})^2}{\lambda_i^{-1}-\lambda_{i+1}^{-1}} = \sum_{i=1}^k \left((\lambda_{i+1}^{-1}-\lambda_i^{-1}) + \frac{4}{\lambda_{i+1}-\lambda_i}\right) = \sum_{i=1}^k \frac{4}{\lambda_{i+1}-\lambda_i},$$

$$\sum_{i=1}^k \frac{\lambda_i^{-1}+\lambda_{i+1}^{-1}}{\lambda_i^{-1}-\lambda_{i+1}^{-1}} = \sum_{i=1}^k \frac{\lambda_{i+1}+\lambda_i}{\lambda_{i+1}-\lambda_i},$$

we have $\bar{t}(\lambda_1^{-1}, \dots, \lambda_k^{-1}) = \bar{t}(\lambda_1, \dots, \lambda_k)$, implying that $\theta(\sigma_2, \sigma_1) = \theta(\sigma_1, \sigma_2)$. Moreover, the generalized eigenvalues of $(-A, B)$ are $-\lambda_1 > \dots > -\lambda_k$. Therefore,

$$\cos\theta(-\sigma_1, \sigma_2) = \bar{t}(-\lambda_1, \dots, -\lambda_k) = -\frac{\sum_{i=1}^k \frac{\lambda_{i+1}+\lambda_i}{\lambda_{i+1}-\lambda_i}}{\sqrt{\left(\sum_{i=1}^k \frac{1}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^k \frac{(\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)}},$$

which implies that $\bar{t}(-\lambda_1, \dots, -\lambda_k) = -\bar{t}(\lambda_1, \dots, \lambda_k)$, i.e., $\theta(-\sigma_1, \sigma_2) = \pi - \theta(\sigma_1, \sigma_2)$.

Finally, we will prove the property (4). If we denote $\theta = \theta(\sigma_1, \sigma_2)$, $\theta_1 = \theta(\sigma_1, \sigma_3)$ and $\theta_2 = \theta(\sigma_3, \sigma_2)$, the property (4) reduces to

$$(*) \qquad \cos(\theta) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2).$$

Similarly to the proof given in Section 3.2, we assume that $\sigma_3 = (A+B)^\perp$ without loss of generality. The generalized eigenvalues of $(A, A+B)$ are $(1+\lambda_i)$ and $(1+\lambda_i^*)$, and the generalized eigenvalues of $(A+B, B)$ are $\frac{\lambda_i}{1+\lambda_i}$ and $\frac{\lambda_i^*}{1+\lambda_i^*}$, where $i = 1, \dots, k$. Both sets of generalized eigenvalues are orientation-preserving Möbius transformations of $\lambda_i$ and $\lambda_i^*$, $i = 1, \dots, k$. Lemma 3.3.0.1 implies that

$$\cos(\theta_1) = \frac{\sum_{i=1}^k \frac{2+\lambda_{i+1}+\lambda_i}{\lambda_{i+1}-\lambda_i}}{\sqrt{\left(\sum_{i=1}^k \frac{1}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^k \frac{(2+\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)}},$$

and

$$\cos(\theta_2) = \frac{\sum_{i=1}^k \frac{\lambda_{i+1}+\lambda_i+2\lambda_i\lambda_{i+1}}{\lambda_{i+1}-\lambda_i}}{\sqrt{\left(\sum_{i=1}^k \frac{(1+\lambda_i)(1+\lambda_{i+1})}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^k \frac{(\lambda_{i+1}+\lambda_i+2\lambda_i\lambda_{i+1})^2}{(1+\lambda_i)(1+\lambda_{i+1})(\lambda_{i+1}-\lambda_i)}\right)}}$$

$$= \frac{\sum_{i=1}^k \frac{(2+\lambda_i+\lambda_{i+1})(\lambda_{i+1}+\lambda_i)}{2(\lambda_{i+1}-\lambda_i)} + \frac{\lambda_i-\lambda_{i+1}}{2}}{\sqrt{\left(\sum_{i=1}^k \frac{(2+\lambda_i+\lambda_{i+1})^2}{4(\lambda_{i+1}-\lambda_i)} + \frac{\lambda_i-\lambda_{i+1}}{4}\right)\left(\sum_{i=1}^k \frac{(\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i} + \frac{\lambda_i^2}{1+\lambda_i} - \frac{\lambda_{i+1}^2}{1+\lambda_{i+1}}\right)}}$$

$$= \frac{\sum_{i=1}^k \frac{(2+\lambda_i+\lambda_{i+1})(\lambda_{i+1}+\lambda_i)}{\lambda_{i+1}-\lambda_i}}{\sqrt{\left(\sum_{i=1}^k \frac{(2+\lambda_i+\lambda_{i+1})^2}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^k \frac{(\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)}}.$$

By applying the Cauchy-Binet identity, we have

$$\sin(\theta_1) = \frac{\sqrt{\left(\sum_{i=1}^k \frac{1}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^k \frac{(2+\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right) - \left(\sum_{i=1}^k \frac{2+\lambda_{i+1}+\lambda_i}{\lambda_{i+1}-\lambda_i}\right)^2}}{\sqrt{\left(\sum_{i=1}^k \frac{1}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^k \frac{(2+\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)}}$$

$$= \frac{\sqrt{\frac{1}{2}\sum_{i\neq j} \frac{(\lambda_{i+1}+\lambda_i-\lambda_{j+1}-\lambda_j)^2}{(\lambda_{i+1}-\lambda_i)(\lambda_{j+1}-\lambda_j)}}}{\sqrt{\left(\sum_{i=1}^k \frac{1}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^k \frac{(2+\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)}},$$

and

$$\sin(\theta_2) = \frac{\sqrt{\left(\sum_{i=1}^k \frac{(2+\lambda_i+\lambda_{i+1})^2}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^k \frac{(\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right) - \left(\sum_{i=1}^k \frac{(2+\lambda_i+\lambda_{i+1})(\lambda_{i+1}+\lambda_i)}{\lambda_{i+1}-\lambda_i}\right)^2}}{\sqrt{\left(\sum_{i=1}^k \frac{(2+\lambda_i+\lambda_{i+1})^2}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^k \frac{(\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)}}$$

$$= \frac{\sqrt{\frac{1}{2}\sum_{i\neq j} \frac{4(\lambda_{i+1}+\lambda_i-\lambda_{j+1}-\lambda_j)^2}{(\lambda_{i+1}-\lambda_i)(\lambda_{j+1}-\lambda_j)}}}{\sqrt{\left(\sum_{i=1}^k \frac{(2+\lambda_i+\lambda_{i+1})^2}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^k \frac{(\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)}}.$$

Inequalities (3.5a) and (3.5b) imply that

$$\sin(\theta_1)\sin(\theta_2) = \frac{\frac{1}{2}\left|\sum_{i\neq j} \frac{2(\lambda_{i+1}+\lambda_i-\lambda_{j+1}-\lambda_j)^2}{(\lambda_{i+1}-\lambda_i)(\lambda_{j+1}-\lambda_j)}\right|}{\left|\sum_{i=1}^k \frac{(2+\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right|\sqrt{\left(\sum_{i=1}^k \frac{1}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^k \frac{(\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)}}$$

$$= \frac{\frac{1}{2}\sum_{i\neq j} \frac{2(\lambda_{i+1}+\lambda_i-\lambda_{j+1}-\lambda_j)^2}{(\lambda_{i+1}-\lambda_i)(\lambda_{j+1}-\lambda_j)}}{\left(\sum_{i=1}^k \frac{(2+\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)\sqrt{\left(\sum_{i=1}^k \frac{1}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^k \frac{(\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)}}.$$

By combining the equations above and using the Cauchy-Binet identity again, we have

$$
\begin{aligned}
&\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)\\
=&\frac{\left(\sum_{i=1}^k \frac{2+\lambda_{i+1}+\lambda_i}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^k \frac{(2+\lambda_i+\lambda_{i+1})(\lambda_{i+1}+\lambda_i)}{\lambda_{i+1}-\lambda_i}\right) - \frac{1}{2}\sum_{i\neq j}\frac{2(\lambda_{i+1}+\lambda_i-\lambda_{j+1}-\lambda_j)^2}{(\lambda_{i+1}-\lambda_i)(\lambda_{j+1}-\lambda_j)}}{\left(\sum_{i=1}^k \frac{(2+\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)\sqrt{\left(\sum_{i=1}^k \frac{1}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^k \frac{(\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)}}\\
=&\frac{\left(\sum_{i=1}^n \frac{\lambda_{i+1}+\lambda_i}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^n \frac{(2+\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)}{\left(\sum_{i=1}^k \frac{(2+\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)\sqrt{\left(\sum_{i=1}^k \frac{1}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^k \frac{(\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)}} = \cos(\theta).
\end{aligned}
$$

This proves the property (4) in Definition 3.1.1. In conclusion, the function $\theta$ given by (3.2) is an invariant angle function. □

### 3.4. Proof of Theorem 3.1.2, case (3)

To prove the statement (3) in Theorem 3.1.2, we begin by establishing the following lemma:

LEMMA 3.4.0.1. *Let $K_l = \sum_{s+t=r+l} \mathbf{e}_s \otimes \mathbf{e}_t \in Mat_r(\mathbb{R})$, $l = 1, \ldots, r$, and define*

$$X = \sum_{l=1}^{r} x_l K_l, \quad \tilde{X} = \sum_{l=1}^{r-1} x_l K_{l+1}.$$

*Then for any $s > 0$ and $t \in \mathbb{R}$, there exists an element $g \in GL^+(n, \mathbb{R})$ satisfying the conditions:*

$$g.\tilde{X} = \tilde{X}, \tag{3.6}$$

$$g.X = sX + t\tilde{X}. \tag{3.7}$$

PROOF. We claim that there exists a matrix $g$ of the form

$$g = \sum_{l \leq j} s^{r/2-j+1} p_l^{(j-l)} \mathbf{e}_l \otimes \mathbf{e}_j \tag{3.8}$$

that satisfies equations (3.6) and (3.7).

First, we note that the entries above the anti-diagonal on both sides of both equations vanish for $g$ that follows equation (3.8). The entries on the anti-diagonal of both sides of (3.6) vanish, as well.

Next, we will prove by induction on $k$ that there exist numbers $p_l^{(k)} \in \mathbb{R}$, where $l = 1, \dots, r-k$, such that all entries under the anti-diagonal of both sides of (3.6) are equal, and those under or on the anti-diagonal of both sides of (3.7) are equal.

We start with the base case $k = 0$. If we set $p_l^{(0)} = 1$ for $l = 1, \dots, r$, then the $(l+1, r+2-l)$ entries of both sides of (3.6) are equal to $x_1$, and the $(l+1, r+1-l)$ entries of both sides of (3.7) are equal to $sx_1$, where $l = 1, \dots, r-1$. For the entries above $(l+1, r+2-l)$ of (3.6) and above $(l+1, r+1-l)$ of (3.7), their expressions involve $p_l^{(1)}$, which will be determined in the $k = 1$ case of the induction. Thus, we do not need to discuss these entries in the $k = 0$ case.

We proceed to the general case $k > 0$. Assume that the solutions $p_l^{(k')}$ are determined for $0 \le k' < k$. For $l = 2, \dots, r-k$, we compare the $(l+k, r+2-l)$ entries of both sides of (3.6). Equality of both sides yields $(r-k-1)$ equations in unknowns $p_1^{(k)}, \dots, p_{r-k}^{(k)}$:

$$s^{-k}\left(x_{k+1} + x_1(p_l^{(k)} + p_{r-k+2-l}^{(k)}) + R_l^{(k)}\right) = x_{k+1}, \tag{3.9}$$

where $R_l^{(k)}$ is a polynomial in terms of $x_1, \dots, x_k$ and $p_j^{(1)}, \dots, p_j^{(k-1)}$. Since $P^{\mathrm{T}}\tilde{X}P$ is symmetric, $R_l^{(k)} = R_{r-k+2-l}^{(k)}$, which implies that the $l$-th equation coincides with the $(r-k+2-l)$-th equation. Thus, the number of distinct equations reduces to $\lfloor \frac{r-k}{2} \rfloor$.

For $l = 1, \dots, r-k$, we compare the $(l+k, r+1-l)$ entries of both sides of (3.7). Equality of both sides yields $(r-k)$ equations in unknowns $p_1^{(k)}, \dots, p_{r-k}^{(k)}$:

$$s^{1-k}\left(x_{k+1} + x_1(p_l^{(k)} + p_{r-k+1-l}^{(k)}) + Q_l^{(k)}\right) = sx_{k+1} + tx_k, \tag{3.10}$$

where $Q_l^{(k)}$ is a polynomial in terms of $x_1, \dots, x_k$ and $p_j^{(1)}, \dots, p_j^{(k-1)}$. Since $P^{\mathrm{T}}XP$ is symmetric, $Q_l^{(k)} = Q_{r-k+1-l}^{(k)}$, which implies that the $l$-th equation coincides with the $(r-k+1-l)$-th equation. Thus, the number of distinct equations reduces to $\lfloor \frac{r-k+1}{2} \rfloor$.

By combining equations (3.9) and (3.10) together, we derive a linear equation system consisting of $\lfloor \frac{r-k+1}{2} \rfloor + \lfloor \frac{r-k}{2} \rfloor = (r-k)$ equations in unknowns $p_1^{(k)}, \dots, p_{r-k}^{(k)}$:

$$\begin{aligned}
p_1^{(k)} + p_{r-k}^{(k)} &= x_1^{-1}(s^k x_{k+1} + ts^{k-1}x_k - x_{k+1} - Q_1^{(k)}) := Q_1'^{(k)},\\
p_2^{(k)} + p_{r-k}^{(k)} &= x_1^{-1}(s^k x_{k+1} - x_{k+1} - R_2^{(k)}) := R_2'^{(k)},\\
p_2^{(k)} + p_{r-k-1}^{(k)} &= x_1^{-1}(s^k x_{k+1} + ts^{k-1}x_k - x_{k+1} - Q_2^{(k)}) := Q_2'^{(k)} \dots
\end{aligned}$$

If $(r-k)$ is even, the last equation is $2p^{(k)}_{\frac{r-k}{2}+1} = R'^{(k)}_{\frac{r-k}{2}+1}$; if $(r-k)$ is odd, the last equation is $2p^{(k)}_{\frac{r-k+1}{2}} = Q'^{(k)}_{\frac{r-k+1}{2}}$.

If we arrange the $(r-k)$ unknowns as $p^{(k)}_1, p^{(k)}_{r-k}, p^{(k)}_2, \ldots, p^{(k)}_{\lfloor \frac{r-k}{2}+1 \rfloor}$, the coefficient matrix for this linear equation system is an invertible Jordan matrix $J_{1,r-k}$. Thus, a unique solution $p^{(k)}_1, \ldots, p^{(k)}_{r-k}$ for (3.9) and (3.10) exists, dependent on $s$, $t$, $x_1, \ldots, x_{k+1}$ and $p^{(k')}_j$, where $1 \leq j \leq r-k'$ and $k' < k$.

By induction, a solution set $p^{(k)}_l$ for (3.9) and (3.10) exists in terms of $x_1, \ldots, x_r$, $s$, and $t$, where $k = 1, \ldots, r-1$ and $l = 1, \ldots, r-k$. That is to say, there exists a matrix $g = \sum_{i \leq j} s^{r/2-j+1} p^{(j-l)}_l \mathbf{e}_l \otimes \mathbf{e}_j$ that satisfies (3.6) and (3.7). $\square$

Lemma 3.4.0.1 implies the following:

Lemma 3.4.0.2. *(1) Suppose that $(A, B)$ is a regular pencil of symmetric $n \times n$ matrices with only one distinct eigenvalue $\lambda \in \mathbb{R}$, and let $C = A - \lambda B$. Then, for any $s > 0$ and $t \in \mathbb{R}$, there is an element $g \in GL^+(n, \mathbb{R})$ such that:*

$$g.C = C, \quad g.B = sB + tC.$$

*(2) Suppose that $(A, B)$ is a regular pencil of symmetric $n \times n$ matrices with only two distinct eigenvalues $\lambda, \lambda' \in \mathbb{R}$, and let $C = A - \lambda B$, $C' = A - \lambda' B$. Then for any $s, s' > 0$, there is an element $g \in GL^+(n, \mathbb{R})$ such that:*

$$g.C = sC, \quad g.C' = s'C'.$$

Proof. (1) Suppose that the pencil $(A, B)$ has one distinct eigenvalue $\lambda \in \mathbb{R}$. According to Lemma 2.3.3.1, we may assume that the matrix pencil is in the normal form:

$$A = \operatorname{diag}(A_1, \ldots, A_k),$$

and

$$B = \operatorname{diag}(B_1, \ldots, B_k),$$

where $A_j = B_j J_{\lambda, r_j}$. Here and after, $J_{\lambda,r}$ denotes the Jordan block matrix of dimension $r$ and eigenvalue $\lambda$. Thus,

$$C = A - \lambda B = \operatorname{diag}(\tilde{B}_1, \ldots, \tilde{B}_k),$$

where $\tilde{B}_j = B_j J_{0,r_j}$.

According to Lemma 3.4.0.1, for any $s > 0$ and $t \in \mathbb{R}$, there exist elements $g_j \in GL^+(r_j, \mathbb{R})$ such that:

$$g_j.\tilde{B}_j = \tilde{B}_j, \quad g_j.B_j = sB_j + t\tilde{B}_j.$$

Let $g = \mathrm{diag}(g_1, \ldots, g_k) \in GL^+(n, \mathbb{R})$, then $g.C = C$ and $g.B = sB + tC$.

(2) Suppose that the pencil $(A, B)$ has exactly two distinct eigenvalues $\lambda, \lambda' \in \mathbb{R}$. We may assume that the matrix pencil is in the normal form:

$$A = \mathrm{diag}(A_1, \ldots, A_k, A'_1, \ldots, A'_l),$$

and

$$B = \mathrm{diag}(B_1, \ldots, B_k, B'_1, \ldots, B'_l),$$

where $A_j = B_j J_{\lambda,r_j}$ and $A'_j = B'_j J_{\lambda',r'_j}$. Thus,

$$C = \mathrm{diag}(\tilde{B}_1, \ldots, \tilde{B}_k, \tilde{B}'_1 + (\lambda' - \lambda)B'_1, \ldots, \tilde{B}'_l + (\lambda' - \lambda)B'_l)$$

and

$$C' = \mathrm{diag}(\tilde{B}_1 + (\lambda - \lambda')B_1, \ldots, \tilde{B}_k + (\lambda - \lambda')B_k, \tilde{B}'_1, \ldots, \tilde{B}'_l),$$

where $\tilde{B}_j = B_j J_{0,r_j}$, and $\tilde{B}'_j = B'_j J_{0,r'_j}$. According to Lemma 3.4.0.1, for any $s, t > 0$, there exist matrices $g_j$, $j = 1, \ldots, k$, such that

$$g_j.\tilde{B}_j = \tilde{B}_j, \quad g_j.B_j = \frac{t}{s}B_j + \frac{t-s}{s(\lambda - \lambda')}\tilde{B}_j,$$

and matrices $g'_j$, $j = 1, \ldots, l$, such that

$$g'_j.\tilde{B}'_j = \tilde{B}'_j, \quad g_j.B'_j = \frac{s}{t}B'_j + \frac{s-t}{t(\lambda' - \lambda)}\tilde{B}'_j.$$

Let $g = \mathrm{diag}(\sqrt{s}g_1, \ldots, \sqrt{s}g_k, \sqrt{t}g'_1, \ldots, \sqrt{t}g'_l) \in GL^+(n, \mathbb{R})$, then $g.C = sC$ and $g.C' = tC'$. $\square$

Lastly, we come back to the proof of the main theorem.

Proof of Theorem 3.1.2, case (3). Recall that a positive scaling of the normal vector does not change the associated co-oriented hyperplane. Therefore, we can replace the part "$g \in$

$SL(n,\mathbb{R})$" in Definition 3.1.1 with "$g \in GL^+(n,\mathbb{R})$". We will prove the statement (3) of the theorem by exhausting all possible cases.

(1). Suppose that $(A, B)$ has only one eigenvalue $\lambda \in \mathbb{R}$, and $\lambda > 0$. By Lemma 3.4.0.2, there is an element $g \in GL^+(n,\mathbb{R})$, such that $g.C = C$ and $g.B = \lambda B + C = A$. For $k \in \mathbb{Z}$, denote $A_k = g^{1-k}.A$ and $\sigma_k = A_k^{\perp}$. Then $A_1 = A$, $A_2 = B$, and

$$\theta(\sigma_k, \sigma_{k+1}) = \theta(A_k^{\perp}, A_{k+1}^{\perp}) = \theta((g^{1-k}.A)^{\perp}, (g^{1-k}.B)^{\perp}) = \theta(A^{\perp}, B^{\perp}) = \theta(\sigma_1, \sigma_2).$$

The relations $g.C = C$ and $g.B = \lambda B + C = A$ imply that $g.(A - \lambda B) = (A - \lambda B)$ and $g.(A - B) = \lambda(A - B)$. Thus,

$$(A_k - \lambda A_{k+1}) = g^{1-k}.(A - \lambda B) = A - \lambda B, \quad (A_k - A_{k+1}) = g^{1-k}.(A - B) = \lambda^{1-k}(A - B),$$

and

$$(\lambda^k - 1)A_k = (\lambda^k - \lambda)A_{k+1} + (\lambda - 1)A_1,$$

for all $k > 1$. Therefore, $A_k$ is a positive linear combination of $A_1$ and $A_{k+1}$, i.e., $\theta_k$ lies between $\theta_1$ and $\theta_{k+1}$. The property (4) of invariant functions implies that

$$\theta(\sigma_1, \sigma_m) = \sum_{k=1}^{m-1} \theta(\sigma_k, \sigma_{k+1}) = (m-1)\theta(\sigma_1, \sigma_2)$$

for any $m > 1$. However, for $m$ large enough, $\theta(\sigma_1, \sigma_m) = (m-1)\theta(\sigma_1, \sigma_2) > \pi$, a contradiction with the property (1) of invariant angle functions.

(2). Suppose that $(A, B)$ has only has only one eigenvalue $\lambda \in \mathbb{R}$, and $\lambda < 0$. The previous case implies that $\theta((-A)^{\perp}, B^{\perp})$ does not exist. If $\theta(A^{\perp}, B^{\perp})$ exists, then $\theta((-A)^{\perp}, B^{\perp}) = \pi - \theta(A^{\perp}, B^{\perp})$, a contradiction.

(3). Suppose that $(A, B)$ has only one eigenvalue $\lambda$ and $\lambda = 0$. In this case, $C = A$. There exists an element $g \in GL^+(n,\mathbb{R})$ such that

$$g.A = A, \quad g.B = B + A.$$

Thus

$$\theta(A^{\perp}, B^{\perp}) = \theta(A^{\perp}, (A+B)^{\perp}) + \theta(B^{\perp}, (A+B)^{\perp}) > \theta(A^{\perp}, (A+B)^{\perp}) = \theta((g.A)^{\perp}, (g.B)^{\perp}),$$

a contradiction with property (2) of invariant angles.

(4). Suppose that the pencil $(A, B)$ has exactly two distinct eigenvalues $\lambda$ and $\lambda'$, both $A$ and $B$ are positive linear combinations of $C = A - \lambda B$ and $C' = A - \lambda' B$. Lemma 3.4.0.2 implies the existence of an element $g \in GL^+(n, \mathbb{R})$ such that $g.A = B$, $g.C \sim C$, and $g.C' \sim C'$. Here, $g.C \sim C$ denotes that $g.C$ differs from $C$ by a positive multiple. Let $A_k = g^{1-k}.A$, and $\theta_k = A_k^\perp$, then

$$\theta(\sigma_1, \sigma_m) = (m-1)\theta(\sigma_1, \sigma_2),$$

similarly to case (1). The angle exceeds $\pi$ for $m$ large enough, leading to a contradiction with the property (1) of invariant angle functions.

(5-7). Suppose that the pencil $(A, B)$ has exactly two distinct eigenvalues $\lambda$ and $\lambda'$, each of $A$ and $B$ is either a positive linear combination of $C = A - \lambda B$ and $C' = A - \lambda' B$, or a positive linear combination of $-C$ and $-C'$. Then either $(A, -B)$, $(-A, B)$, or $(-A, -B)$ is in case (4). Therefore, $\theta(A^\perp, B^\perp)$ does not exist.

(8). Suppose that the pencil $(A, B)$ has exactly two distinct eigenvalues $\lambda$ and $\lambda'$, where $A$ is a positive linear combination of $C$ and $C'$, while $B$ is a positive linear combination of $-C$ and $C'$. Note that $C'$ is a positive linear combination of $A$ and $B$. Lemma 3.4.0.2 yields an element $g \in GL^+(n, \mathbb{R})$ such that $g.C = sC$, $g.C' = s'C'$, where $s' > s$. The latter assumption implies that $g.A$ is a positive linear combination of $A$ and $C'$, and $g.B$ is a positive linear combination of $B$ and $C'$. Consequently,

$$\theta(A^\perp, B^\perp) = \theta(A^\perp, (g.A)^\perp) + \theta((g.A)^\perp, (g.B)^\perp) + \theta(B^\perp, (g.B)^\perp) > \theta((g.A)^\perp, (g.B)^\perp), \tag{3.11}$$

which is a contradiction with property (2) of invariant angles.

(9-11). Suppose that the pencil $(A, B)$ has exactly two distinct eigenvalues $\lambda$ and $\lambda'$, where $A$ is either a positive linear combination of $C$ and $C'$, or a positive linear combination of $-C$ and $-C'$, and $B$ is either a positive linear combination of $C$ and $-C'$, or a positive linear combination of $-C$ and $C'$. Then either $(A, -B)$, $(-A, B)$, or $(-A, -B)$ is in case (8). Hence, $\theta(A^\perp, B^\perp)$ does not exist.

In conclusion, for all symmetric matrix pencils $(A, B)$ of type (3), $(A^\perp, B^\perp)$ is not in the domain of any invariant angle function. $\square$

CHAPTER 4

# Criteria for Disjoint Hyperplanes

To apply the ridge cycle condition introduced in Chapter 3 to a given convex polyhedron $P$ in $\mathcal{P}(n)$, we need to identify the ridges of $P$. We propose an algorithm to ascertain that a Dirichlet-Selberg domain $DS(X, \Gamma_l)$ satisfies the ridge cycle condition, where $\Gamma_l \subset SL(n, \mathbb{R})$ is a finite subset:

(1) We initiate by taking a pair of elements $g, g' \in \Gamma_l$ and perform a sub-algorithm, detailed in Section 4.2, to determine whether the bisectors $Bis(X, g.X)$ and $Bis(X, g'.X)$ are disjoint.

(1') As an alternative to step (1), we check the condition given in Section 4.3, which is formulated in terms of the length and angles between $X$, $g.X$, and $g'.X$. This condition implies that the bisectors $Bis(X, g.X)$ and $Bis(X, g'.X)$ are disjoint.

(2) For any pair $(g, g')$ in $\Gamma_l$, if either the sub-algorithm in step (1) shows that $Bis(X, g.X)$ and $Bis(X, g'.X)$ intersect, or the condition in step (1') does not hold, we assume that a ridge $r$ of $DS(X, \Gamma_l)$ is contained in $Bis(X, g.X) \cap Bis(X, g'.X)$. Subsequently, we verify the angle-sum condition for the ridge cycle $[r]$.

(3) If the angle-sum condition is satisfied for all possible ridge cycles as in step (2), then $DS(X, \Gamma_l)$ satisfies the ridge-cycle condition.

We note that not every finitely-sided Dirichlet-Selberg domain meeting the ridge-cycle condition can be conclusively identified by this algorithm. Nevertheless, the above algorithm gives a more efficient approach than the thorough algorithm described in Chapter 5.

We will describe and prove the results mentioned in steps (1) and (1') in the sections below.

## 4.1. A criterion for intersecting hyperplanes

We seek an equivalent condition for $\bigcap_{i \in \mathcal{I}} \sigma_i \neq \varnothing$, where $\mathcal{I}$ is a finite set. Let $A_i \in Sym_n(\mathbb{R})$ be the normal vector of $\sigma_i$ for $i \in \mathcal{I}$, we denote the collection of symmetric matrices,

$$\mathcal{A} = \{A_i \in Sym_n(\mathbb{R}) | i \in \mathcal{I}\}.$$

Moreover, we denote the collection of hyperplanes,

$$\Sigma = \{\sigma_i \subset \mathcal{P}(n) | i \in \mathcal{I}\}.$$

The definiteness of a collection of symmetric $n \times n$ matrices is defined as follows:

DEFINITION 4.1.1. *We say the collection $\mathcal{A} = \{A_i \in Sym_n(\mathbb{R}) | i \in \mathcal{I}\}$ is (semi-) definite if there exist numbers $c_i \in \mathbb{R}$ for $i \in \mathcal{I}$ such that*

$$A = \sum_{i \in \mathcal{I}} c_i A_i$$

*is a non-zero positive (semi-) definite matrix.*

REMARK 4.1.1. *The collection $\{A\}$ consisting of a single $n \times n$ symmetric matrix is definite if and only if $A$ is either positive or negative definite. The collection $\{A, B\}$ is definite if and only if the symmetric matrix pencil $(A, B)$ is definite.*

We will need more notations, related to the Satake compactification $\overline{\mathcal{P}(n)} \subset \mathbf{P}(Sym_n(\mathbb{R}))$ (See Definition 2.1.7):

DEFINITION 4.1.2. *For $A \in Sym_n(\mathbb{R})$, define*

$$N(A) = \{X \in \mathbf{P}(Sym_n(\mathbb{R})) | \operatorname{tr}(A \cdot X) = 0\},$$

*and define $\overline{A^\perp} = \overline{\mathcal{P}(n)} \cap N(A)$.*

The main theorem of this section establishes a relationship between the definiteness of $\mathcal{A} = \{A_i\}$ and the emptiness of the intersection $\bigcap A_i^\perp$:

THEOREM 4.1.3. *The collection $\mathcal{A} = \{A_i\}_{i=1}^k$ of $n \times n$ symmetric matrices is semi-definite if and only if the intersection $\bigcap_{i=1}^k A_i^\perp$ is empty. Furthermore, $\mathcal{A}$ is (strictly) definite if and only if $\bigcap \overline{A_i^\perp} = \varnothing$.*

PROOF. Throughout the proof, we use the projective model of the space $\mathcal{P}(n)$.
First, we prove the statement for $k = 1$, i.e., $\mathcal{A}$ consists of a single matrix $\{A\}$. By applying the $SL(n, \mathbb{R})$-action, we assume that $A$ is a diagonal matrix $\operatorname{diag}(I_p, -I_q, O_s)$ without loss of generality.

We begin by assuming that $A$ is definite, i.e., $A = I_n$ or $A = -I_n$. For any point in $\overline{\mathcal{P}(n)}$ represented by an $n \times n$ semi-definite matrix $X$, either $\mathrm{tr}(A.X) = \mathrm{tr}(X) > 0$ or $\mathrm{tr}(A.X) = -\mathrm{tr}(X) < 0$ holds. In both cases, $\overline{A^\perp} = \varnothing$.

Next, assume that $A$ is semi-definite but not definite, i.e., $A = \pm\mathrm{diag}(I_p, O_s)$ with $s > 0$. In this case, $A^\perp = \varnothing$ for the same reason. However, the matrix $X = \mathrm{diag}(O_p, I_s)$ represents a point in $\partial_S\mathcal{P}(n)$ that satisfies $\mathrm{tr}(A.X) = 0$, implying that $\overline{A^\perp}$ is non-empty.

Lastly, assume that $A$ is indefinite, i.e., $A = \mathrm{diag}(I_p, -I_q, O_s)$ with $p, q > 0$. A matrix $X = \mathrm{diag}(I_p/q, I_q/p, I_s)$ represents a point in $\mathcal{P}(n)$ that satisfies $\mathrm{tr}(A.X) = 0$, implying that $A^\perp$ is non-empty.

Having established the statement for $k = 1$, we now extend it to general $k \in \mathbb{N}$. We first assume that $\mathcal{A} = \{A_1, \ldots, A_k\}$ is definite. By definition, there exist real numbers $c_i \in \mathbb{R}$ for $i = 1, \ldots, k$ such that $\sum c_i A_i$ is positive definite. If $\bigcap \overline{A_i^\perp}$ is non-empty, say $X \in \bigcap \overline{A_i^\perp}$, then $\mathrm{tr}(X.(\sum c_i A_i)) = \sum c_i.\mathrm{tr}(X.A_i) = 0$. However, the statement for $k = 1$ implies that $\mathrm{tr}(X \cdot (\sum c_i A_i)) > 0$, leading to a contradiction.

Next, assume that $\mathcal{A}$ is semi-definite but not definite. Then $\bigcap A_i^\perp = \varnothing$, for the same reason as in the previous case. On the other hand, if $\bigcap \overline{A_i^\perp}$ is empty, then the subspace $\bigcap N(A_i) \subset \mathbf{P}(Sym(n))$ is disjoint from the closed convex region $\overline{\mathcal{P}(n)} \subset \mathbf{P}(Sym(n))$. Therefore, there exists a support hyperplane $N(B) \subset \mathbf{P}(Sym(n))$ such that $\bigcap N(A_i) \subseteq N(B)$ and $N(B) \cap \overline{\mathcal{P}(n)} = \varnothing$. The first condition implies that $B \in \mathit{span}(A_1, \ldots, A_k)$, while the second condition, together with the statement for $k = 1$, implies that $B$ is definite. However, our assumption that $\mathcal{A}$ is indefinite contradicts this conclusion.

Lastly, assume that $\mathcal{A}$ is not semi-definite. Analogously to the previous case, if $\bigcap A_i^\perp$ is empty, there exists a supporting hyperplane $N(B) \subset \mathbf{P}(Sym(n))$ such that $\bigcap N(A_i) \subseteq N(B)$ and $N(B) \cap \mathcal{P}(n) = \varnothing$. This leads to a contradiction for a similar reason. $\square$

We focus on the case $k = 2$. If two hyperplanes $A^\perp$ and $B^\perp$ are disjoint, we have the following supplement to Theorem 4.1.3:

Lemma 4.1.3.1. *If two hyperplanes $A^\perp$ and $B^\perp$ in $\mathcal{P}(n)$ are disjoint and $(A, B)$ is regular, then all generalized eigenvalues of $(A, B)$ are real numbers.*

This result is needed to prove the classification theorem in Section 4.2. The proof of Lemma 4.1.3.1 requires some algebraic results.

LEMMA 4.1.3.2. *Suppose that $t_0$ is a real generalized eigenvalue of a symmetric $n \times n$ matrix pencil $(A, B)$. We define a continuous function $\lambda(t)$ in a neighborhood of $t = t_0$ such that $\lambda(t)$ is an eigenvalue of $A - Bt$ and $\lambda(t_0) = 0$.*
*Then, in a neighborhood of $t = t_0$, the function $\lambda(t)$ can be expressed as a product:*

$$\lambda(t) = (t - t_0)^s \varphi(t),$$

*where $s \in \mathbb{N}_+$ and $\varphi(t)$ is a continuous function with $\varphi(t_0) \neq 0$.*

PROOF. The graph of $\lambda = \lambda(t)$ locally represents a branch of the algebraic curve $\{(\lambda, t) | \det(\lambda I + tB - A) = 0\}$. Thus, $\lambda(t)$ has a Puiseux series expansion

$$\lambda(t) = \sum_{n \geq s} l_n (t - t_0)^{n/d}$$

in a neighbourhood of $(\lambda, t) = (0, t_0)$, where $s$ and $d$ are positive integers, and $l_s \neq 0$.
If $d \geq 2$, then this algebraic curve has a ramification of index $d$, denoted by $\lambda^{(k)}(t) = \sum_{n \geq s} l_n e^{2k\pi \mathrm{i}/d}(t - t_0)^{n/d}$, $k = 0, \ldots, d-1$ (see, e.g., [BK86]). Some of the branches take nonreal values in a punctured neighborhood of $t = t_0$. However, for any $t \in \mathbb{R}$, all eigenvalues of the real symmetric matrix $A - Bt$ are real, which leads to a contradiction. Hence $d = 1$, and

$$\lambda(t) = (t - t_0)^s \sum_{n \geq s} l_n (t - t_0)^{n-s} := (t - t_0)^s \varphi(t)$$

in a neighborhood of $t = t_0$. Moreover, $\varphi(t_0) = l_s \neq 0$. □

PROOF OF LEMMA 4.1.3.1. First, we assume that $\overline{A^\perp}$ and $\overline{B^\perp}$ are disjoint. Theorem 4.1.3 implies that the pencil $(A, B)$ is (strictly) definite.
Let $A'$ and $B'$ be another basis of $span(A, B)$, such that $B'$ is a positive definite matrix. By Lemma 2.3.3.1, it suffices to prove that all generalized eigenvalues of $(A', B')$ are real.
Suppose that the polynomial $\det(A' - tB')$ has distinct real zeroes $t_i$ of multiplicity $r_i$, where $i = 1, \ldots, k$. For each $i$, there is a neighborhood $U_i \supset t_i$, on which the eigenvalues of $(A' - tB')$ are

smooth functions $\lambda_j(t)$ of $t$, $j = 1, \ldots, n$. Thus,

$$\det(A' - tB') = \prod_{j=1}^{n} \lambda_j(t).$$

By Lemma 4.1.3.2, if $\lambda_j(t)$ changes sign at $t = t_i$, then $\lambda_j(t)$ has a factor $(t - t_i)$. Since the multiplicity of the zero $t = t_i$ of $\det(A' - tB')$ is $r_i$, at most $r_i$ eigenvalues change their signs at $t = t_i$, meaning the signature of $A' - tB'$ changes by at most $2r_i$ at $t = t_i$.

As $B'$ is positive definite, there exists a number $M < \infty$ such that the matrix $(A' + MB')$ is positive definite, and $(A' - MB')$ is negative definite. Therefore, as $t$ changes from $-M$ to $M$, the signature of $(A' - tB')$ increases by $2n$. Hence,

$$2n \leq \sum_{i=1}^{k} (2r_i) \leq 2 \sum_{i=1}^{k} r_i \leq 2n.$$

Consequently, $\sum r_i = n$, implying that all zeroes of $(A' - tB')$ are real. Thus, all the generalized eigenvalues of $(A', B')$ are real.

Now we assume that $A^\perp$ and $B^\perp$ are disjoint. We approximate $(A, B) \in (Sym_n(\mathbb{R}))^2$ by a sequence $\{(A_i, B_i)\}_{i=1}^{\infty}$ consisting of strictly definite matrix pencils. As discussed earlier, all generalized eigenvalues of $(A_i, B_i)$ are real. Since the generalized eigenvalues of $(A, B)$ are the limits of those of $(A_i, B_i)$ as $i \to \infty$, all the generalized eigenvalues of $(A, B)$ are real as well. □

## 4.2. Classification and algorithm for disjoint hyperplanes

The following theorem is the main result of this section, which characterizes pairs $(A, B)$ of symmetric matrices such that the hyperplanes $A^\perp$ and $B^\perp$ are disjoint in $\mathcal{P}(n)$.

Theorem 4.2.1. *Hyperplanes $A^\perp$ and $B^\perp$ in $\mathcal{P}(n)$ are disjoint if and only if either of the following holds, up to a congruence transformation of $(A, B)$:*

*(1) The matrix pencil $(A, B)$ is diagonal and semi-definite.*

*(2) The matrix pencil $(A, B)$ is block-diagonal, where the blocks are at most 2-dimensional. Moreover, all blocks $(A_i, B_i)$ of dimension 2 share the same generalized eigenvalue $\lambda$, while $A - \lambda B$ is semi-definite.*

Lemma 4.2.1.1. *Suppose that $A_0, B_0 \in Sym_m(\mathbb{R})$ and $A = \operatorname{diag}(A_0, O), B = \operatorname{diag}(B_0, O) \in Sym_n(\mathbb{R})$. Then $A^\perp \cap B^\perp = \varnothing$ if and only if $A_0^\perp \cap B_0^\perp = \varnothing$ (in $\mathcal{P}(m)$).*

Proof. On the one hand, if $X_0 \in A_0^\perp \cap B_0^\perp$, then $X = \mathrm{diag}(X_0, I_{n-m}) \in A^\perp \cap B^\perp$. On the other hand, if $X \in A^\perp \cap B^\perp = \mathrm{diag}(A_0, O_{n-m})^\perp \cap \mathrm{diag}(B_0, O_{n-m})^\perp$, suppose that

$$X = \begin{pmatrix} X_1 & X_2^{\mathrm{T}} \\ X_2 & X_3 \end{pmatrix},$$

where $X_1$ is a $m \times m$ matrix. Then $X_1$ is positive definite, and $c \cdot X_1 \in A_0^\perp \cap B_0^\perp$ for certain $c > 0$. □

Proof of Theorem 4.2.1. The "if" part is straightforward. The "only if" part of the proof is divided in two cases, depending on whether $(A, B)$ is regular.

**Case (1).** Suppose that $(A, B)$ is a regular pencil. Without loss of generality, assume that $B$ is invertible. Lemma 4.1.3.1 implies that all generalized eigenvalues of $(A, B)$ are real. By Lemma 2.3.3.1, up to a congruence transformation, we can assume that $(A, B)$ is a real block-diagonal matrix pencil, and $B^{-1}A$ is a real matrix in Jordan normal form. Moreover, the dimensions of the blocks of $(A, B)$ are the same as those of the Jordan normal form $B^{-1}A$.

Suppose that the Jordan normal form $B^{-1}A$ contains a block $J_i = J_{\lambda_i, d_i}$ of dimension 3. Lemma 2.3.6.1 implies that

$$B_i = \begin{pmatrix} 0 & 0 & a \\ 0 & a & b \\ a & b & c \end{pmatrix}, \quad A_i - \lambda B_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & a & b \end{pmatrix},$$

for real numbers $a, b, c$. Moreover, $a \neq 0$ since $B$ is invertible. Therefore, all elements in $span(A_i, B_i) = span(B_i, A_i - \lambda_i B_i)$ are indefinite, i.e., the pencil $(A, B)$ is indefinite. By Theorem 4.1.3, $A^\perp$ and $B^\perp$ intersect. Similarly, $A^\perp$ and $B^\perp$ intersect if the Jordan normal form $B^{-1}A$ contains a block of dimension greater than 3.

Suppose that the Jordan normal form $B^{-1}A$ contains a block $J_i = J_{\lambda_i, d_i}$ of dimension 2. Similarly to the previous case, elements other than $B_i - \lambda_i A_i$ in $(A_i, B_i)$ are indefinite. Thus, if $A^\perp$ and $B^\perp$ are disjoint, i.e., $(A, B)$ is semi-definite, then $B_i - \lambda_i A_i$ is the unique semi-definite element in the pencil $(A_i, B_i)$. Therefore if $(A, B)$ is semi-definite, all blocks of dimension 2 share the same eigenvalue $\lambda$, and $B - \lambda A$ is a semi-definite matrix. In this case, the matrix $B - \lambda A$ is diagonal since all 2-dimensional blocks $B_i - \lambda_i A_i$ are diagonal.

Suppose that the Jordan normal form $B^{-1}A$ is diagonal, i.e., both $A$ and $B$ are diagonal. Hyperplanes $A^{\perp}$ and $B^{\perp}$ are disjoint if and only if $(A, B)$ is semi-definite.

**Case (2).** Now suppose that the matrix pencil $(A, B)$ is singular. According to Lemma 2.3.6.2, $(A, B)$ is congruent to both

$$P^{\mathrm{T}}AP = \begin{pmatrix} A_1 & O & O \\ O & O & O \\ O & O & O \end{pmatrix}, \quad P^{\mathrm{T}}BP = \begin{pmatrix} B_1 & B_2 & O \\ B_2^{\mathrm{T}} & O & O \\ O & O & B_3 \end{pmatrix}, \tag{4.1}$$

and

$$P'^{\mathrm{T}}AP' = \begin{pmatrix} A'_1 & A'_2 & O \\ A'^{\mathrm{T}}_2 & O & O \\ O & O & A_3 \end{pmatrix}, \quad P'^{\mathrm{T}}BP' = \begin{pmatrix} B'_1 & O & O \\ O & O & O \\ O & O & O \end{pmatrix}, \tag{4.2}$$

where $A_1$, $B_3$, $A'_3$ and $B'_1$ are invertible.

Suppose that both $A'_2$ and $B_2 \neq O$. The nonzero $A'_2$ implies that $A$ contains an indefinite principal minor, thus $A$ is indefinite. Consequently, $A_1$ is also indefinite.

We proceed to construct a positive definite matrix that is orthogonal to both $A$ and $B$. According to Theorem 4.1.3, $A_1^{\perp}$ is nonempty. Let $X_1 \in A_1^{\perp}$ and choose $X_3$ to be an arbitrary positive definite matrix of the same size as $B_3$. As $B_2 \neq O$, there exists a matrix $X_2$ of the same size as $B_2$ such that $2\mathrm{tr}(X_2 \cdot B_2^{\mathrm{T}}) + \mathrm{tr}(X_1 \cdot B_1) + \mathrm{tr}(X_3 \cdot B_3) = 0$. Since $X_1$ is positive definite, there exists a positive definite matrix $X_4$ such that

$$\begin{pmatrix} X_1 & X_2 \\ X_2^{\mathrm{T}} & X_4 \end{pmatrix}$$

is positive definite. Hence,

$$\begin{pmatrix} X_1 & X_2 & O \\ X_2^{\mathrm{T}} & X_4 & O \\ O & O & X_3 \end{pmatrix} > 0, \ X = P \cdot \begin{pmatrix} X_1 & X_2 & O \\ X_2^{\mathrm{T}} & X_4 & O \\ O & O & X_3 \end{pmatrix} \cdot P^{\mathrm{T}} \in A^{\perp} \cap B^{\perp}.$$

For the reason above, $A^{\perp}$ and $B^{\perp}$ are disjoint only if either $A'_2 = O$ or $B_2 = O$. Without loss of generality, suppose that $B_2 = O$, then $(A, B)$ is congruent to $(\mathrm{diag}(A_0, O_{n-m}), \mathrm{diag}(B_0, O_{n-m}))$, where $(A_0, B_0) := (\mathrm{diag}(A_1, O), \mathrm{diag}(B_1, B_3))$ is an invertible pencil of dimension $m$. According to Lemma 4.2.1.1, the condition $A^\perp \cap B^\perp = \varnothing$ is equivalent to $A_0^\perp \cap B_0^\perp = \varnothing$.

Therefore, $A^\perp \cap B^\perp = \varnothing$ only if either of the two cases in Theorem 4.2.1 holds for the regular pencil $(A_0, B_0)$. Consequently, either of the two cases holds for $(A, B)$ as well. □

To check if two hyperplanes $A^\perp$ and $B^\perp$ are disjoint, we describe the following algorithm derived from the proof of Theorem 4.2.1.

**Algorithm for certifying disjointness of two hyperplanes.** For given normal vectors $A, B \in Sym_n(\mathbb{R})$ of hyperplanes in $\mathcal{P}(n)$, the following steps ascertain if $A^\perp \cap B^\perp = \varnothing$.

(1) Determine if $(A, B)$ is regular by computing the coefficients of the polynomial $\det(A - tB)$.

(2) If $(A, B)$ is regular, assume that $A$ is invertible without loss of generality. Compute the Jordan normal form of $A^{-1}B = PJP^{-1}$ using the standard algorithm.

(3) If any Jordan block of $J$ has dimension $\geq 3$, then $A^\perp$ and $B^\perp$ are not disjoint.

(4) Otherwise, compute $A_0 = P^{\mathrm{T}}AP$ and $B_0 = P^{\mathrm{T}}BP$. If $J$ has blocks of dimension 2, check if all these blocks share the same eigenvalue $\lambda$ and if the diagonal matrix $A_0 - \lambda B_0$ is semi-definite. This condition holds if and only if $A^\perp \cap B^\perp = \varnothing$.

(5) If $J$ is diagonal, both $A_0$ and $B_0$ are diagonal. Check if $A_0$ and $B_0$ have a positive semi-definite linear combination. This condition holds if and only if $A^\perp \cap B^\perp = \varnothing$.

(6) If $(A, B)$ is singular, compute the standard form of $(A, B)$ as in equations (4.1) and (4.2) following the algorithm described in [JL16].

(7) In the standard form mentioned above, if both matrices $B_2$ and $A_2'$ are nonzero, then $A^\perp$ and $B^\perp$ are not disjoint.

(8) Otherwise, assume that $B_2 = O$. Let $A_0 = \mathrm{diag}(A_1, O)$ and $B_0 = \mathrm{diag}(B_1, B_3)$, then the matrix pencil $(A_0, B_0)$ is regular. Check if $A_0^\perp \cap B_0^\perp = \varnothing$ by performing steps (2) to (5). According to Lemma 4.2.1.1, this is equivalent to $A^\perp \cap B^\perp = \varnothing$.

### 4.3. A sufficient condition for intersecting bisectors

In this section, we formulate a sufficient condition for disjointness of bisectors $Bis(X, Y)$ and $Bis(Y, Z)$ in terms of the distances and angle between $X, Y$ and $Z \in \mathcal{P}(n)$. A similar condition was proven in the context of hyperbolic spaces [KL19]:

Proposition 4.3.1. *Bisectors $Bis(x, y)$ and $Bis(y, z)$ in $\mathbf{H}^n$ are disjoint if* $\cosh \frac{L}{2} \sin \frac{\alpha}{2} > 1$, *where* $L = \min(d(x, y), d(y, z))$, *and* $\alpha = \angle xyz$.

This disjointness condition implies a quasi-geodesic property of piecewise geodesic paths in $\mathbf{H}^n$ which is utilized in the Kapovich-Leeb-Porti (KLP) algorithm [**KLP17**] for hyperbolic spaces. For symmetric spaces of higher rank, the quasi-geodesic property in the KLP algorithm requires that the vector-valued distances $d_\Delta(Y, X)$ and $d_\Delta(Y, Z)$ are away from the boundary of the model Weyl chamber $\Delta$ of $\mathcal{P}(n)$ (see Subsection 2.1.4).

To formulate a sufficient condition for the disjointness of Selberg's bisectors $Bis(X, Y)$ and $Bis(Y, Z)$, it appears that the vector-valued distances $d_\Delta(Y, X)$ and $d_\Delta(Y, Z)$ must be away from certain hyperplanes in $\Delta$, in contrast with the result mentioned above:

Definition 4.3.1. *We divide the model flat $F_{mod}$ of $\mathcal{P}(n)$ into $(2^n - 2)$ chambers, denoted by*

$$\Delta^{\mathcal{I}} = \{X = \operatorname{diag}(x_i) \in F_{mod} | 0 < x_i < 1,\ \forall i \in \mathcal{I}; \quad x_i > 1,\ \forall i \notin \mathcal{I}\}.$$

*For any number $t \in (0, 1)$, define*

$$\Delta_t^{\mathcal{I}} = \left\{ X \in \Delta^{\mathcal{I}} \,\middle|\, \frac{\min|\log x_i|}{\max|\log x_i|} \geq t \right\}.$$

*$\Delta_t^{\mathcal{I}}$ is a cone contained in the chamber $\Delta^{\mathcal{I}}$ and is away from the chamber boundary.*

Consider a given point $Y \in \mathcal{P}(n)$ and a given maximal totally-geodesic flat submanifold $F \ni Y$ in $\mathcal{P}(n)$. Subsection 2.1.4 implies that there exists an element $g \in SL(n, \mathbb{R})$ that maps $F$ to the model flat $F_{mod}$ and maps $Y$ to the identity. As $I = g.Y = (Y^{1/2}g)^T(Y^{1/2}g)$, we have that $Y^{1/2}g \in SO(n)$.

Theorem 4.3.2. *Let $X, Y, Z$ be points in $\mathcal{P}(n)$, and $L = \min(s(Y, X), s(Y, Z))$. There exist elements $g_X$ and $g_Z \in SL(n, \mathbb{R})$ that map $Y$ to the identity and map $X$ and $Z$ into $F_{mod}$, respectively. Let $\theta$ be the maximum angle between the corresponding column vectors of $Y^{1/2}g_X$ and $Y^{1/2}g_Z \in SO(n)$. Suppose that there is a number $t \in (0, 1)$ and a subset $\mathcal{I} \subset \{1, \ldots, n\}$ such that the points $g_X.X \in \Delta_t^{\mathcal{I}}$, $g_Z.Z \in \Delta_t^{\mathcal{I}^c}$, and*

$$\frac{1 + \sqrt{n-2}\sin\theta}{\cos\theta - \sqrt{n-2}\sin\theta} \leq \sqrt{t} \cdot \left(\frac{L-1}{n-1}\right)^{t/2}. \tag{4.3}$$

*Then the bisectors $Bis(X, Y)$ and $Bis(Y, Z)$ in $\mathcal{P}(n)$ are disjoint.*

Lemmas for proving Theorem 4.3.2 are presented below.

LEMMA 4.3.2.1. *Let* $X = \operatorname{diag}(x_i) \in \Delta_t^{\mathcal{I}}$ *and* $s(I, X) \geq L$. *For any* $i \in \mathcal{I}$ *and* $j \in \mathcal{I}^{\mathsf{c}}$,

$$\frac{|x_i^{-1} - 1|}{|x_j^{-1} - 1|} \geq t \cdot \left(\frac{L-1}{n-1}\right)^t.$$

PROOF. Without loss of generality, assume that $\mathcal{I} = \{1, \ldots, k\}$, where $k < n$. Since $X \in \Delta_t^{\mathcal{I}}$, there exists $u > 0$ such that

$$e^{tu} \leq x_i \leq e^u, \ \forall i > k; \quad e^{-u} \leq x_i \leq e^{-tu}, \ \forall i \leq k.$$

Since $s(I, X) = \sum x_i \geq L$,

$$(n-k)(e^u - 1) \geq L - ke^{-tu} - (n-k) \geq L - n.$$

Let $e^u - 1 = v$, then

$$v \geq \frac{L-n}{n-k} \geq \frac{L-n}{n-1}.$$

For any $i \in \mathcal{I}$ and $j \in \mathcal{I}^{\mathsf{c}}$,

$$\frac{|x_i^{-1} - 1|}{|x_j^{-1} - 1|} \geq \frac{e^{tu} - 1}{1 - e^{-u}} = \frac{(1+v)^t - 1}{1 - (1+v)^{-1}}.$$

It is self-evident that

$$\frac{(1+v)^t - 1}{1 - (1+v)^{-1}} = (1+v) \cdot \frac{(1+v)^t - 1}{(1+v) - 1} \geq (1+v)\frac{d}{dv}(1+v)^t = t(1+v)^t.$$

Therefore,

$$\frac{|x_i^{-1} - 1|}{|x_j^{-1} - 1|} \geq t(1+v)^t \geq t \cdot \left(\frac{L-1}{n-1}\right)^t.$$

□

LEMMA 4.3.2.2. *Suppose that* $g = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \in SO(n)$, *where* $g_1 \in Mat_k(\mathbb{R})$. *Then,* $g = g_+ g_-^{-1}$, *where*

$$g_+ = \begin{pmatrix} (g_1^{-1})^{\mathrm{T}} & -(g_1^{-1})^{\mathrm{T}} g_3^{\mathrm{T}} \\ O & I \end{pmatrix}, \quad g_- = \begin{pmatrix} I & O \\ -g_4^{-1} g_3 & g_4^{-1} \end{pmatrix}.$$

PROOF. Notice that

$$\begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \begin{pmatrix} I & O \\ -g_4^{-1}g_3 & g_4^{-1} \end{pmatrix} = \begin{pmatrix} g_1 - g_2g_4^{-1}g_3 & g_2g_4^{-1} \\ O & I \end{pmatrix}.$$

It suffices to prove that $(g_1^{-1})^{\mathrm{T}} = g_1 - g_2g_4^{-1}g_3$ and $(g_1^{-1})^{\mathrm{T}}g_3^{\mathrm{T}} = -g_2g_4^{-1}$. Indeed, since $g^{\mathrm{T}}g = I$, we have

$$g_1^{\mathrm{T}}g_2 + g_3^{\mathrm{T}}g_4 = O,$$

therefore

$$(g_1^{-1})^{\mathrm{T}}g_3^{\mathrm{T}} = -(g_1^{-1})^{\mathrm{T}}(g_1^{\mathrm{T}}g_2g_4^{-1}) = -g_2g_4^{-1}.$$

Since $gg^{\mathrm{T}} = I$, we have

$$g_1g_1^{\mathrm{T}} + g_2g_2^{\mathrm{T}} = I, \ \ g_3g_1^{\mathrm{T}} + g_4g_2^{\mathrm{T}} = O,$$

therefore

$$(g_1^{-1})^{\mathrm{T}} = g_1 + g_2g_2^{\mathrm{T}}(g_1^{-1})^{\mathrm{T}} = g_1 - g_2(g_4^{-1}g_3g_1^{\mathrm{T}})(g_1^{-1})^{\mathrm{T}} = g_1 - g_2g_4^{-1}g_3.$$

□

LEMMA 4.3.2.3. *Define*

$$\sigma_r(A) = \max_i \sum_{j=1}^n |a_{ij}|, \quad \sigma_c(A) = \max_j \sum_{i=1}^n |a_{ij}|$$

*for a matrix $A = (a_{ij}) \in Mat_n(\mathbb{R})$. If there exist elements $A, B \in Mat_n(\mathbb{R})$ such that $\sigma_r(A) \leq a$ and $\sigma_r(B) \leq b$, then $\sigma_r(AB) \leq ab$. A similar conclusion holds for $\sigma_c$.*

PROOF. We present the proof for $\sigma_r$ only. For any $1 \leq i \leq n$,

$$\sum_{j=1}^n \left| \sum_{k=1}^n a_{ik}b_{kj} \right| \leq \sum_{j,k=1}^n |a_{ik}||b_{kj}| = \sum_{k=1}^n |a_{ik}| \left( \sum_{j=1}^n |b_{kj}| \right) \leq \sum_{k=1}^n |a_{ik}| \cdot b \leq ab.$$

Thus, $\sigma_r(AB) \leq ab$. □

LEMMA 4.3.2.4. *Consider a matrix $A = (a_{ij}) \in Mat_n(\mathbb{R})$, where $a_{ii} \geq a$ and $\sum_{j \neq i} |a_{ij}| \leq a'$ for all $i = 1, \ldots, n$, and $a > a'$ are real numbers. Then $A$ is invertible, with $\sigma_r(A^{-1}) \leq \frac{1}{a-a'}$. A similar conclusion holds for $\sigma_c$.*

PROOF. We provide the proof for $\sigma_r$ only. It is well-known that diagonally dominant matrices are invertible. For such a matrix $A$, we have

$$A^{-1} = A_1 A_2^{-1},$$

where

$$A_1 = \operatorname{diag}(a_{ii}^{-1}),\ \ A_2 = (a_{ij}/a_{ii})_{i,j=1}^{n}.$$

The entries of $A_1$ are bounded by $a^{-1}$, thus $\sigma_r(A_1) \le a^{-1}$. Moreover, note that

$$A_2^{-1} = \sum_{k=0}^{\infty} (I - A_2)^k,$$

while the lemma assumption implies that $\sigma_r(I - A_2) \le a'/a$. By Lemma 4.3.2.3, $\sigma_r(I - A_2)^k \le (a'/a)^k$ for any $k \in \mathbb{N}$. Therefore,

$$\sigma_r(A_2^{-1}) \le \sum_{k=0}^{\infty} (a'/a)^k = 1/(1 - a'/a).$$

Using Lemma 4.3.2.3 again, we conclude that

$$\sigma_r(A^{-1}) \le \sigma_r(A_1)\sigma_r(A_2^{-1}) \le \frac{1}{a - a'}.$$

□

PROOF OF THEOREM 4.3.2. Applying the $SL(n, \mathbb{R})$ action, we can assume that $Y = I$, $X$ is diagonal, and $\mathcal{I} = \{k+1, \dots, n\}$, where $1 \le k < n$. Then $g_Z := g = (g_{ij}) \in SO(n)$. Lemma 4.3.2.2 implies a decomposition $g = g_+ g_-^{-1}$. Since $g \in SO(n)$, i.e., $g.I = I$, we have $g_-.I = g_+.I$. Denote the diagonal matrices $X = X_0$ and $g.Z = Z_0$, then $(g_+^{-1})^{\mathrm{T}} \in GL^+(n, \mathbb{R})$ takes $X$ to $(g_+^{-1})^{\mathrm{T}}.X_0$, and takes $Z = ((g_-^{-1})^{\mathrm{T}} g_+^{\mathrm{T}}).Z_0$ to $(g_-^{-1})^{\mathrm{T}}.Z_0$.
The theorem reduces to

$$Bis((g_+^{-1})^{\mathrm{T}}.X_0, (g_+^{-1})^{\mathrm{T}}.I) \cap Bis((g_-^{-1})^{\mathrm{T}}.Z_0, (g_-^{-1})^{\mathrm{T}}.I) = \varnothing,$$

or equivalently,

$$(g_+.(X_0^{-1} - I))^{\perp} \cap (g_-.(Z_0^{-1} - I))^{\perp} = \varnothing, \tag{*}$$

under the assumption (4.3).

Let $X_0 = \mathrm{diag}(x_i)$ and $Z_0 = \mathrm{diag}(z_i)$. Then,

$$X_0^{-1} - I = \mathrm{diag}(x_i^{-1} - 1), \quad Z_0^{-1} - I = \mathrm{diag}(z_i^{-1} - 1).$$

Since $s(I, X), s(I, Z) \geq L$, Lemma 4.3.2.1 implies that for any $i \leq k$ and $j > k$,

$$\frac{|x_j^{-1} - 1|}{|x_i^{-1} - 1|} \geq t \cdot \left(\frac{L-1}{n-1}\right)^t, \ \frac{|z_i^{-1} - 1|}{|z_j^{-1} - 1|} \geq t \cdot \left(\frac{L-1}{n-1}\right)^t.$$

Thus, there exist positive constants $c_x$ and $c_z$ such that for any $i \leq k$ and $j > k$,

$$\begin{aligned} c_x(x_j^{-1} - 1) &\geq t \cdot \left(\frac{L-1}{n-1}\right)^t, \quad -1 \leq c_x(x_i^{-1} - 1) < 0. \\ c_z(z_i^{-1} - 1) &\geq t \cdot \left(\frac{L-1}{n-1}\right)^t, \quad -1 \leq c_z(z_j^{-1} - 1) < 0. \end{aligned} \tag{4.4}$$

If we let

$$h = (h_{ij}) = \begin{pmatrix} (g_1^{-1})^{\mathrm{T}} & -(g_1^{-1})^{\mathrm{T}} g_3^{\mathrm{T}} \\ -g_4^{-1} g_3 & g_4^{-1} \end{pmatrix},$$

then $h$ is decomposed as $h = h_a^{-1} h_b$, where

$$h_a = \begin{pmatrix} g_1^{\mathrm{T}} & O \\ O & g_4 \end{pmatrix}, \ h_b = \begin{pmatrix} I & -g_3^{\mathrm{T}} \\ -g_3 & I \end{pmatrix}.$$

The assumption of the theorem implies that the angle between $\mathbf{e}_i$ and the $i$-th column vector of $g$ is at most $\theta$, i.e., the diagonal elements of $h_a$ are no less than $\cos\theta$. For $i \leq k$,

$$\sum_{j \neq i, j \leq k} |g_{ij}| \leq \sqrt{(k-1) \sum_{j \neq i, j \leq k} g_{ij}^2} \leq \sqrt{(k-1)} \sin\theta \leq \sqrt{(n-2)} \sin\theta.$$

Similarly, for $i > k$, $\sum_{j \neq i, j > k} |g_{ij}| \leq \sqrt{(n-2)} \sin\theta$. Hence, by Lemma 4.3.2.4:

$$\sigma_r(h_a^{-1}), \sigma_c(h_a^{-1}) \leq \frac{1}{\cos\theta - \sqrt{n-2}\sin\theta}.$$

Moreover, the assumption of the theorem implies that $\sigma_r(h_b), \sigma_c(h_b) \leq 1 + \sqrt{n-2}\sin\theta$. Applying Lemma 4.3.2.3, we deduce that

$$\sigma_r(h), \sigma_c(h) \leq \frac{1 + \sqrt{n-2}\sin\theta}{\cos\theta - \sqrt{n-2}\sin\theta}.$$

We establish the condition (*) by proving the positive definiteness of the linear combination $c_x \cdot g_+.(X_0^{-1} - I) + c_z \cdot g_-.(Z_0^{-1} - I)$. Let $c_x \cdot g_+.(X_0^{-1} - I) = (\xi_{ij})$ and $c_z \cdot g_-.(Z_0^{-1} - I) = (\zeta_{ij})$. For $i \leq k$, we have the following inequalities:

$$\xi_{ii} = \sum_{l \leq k} h_{li}^2 (x_l^{-1} - 1) \geq -\sum_{l \leq k} h_{li}^2,$$

$$\sum_{j \neq i} |\xi_{ij}| \leq \sum_{j \neq i,\ l \leq k} |h_{li}||h_{lj}||x_l^{-1} - 1| \leq \sum_{j \neq i,\ l \leq k} |h_{li}||h_{lj}|,$$

$$\zeta_{ii} = (z_i^{-1} - 1) + \sum_{l > k} h_{li}^2 (z_l^{-1} - 1) \geq t((L-1)/(n-1))^t - \sum_{l > k} h_{li}^2,$$

$$\sum_{j \neq i} |\zeta_{ij}| \leq \sum_{j \neq i,\ l > k} |h_{li}||h_{lj}||z_l^{-1} - 1| \leq \sum_{j \neq i,\ l > k} |h_{li}||h_{lj}|.$$

Hence,

$$\xi_{ii} + \zeta_{ii} \geq t((L-1)/(n-1))^t - \sum_{l=1}^{n} h_{li}^2 \geq \left( \frac{1 + \sqrt{n-2}\sin\theta}{\cos\theta - \sqrt{n-2}\sin\theta} \right)^2 - \sum_{l=1}^{n} h_{li}^2 = \sigma_r(h)\sigma_c(h) - \sum_{l=1}^{n} h_{li}^2$$

$$\geq \sum_{l=1}^{n} \sigma_r(h)|h_{li}| - \sum_{l=1}^{n} h_{li}^2 \geq \sum_{l,j} |h_{lj}||h_{li}| - \sum_{l=1}^{n} h_{li}^2 = \sum_{j \neq i, 1 \leq l \leq n} |h_{li}||h_{lj}| \geq \sum_{j \neq i} |\xi_{ij} + \zeta_{ij}|.$$

For $i > k$, the inequality $\xi_{ii} + \zeta_{ii} \geq \sum_{j \neq i} |\xi_{ij} + \zeta_{ij}|$ holds analogously. This implies that $c_x \cdot g_+.(X_0^{-1} - I) + c_z \cdot g_-.(Z_0^{-1} - I)$ is diagonally dominant and hence positive definite. According to Theorem 4.1.3, $Bis(X, Y)$ and $Bis(Y, Z)$ are disjoint. $\square$

CHAPTER 5

# Algorithm for Computing Dirichlet-Selberg Domains

This chapter aims to present a sub-algorithm for step (2) of Poincaré's algorithm for $SL(n,\mathbb{R})$, as described in Subsection 2.2.4. Specifically, for a given point $X \in \mathcal{P}(n)$ and a given finite set $\Gamma_l = \{g_1, \ldots, g_k\} \subset SL(n,\mathbb{R})$, we seek an algorithm that computes the poset structure of faces of the Dirichlet-Selberg domain

$$P = DS(X, \Gamma_l) = \{Y \in \mathcal{P}(n) | s(Y, X) \leq s(Y, g_i.X),\ \forall 1 \leq i \leq k\},$$

where the inclusion relation between faces serves as the partial relation on $\mathcal{F}(P)$. The Dirichlet-Selberg domain $DS(X, \Gamma_l)$ can be expressed as the intersection of $k$ half-spaces in $\mathcal{P}(n)$:

$$DS(X, \Gamma_l) = \bigcap_{i=1}^{n} H_i,$$

where

$$H_i = \{Y \in \mathcal{P}(n) | \mathrm{tr}(((g_i.X)^{-1} - X^{-1}) \cdot Y) \geq 0\}.$$

Thus, we pose a more general question:

QUESTION 5.0.1. *Describe an algorithm that computes the poset structure of the $\mathcal{P}(n)$-polyhedron*

$$P = \bigcap_{i=1}^{k} H_i,\ \ H_i = \{Y \in \mathcal{P}(n) | \mathrm{tr}(A_i Y) \geq 0\},$$

*for the given elements $A_1, \ldots, A_k \in Sym_n(\mathbb{R})$.*

We need to choose an appropriate computational model for such an algorithm. Following the work of Epstein and Petronio [EP94], we adopt the **Blum-Shub-Smale (BSS) computational model** [BSS89] for our algorithm. In the BSS model, arbitrarily many real numbers can be stored, and rational functions over real numbers can be computed in a single step. We assert that a BSS algorithm exists that computes the poset structure of finitely-sided convex polyhedra in $\mathcal{P}(n)$:

THEOREM 5.0.2. *There is a BSS algorithm with an input consisting of a point $X \in \mathcal{P}(n)$ and a finite list of elements $A'_1, \ldots, A'_{k'} \in Sym_n(\mathbb{R})$ which yields an output describing the poset structure for the face set $\mathcal{F}(P)$, where*

$$P = \bigcap_{i=1}^{k'} \{Y | \mathrm{tr}(A'_i Y) \geq 0\}.$$

*Specifically, the output consists of the data $\mathcal{A}$, $L^{face}$, $L^{pos}$, and $L^{samp}$, where*

(1) *$\mathcal{A} = \{A_1, \ldots, A_k\}$ is a subset of the input set $\{A'_1, \ldots, A'_{k'}\}$.*

(2) *$L^{face}$ is a two-dimensional array comprised of numbers from the set $\{1, \ldots, k'\}$, describing the set $\{F_1, \ldots, F_m\}$ of faces of $P$. Specifically, $L^{face}$ is a 2D array $\{L_1^{face}, \ldots, L_m^{face}\}$, where $m = |\mathcal{F}(P)|$, and such that*

$$span(F_j) = \bigcap_{i \in L_j^{face}} A_i, \ j = 1, \ldots, m.$$

(3) *$L^{pos}$ is a two-dimensional array comprised of numbers from the set $\{1, \ldots, m\}$, describing the inclusion relation among the faces of $P$, namely*

$$L_j^{pos} = \{1 \leq l \leq m | F_l \subsetneq F_j\}, \ j = 1, \ldots, m.$$

(4) *$L^{samp}$ is an array of elements in $\mathcal{P}(n)$ serving to describe sample points associated with the faces of $P$:*

$$L_j^{samp} \in F_j, \ j = 1, \ldots, m.$$

We will describe the algorithm claimed by Theorem 5.0.2 in the subsequent sections.

## 5.1. Sample points for planes of $\mathcal{P}(n)$

In this section, we describe an essential step of the algorithm claimed in Theorem 5.0.2. This sub-algorithm is designed to check the emptiness of the intersection of the given hyperplanes and to yield a sample point in this intersection.

LEMMA 5.1.0.1. *There is a numerical algorithm with an input consisting of matrices $A_1, \ldots, A_l \in Sym_n(\mathbb{R})$, yielding the following outcome:*

- *If the intersection $\bigcap_{i=1}^{l} A_i^{\perp} = \varnothing$, the algorithm outputs **false**.*

- *If $\bigcap_{i=1}^{l} A_i^{\perp}$ is non-empty, the algorithm outputs **true** and provides a sample point in $\bigcap_{i=1}^{l} A_i^{\perp}$.*

We utilize the following lemma to prove Lemma 5.1.0.1:

LEMMA 5.1.0.2. *Suppose that $B_1, \ldots, B_l \in Sym_n(\mathbb{R})$ are linearly independent matrices, and that $span(B_1, \ldots, B_l)$ contains an invertible element. Then $span(B_1, \ldots, B_l)$ contains a positive definite element if and only if*

$$\sum x_0^i B_i > 0 \tag{5.1}$$

*holds for a real and isolated critical point $(x_0^1, \ldots, x_0^l)$ of the homogeneous polynomial $P(x^1, \ldots, x^l) = \det(\sum x^i B_i)$ restricted to the unit sphere $\mathbf{S}^{l-1}$.*

PROOF. The "if" part is self-evident. To prove the "only if" part, we assume that $X' = \sum x'^i B_i$ is a positive definite element in $span(B_1, \ldots, B_l)$, where $(x'^i) := \mathbf{x'} \neq \mathbf{0} \in \mathbb{R}^l$.
We first show the existence of a critical point of $P|_{\mathbf{S}^{l-1}}$ satisfying (5.1). Without loss of generality, we assume that $\mathbf{x'}$ is a unit vector. Let $\Sigma$ be the connected component of $\mathbf{S}^{l-1} \backslash \{P(x^1, \ldots, x^l) = 0\}$ containing $\mathbf{b}$. Since $X' = \sum x'^i B_i$ is positive definite, $\sum x^i B_i$ is also positive definite for all $\mathbf{x} = (x^i) \in \Sigma$. Furthermore, as $P|_{\partial \Sigma} = 0$, $\Sigma$ contains a point $\mathbf{x_0} = (x_0^i)$ which is a local maximum point of $P|_{\mathbf{S}^{l-1}}$. Consequently, $\mathbf{x_0}$ is a critical point of $P|_{\mathbf{S}^{l-1}}$ with $\sum x_0^i B_i$ being positive-definite.
We proceed to show that the critical point $\mathbf{x_0} = (x_0^1, \ldots, x_0^l)$ is isolated. Suppose to the contrary that the critical set of $P|_{\mathbf{S}^{l-1}}$ contains a subset $S \subset \mathbf{S}^{l-1}$, which is an algebraic variety with $\dim(S) \geq 1$ and $\mathbf{x_0} \in S$. By replacing $\mathbf{x_0}$ with another point in $S$, we assume that $\mathbf{x_0}$ is a regular point in $S$. Consequently, $\mathbf{x_0}$ is contained in a smooth curve $\mathbf{x} : (-\epsilon, \epsilon) \to \mathbf{S}^{l-1}$, where $\mathbf{x}(t)$ is a critical point of $P|_{\mathbf{S}^{l-1}}$ for each $t \in (-\epsilon, \epsilon)$.
The smooth function $\mathbf{x}$ admits an expansion:

$$\mathbf{x}(t) = \mathbf{x_0} + t\mathbf{y_0} + t^2 \mathbf{z_0} + O(t^3),$$

where $\mathbf{y_0} \neq \mathbf{0}$. Since the curve $\mathbf{x}$ lies in the unit sphere,

$$\sum_{i=1}^{l} x_0^i y_0^i = 0, \quad \sum_{i=1}^{l} x_0^i z_0^i + \frac{1}{2} \sum_{i=1}^{l} (y_0^i)^2 = 0,$$

implying that both $\mathbf{y_0}$ and $\mathbf{z_0} + \frac{||\mathbf{y_0}||^2}{2}\mathbf{x_0}$ lie in $T_{\mathbf{x_0}}\mathbf{S}^{l-1}$. Since $\mathbf{x_0}$ is a critical point of $P|_{\mathbf{S}^{l-1}}$, the derivatives of $P$ at $\mathbf{x_0}$ along both $\mathbf{y_0}$ and $\mathbf{z_0} + \frac{||\mathbf{y_0}||^2}{2}\mathbf{x_0}$ vanish. By letting $X(t) = \sum x^i(t)B_i$, $X_0 = \sum x_0^i B_i$, $Y_0 = \sum y_0^i B_i$, and $Z_0 = \sum z_0^i B_i$, the vanishing of these directional derivatives is formulated as

$$\operatorname{tr}(X_0^{-1}Y_0) = 0, \quad \operatorname{tr}(X_0^{-1}Z_0) = -\frac{||\mathbf{y_0}||^2}{2}\operatorname{tr}(X_0^{-1}X_0) = -\frac{n}{2}||\mathbf{y_0}||^2.$$

On the other hand, since the points $\mathbf{x}(t) = (x^1(t), \ldots, x^l(t))$ are critical points for $-\epsilon < t < \epsilon$, Sard's Theorem implies that $\det(X(t)) = P(\mathbf{x}(t)) \equiv P(\mathbf{x_0}) = \det(X_0)$, leading to:

$$\sum_{i=1}^{n} \lambda_i = 0, \qquad \sum_{1 \le i < j \le n} \lambda_i \lambda_j + \sum_{i=1}^{n} \mu_i = 0,$$

where $\lambda_i$ and $\mu_i$, $i = 1, \ldots, n$ are eigenvalues of $X_0^{-1}Y_0$ and $X_0^{-1}Z_0$, respectively. Since $X_0, Y_0$ and $Z_0$ are real symmetric matrices, $\lambda_i$ and $\mu_i$ are real numbers. Combining the equations above, we obtain that

$$0 \le \sum_{i=1}^{n} \lambda_i^2 = (\sum_{i=1}^{n} \lambda_i)^2 - 2(\sum_{i<j} \lambda_i \lambda_j) = 2\sum_{i=1}^{n} \mu_i = 2\operatorname{tr}(X_0^{-1}Z_0) = -n \sum_{i=1}^{l} ||\mathbf{y_0}||^2 < 0,$$

which is a contradiction. □

Proof of Lemma 5.1.0.1. Suppose that $A_1, \ldots, A_l \in Sym_n(\mathbb{R})$ is the input, and $\{B_1, \ldots, B_{l'}\}$ is a basis for the orthogonal complement of $\operatorname{span}(A_1, \ldots, A_l)$ in $Sym_n(\mathbb{R})$. Then,

$$\bigcap_{i=1}^{l} A_i^{\perp} = span(B_1, \ldots, B_{l'}) \cap \mathcal{P}(n).$$

If $P(x_1, \ldots, x_{l'}) = \det(\sum x^i B_i) \equiv 0$, neither matrix in $span(B_1, \ldots, B_{l'})$ is strictly definite, implying that $\bigcap_{i=1}^{l} A_i^{\perp}$ is empty.

Otherwise, $P(x_1, \ldots, x_{l'})$ is a homogeneous polynomial of degree $n$ in variables $x_1, \ldots, x_{l'}$. Since $\mathbf{S}^{l'-1}$ is compact, the restriction of the polynomial $P(x_1, \ldots, x_{l'})|_{\mathbf{S}^{l'-1}}$ has finitely many isolated critical points, and there exist well-known numerical BSS algorithms to find them, e.g., [BC11]. Let $\mathbf{x}_1, \ldots, \mathbf{x}_m$ denote these isolated critical points, where $\mathbf{x}_j = (x_j^1, \ldots, x_j^l)$. If $\sum x_j^i B_i$ is positive definite for a certain $j \in \{1, \ldots, m\}$, Lemma 5.1.0.2 implies that $\bigcap_{i=1}^{l} A_i^{\perp}$ is non-empty with a sample point $\sum x_j^i B_i$. Conversely, if $\sum x_j^i B_i$ is not strictly definite for all $j$, Lemma 5.1.0.2 implies that $\bigcap_{i=1}^{l} A_i^{\perp}$ is empty. The algorithm we described terminates within a finite number of steps. $\square$

## 5.2. Situations of face and half-space pairs

Similarly to the case of hyperbolic spaces (cf. [**EP94**]), the algorithm claimed in Theorem 5.0.2 will involve a step determining the "situation" of pairs $(F, H)$. Here, $F$ is a face of a given convex polyhedron $P$, and $H$ is a given half-space in $\mathcal{P}(n)$. We begin by defining the relative positions of such pairs:

LEMMA 5.2.0.1. *Let $P$ be a polyhedron in $\mathcal{P}(n)$, and $H$ be a half space in $\mathcal{P}(n)$. For any face $F \in \mathcal{F}(P)$, one of the following relative positions holds for the pair $(F, H)$:*

*(1) The face $F$ lies on the boundary of $H$, i.e., $F \subset \partial H$.*

*(2) The face $F$ lies in the interior of $H$, i.e., $F \subset \operatorname{int}(H)$.*

*(3) The face $F$ lies in $H$ and meets its boundary, i.e., $F \subset H$, $F \cap \partial H \neq \varnothing$, and $F \cap \operatorname{int}(H) \neq \varnothing$.*

*(4) The face $F$ lies in $H^{\mathsf{c}}$, i.e., $F \cap H = \varnothing$.*

*(5) The face $F$ lies in $(\operatorname{int}(H))^{\mathsf{c}}$ and meets its boundary, i.e., $F \cap \operatorname{int}(H) = \varnothing$, $F \cap \partial H \neq \varnothing$, and $F \cap H^{\mathsf{c}} \neq \varnothing$.*

*(6) The face $F$ crosses $\partial H$, i.e., $F \cap \operatorname{int}(H) \neq \varnothing$ and $F \cap H^{\mathsf{c}} \neq \varnothing$.*

*Here $H^{\mathsf{c}}$ refers to the complement of $H$.*

PROOF. There are 8 cases regarding whether $F \cap H^{\mathsf{c}}$, $F \cap \partial H$, and $F \cap \operatorname{int}(H)$ are empty. Among these cases, it is impossible for all three intersections to be simultaneously empty. Furthermore, if both $F \cap H^{\mathsf{c}}$ and $F \cap \operatorname{int}(H)$ are nonempty, then $F \cap \partial H$ must also be nonempty. The remaining six cases correspond to the six relative positions listed in the lemma. $\square$

DEFINITION 5.2.1. *Let $H$ be a half-space and $P$ be a convex polyhedron in $\mathcal{P}(n)$. For $i = 1, \ldots, 6$, we denote $\mathcal{F}_H^{(i)}(P)$ as the set of faces $F \in \mathcal{F}(P)$ such that $(F, H)$ belongs to relative position (i).*

As described in the following lemma, the relative position of a pair $(F, H)$ is determined by the relative positions of pairs $(F', H)$, where $F'$ are all proper faces of $F$.

LEMMA 5.2.1.1. *Consider a half-space $H$ and a polyhedron $P$ in $\mathcal{P}(n)$. Let $F \in \mathcal{F}(P)$, such that $\partial F \neq \varnothing$, $F \neq H$, and $F \neq \overline{H^{\mathsf{c}}}$. Then, the relative position of $(F, H)$ is determined as follows:*

*(1) If $F$ has a proper face $F' \in \mathcal{F}_H^{(6)}(P)$, then $F \in \mathcal{F}_H^{(6)}(P)$.*

*(2) If $F$ has a proper face $F_1' \in \mathcal{F}_H^{(2)}(P) \cup \mathcal{F}_H^{(3)}(P)$ and another proper face $F_2' \in \mathcal{F}_H^{(4)}(P) \cup \mathcal{F}_H^{(5)}(P)$, then $F \in \mathcal{F}_H^{(6)}(P)$.*

*(3) If the previous two cases do not apply and $F$ has a proper face $F' \in \mathcal{F}_H^{(1)}(P) \cup \mathcal{F}_H^{(3)}(P) \cup \mathcal{F}_H^{(5)}(P)$, then $F \in \mathcal{F}_H^{(1)}(P) \cup \mathcal{F}_H^{(3)}(P) \cup \mathcal{F}_H^{(5)}(P)$.*

*(4) If $F' \in \mathcal{F}_H^{(2)}(P)$ for all proper faces $F'$ of $F$, then $F \in \mathcal{F}_H^{(6)}(P)$ if $F \cap \partial H \neq \varnothing$, and $F \in \mathcal{F}_H^{(2)}(P)$ if $F \cap \partial H = \varnothing$.*

*(5) If $F' \in \mathcal{F}_H^{(4)}(P)$ for all proper faces $F'$ of $F$, then $F \in \mathcal{F}_H^{(6)}(P)$ if $F \cap \partial H \neq \varnothing$, and $F \in \mathcal{F}_H^{(4)}(P)$ if $F \cap \partial H = \varnothing$.*

PROOF. **Case (1).** If a proper face $F' \in \mathcal{F}_H^{(6)}(P)$, then $F \cap \mathrm{int}(H)$ contains $F' \cap \mathrm{int}(H)$, which is non-empty, and $F \cap H^{\mathsf{c}}$ contains $F' \cap H^{\mathsf{c}}$, also non-empty. Therefore, $F \in \mathcal{F}_H^{(6)}(P)$.

**Case (2).** If a proper face $F_1' \in \mathcal{F}_H^{(2)}(P) \cup \mathcal{F}_H^{(3)}(P)$, then $F \cap \mathrm{int}(H)$ contains $F_1' \cap \mathrm{int}(H)$, which is non-empty. Additionally, if another proper face $F_2' \in \mathcal{F}_H^{(4)}(P) \cup \mathcal{F}_H^{(5)}(P)$, then $F \cap H^{\mathsf{c}}$ contains $F_2' \cap H^{\mathsf{c}}$, which is also non-empty. Therefore, $F \in \mathcal{F}_H^{(6)}(P)$.

**Case (3).** Suppose that neither case (1) nor case (2) occurs. If a proper face $F' \in \mathcal{F}_H^{(1)}(P) \cup \mathcal{F}_H^{(3)}(P) \cup \mathcal{F}_H^{(5)}(P)$, we claim that $F \notin \mathcal{F}_H^{(6)}(P)$.

If $F' \in \mathcal{F}_H^{(3)}(P)$, then $\partial H$ meets $F'$ at $\partial F'$, thus $\partial H \cap F'$ is a proper face of $F'$ in $\mathcal{F}_H^{(1)}(P)$. Similarly, if $F' \in \mathcal{F}_H^{(5)}(P)$, then $\partial H \cap F'$ is also in $\mathcal{F}_H^{(1)}(P)$. This implies that $\partial F \cap \partial H \neq \varnothing$ holds. Since $F \neq H$ and $F \neq \overline{H^{\mathsf{c}}}$, neither $F \supset \mathrm{int}(H)$ nor $F \supset H^{\mathsf{c}}$ is satisfied. Therefore, if $F \in \mathcal{F}_H^{(6)}(P)$, then $\partial F \cap \mathrm{int}(H) \neq \varnothing$ and $\partial F \cap H^{\mathsf{c}} \neq \varnothing$. However, this condition has been excluded by the two cases above.

Since $F \cap \partial H$ contains $F' \cap \partial H$, which is non-empty, $F \in \mathcal{F}_H^{(1)}(P) \cup \mathcal{F}_H^{(3)}(P) \cup \mathcal{F}_H^{(5)}(P)$.

**Case (4).** If $F' \in \mathcal{F}_H^{(2)}(P)$ for all proper faces of $F$, we have that $\partial F \cap \partial H = \bigcup_{F_i' \in \mathcal{F}(F)} (F_i' \cap \partial H) = \varnothing$. If $F \in \mathcal{F}_H^{(3)}(P)$, then the conditions $F \subset H$ and $F \cap \partial H \neq \varnothing$ imply that $\partial H$ meets $F$ on the boundary of $F$, which is a contradiction.

For any proper face $F' \subset F$, $F \cap \mathrm{int}(H)$ contains $F' \cap \mathrm{int}(H)$, which is non-empty. Knowing that $F \notin \mathcal{F}_H^{(3)}(P)$, we have either $F \in \mathcal{F}_H^{(2)}(P)$ or $F \in \mathcal{F}_H^{(6)}(P)$, depending on whether $F \cap \partial H$ is empty. Similarly, the claim holds if we replace $\mathcal{F}_H^{(2)}(P)$ with $\mathcal{F}_H^{(4)}(P)$. $\square$

## 5.3. Output data for the new polyhedron

Suppose that we have determined the relative position of $(F, H)$ for all $F \in \mathcal{F}(P)$ for a given polyhedron $P$ and a half-space $H$ in $\mathcal{P}(n)$. We will then compute the output data required by Theorem 5.0.2 for the intersection $P \cap H$, namely the lists $L^{face}$, $L^{pos}$ and $L^{samp}$ describing the face set $\mathcal{F}(P \cap H)$, the poset structure on $\mathcal{F}(P \cap H)$, and sample points of the faces in $\mathcal{F}(P \cap H)$, respectively. The set of faces $\mathcal{F}(P \cap H)$ is characterized by the lemma below:

Lemma 5.3.0.1. *Let $P$ be a polyhedron and $H$ be a half-space in $\mathcal{P}(n)$. Then $\mathcal{F}(P \cap H)$ consists of:*

- *Faces $F \in \mathcal{F}_H^{(1)}(P) \cup \mathcal{F}_H^{(2)}(P) \cup \mathcal{F}_H^{(3)}(P)$, and*
- *Intersections $F \cap H$ and $F \cap \partial H$, where $F \in \mathcal{F}_H^{(6)}(P)$.*

Proof. The polyhedron $P$ in $\mathcal{P}(n)$ is described as the intersection $\bigcap_{i=1}^{k-1} H_i$, where $H_i = \{\mathrm{tr}(Y \cdot A_i) \geq 0\}$ for $i = 1, \ldots, k-1$, each $H_i$ is a half-space in $\mathcal{P}(n)$. Let $H = H_k = \{\mathrm{tr}(Y \cdot A_k) \geq 0\}$. A face $F'$ of $P \cap H = \bigcap_{i=1}^{k} H_i$ can be expressed as

$$F' = \{Y \in \mathcal{P}(n) | \mathrm{tr}(Y \cdot A_i) = 0,\ \forall i \in \mathcal{I},\ \mathrm{tr}(Y \cdot A_j) \geq 0, \forall j \in \mathcal{J}\},$$

where $\mathcal{I}$ and $\mathcal{J}$ are disjoint subsets of $\{1, \ldots, k\}$. Here, we assume that the matrices $A_i$ for $i \in \mathcal{I}$ are linearly independent, and the set of inequalities $\mathrm{tr}(Y \cdot A_j) \geq 0$ is irredundant. We consider three cases: when $k \notin \mathcal{I} \cup \mathcal{J}$, when $k \in \mathcal{I}$, and when $k \in \mathcal{J}$.
If $k \notin \mathcal{I} \cup \mathcal{J}$, then $F'$ is a face of $P$. Since $F' \subset P \cap H \subset H$, $F' \in \mathcal{F}_H^{(1)}(P) \cup \mathcal{F}_H^{(2)}(P) \cup \mathcal{F}_H^{(3)}(P)$.
If $k \in \mathcal{I}$, the set

$$F = \{Y \in \mathcal{P}(n) | \mathrm{tr}(Y \cdot A_i) = 0, \forall i \in \mathcal{I} \backslash \{k\},\ \mathrm{tr}(Y \cdot A_j) \geq 0, \forall j \in \mathcal{J}\}$$

is a face of $P$, and $F' = F \cap \partial H$. Since $F \cap \partial H$ is non-empty, $F \notin \mathcal{F}_H^{(2)}(P) \cup \mathcal{F}_H^{(4)}(P)$. Since the equations defining $F'$ are irredundant, $F' = F \cap \partial H$ is not a face of $P$, implying that $F \in \mathcal{F}_H^{(6)}(P)$.
If $k \in \mathcal{J}$, the set

$$F = \{Y \in \mathcal{P}(n) | \mathrm{tr}(Y \cdot A_i) = 0, \forall i \in \mathcal{I},\ \mathrm{tr}(Y \cdot A_j) \geq 0, \forall j \in \mathcal{J} \backslash \{k\}\}$$

is a face of $P$, and $F' = F \cap H$. Since the inequalities are irredundant, $F' \subsetneq F$, thus $F \cap H^{\mathsf{c}}$ is non-empty. Moreover, $F \cap H \neq F \cap \partial H$. These together imply that $F \in \mathcal{F}_H^{(6)}(P)$.

On the other hand, if $F \in \mathcal{F}_H^{(1)}(P) \cup \mathcal{F}_H^{(2)}(P) \cup \mathcal{F}_H^{(3)}(P)$, then $F \subset P \cap H$, implying that $F \in \mathcal{F}(P \cap H)$. If $F \in \mathcal{F}_H^{(6)}(P)$, both $F \cap H$ and $F \cap \partial H$ are subsets of $P \cap H$, implying that $F \cap H$ and $F \cap \partial H \in \mathcal{F}(P \cap H)$. $\square$

The poset structure of $\mathcal{F}(P \cap H)$ is described in the following lemma:

LEMMA 5.3.0.2. *Let $P$ be a polyhedron and $H$ be a half-space in $\mathcal{P}(n)$. For a given face of $P \cap H$, we categorize its proper faces:*

- *For $F \in \mathcal{F}_H^{(1)}(P) \cup \mathcal{F}_H^{(2)}(P) \cup \mathcal{F}_H^{(3)}(P)$, the set of its proper faces in $\mathcal{F}(P \cap H)$ remains unchanged compared to those in $\mathcal{F}(P)$.*
- *For $F \in \mathcal{F}_H^{(6)}(P)$, the proper faces of $F \cap H$ include:*
  - *Proper faces $F'$ of $F$ where $F' \in \mathcal{F}_H^{(1)}(P) \cup \mathcal{F}_H^{(2)}(P) \cup \mathcal{F}_H^{(3)}(P)$,*
  - *Intersections $F' \cap H$ and $F' \cap \partial H$, where $F' \in \mathcal{F}_H^{(6)}(P)$ is a proper face of $F$, and*
  - *The intersection $F \cap \partial H$.*
- *For $F \in \mathcal{F}_H^{(6)}(P)$, the proper faces of $F \cap \partial H$ include:*
  - *Proper faces $F'$ of $F$ where $F' \in \mathcal{F}_H^{(1)}(P)$, and*
  - *Intersections $F' \cap \partial H$, where $F' \in \mathcal{F}_H^{(6)}(P)$ is a proper face of $F$.*

PROOF. A face of $F \cap P$ falls into one of the three cases listed in Lemma 5.3.0.1. On the one hand, it is evident that the faces of $P \cap H$ enumerated in Lemma 5.3.0.2 are proper faces of $F$, $F \cap H$, or $F \cap \partial H$, respectively. On the other hand, we aim to show that these are indeed all possible proper faces of $F$, $F \cap H$, or $F \cap \partial H$, respectively.

**Proper faces of $F \in \mathcal{F}_H^{(1)}(P) \cup \mathcal{F}_H^{(2)}(P) \cup \mathcal{F}_H^{(3)}(P)$.** If $F'$ is a proper face of $F$ in $\mathcal{F}(P)$, it follows that $F'$ is also in $\mathcal{F}(P \cap H)$.

**Proper faces of $F \cap H$, $F \in \mathcal{F}_H^{(6)}(P)$.** Proper faces $F''$ of $F \cap H$ are faces of $P \cap H$. We consider three cases according to Lemma 5.3.0.1:

(1) If $F''$ is a face of $P$, then $F'' \in \mathcal{F}_H^{(1)}(P) \cup \mathcal{F}_H^{(2)}(P) \cup \mathcal{F}_H^{(3)}(P)$. Since $F'' \subsetneq F \cap H \subsetneq F$, $F''$ is a proper face of $F$ in $\mathcal{F}(P)$.

(2) Suppose that $F'' = F' \cap H$, where $F' \in \mathcal{F}_H^{(6)}(P)$. Since $(F' \cap F) \cap H \subset F' \cap H = (F' \cap H) \cap (F \cap H) = (F' \cap F) \cap H$, replacing $F'$ with $F' \cap F$ maintains $F' \cap H$. Hence, we can assume that $F'$ is a face of $F$. Furthermore, $F' \cap H \subsetneq F \cap H$, implying that $F'$ is a proper face of $F$.

(3) Suppose that $F'' = F' \cap \partial H$, where $F' \in \mathcal{F}_H^{(6)}(P)$. Analogously, $(F' \cap F) \cap \partial H \subset F' \cap \partial H = (F' \cap \partial H) \cap (F \cap H) = (F' \cap F) \cap \partial H$, thus replacing $F'$ with $F' \cap F$ maintains $F' \cap \partial H$. Hence, we can assume that $F'$ is a face (not necessarily proper) of $F$.

**Proper faces of $F \cap \partial H$, $F \in \mathcal{F}_H^{(6)}(P)$.** Proper faces $F''$ of $F \cap \partial H$ are faces of $P \cap H$ with $F'' \subset \partial H$. Thus, we consider two cases:

(1) If $F''$ is a face of $P$, it follows that $F'' \in \mathcal{F}_H^{(1)}(P)$. Since $F'' \subsetneq F \cap \partial H \subsetneq F$, $F''$ is a proper face of $F$ in $\mathcal{F}(P)$.

(2) Suppose that $F'' = F' \cap \partial H$, where $F' \in \mathcal{F}_H^{(6)}(P)$. Since $(F' \cap F) \cap \partial H \subset F' \cap \partial H = (F' \cap \partial H) \cap (F \cap \partial H) = (F' \cap F) \cap \partial H$, replacing $F'$ with $F' \cap F$ maintains $F' \cap \partial H$. Hence, we can assume that $F'$ is a face of $F$. Furthermore, $F' \cap \partial H \subsetneq F \cap \partial H$, implying that $F'$ is a proper face of $F$. □

Lastly, we describe a sub-algorithm that obtains sample points for faces of $P \cap H$:

LEMMA 5.3.0.3. *There exists a BSS algorithm with inputs consisting of the lists $L^{face}$, $L^{pos}$ and $L^{samp}$ for a polyhedron $P$ in $\mathcal{P}(n)$, along with the equation for a half-space $H$ in $\mathcal{P}(n)$, and yields sample points for the faces of $P \cap H$.*

PROOF. By Lemma 5.3.0.1, $\mathcal{F}(P \cap H)$ consists of faces $F_j \in \mathcal{F}_H^{(1)}(P) \cup \mathcal{F}_H^{(2)}(P) \cup \mathcal{F}_H^{(3)}(P)$ and intersections $F_j \cap H$ and $F_j \cap \partial H$ for $F_j \in \mathcal{F}_H^{(6)}(P)$. A sample point of the face $F_j \in \mathcal{F}_H^{(1)}(P) \cup \mathcal{F}_H^{(2)}(P) \cup \mathcal{F}_H^{(3)}(P)$ is given in the list $L^{samp}$. Thus, our task reduces to computing sample points of $F_j \cap H$ and $F_j \cap \partial H$, where $F_j \in \mathcal{F}_H^{(6)}(P)$.

We begin by obtaining a sample point of $F' = F_j \cap \partial H$ by induction on the dimension of $F'$. If $F'$ is a minimal face, we compute its sample point using the algorithm described in Lemma 5.1.0.1. If $F'$ is not minimal, we assume by induction that we have obtained all sample points for the proper faces of $F'$. If $|\mathcal{F}(F')| \geq 2$, let $X_0'$ denote the barycenter of the sample points of the proper faces of $F'$. Note that $X_0' \in F'$ due to the convexity of $F'$, and $X_0' \notin \partial F'$ due to the disjointness of the interiors of its proper faces. Thus, the barycenter $X_0'$ lies in the interior of $F'$.

If $\mathcal{F}(F') = \{F''\}$, then $F''$ is minimal, and $F'$ is a half-space in *span*$(F')$. Let $X_0'' \in F''$ be the sample point of $F''$ and take an orthogonal basis $B_1, \ldots, B_m$ of

$$T_{X_0''} span(F') = \{B \in Sym_n(\mathbb{R}) | B \in span_{Sym_n(\mathbb{R})}(F'),\ \mathrm{tr}((X_0'')^{-1}B) = 0\}.$$

For at least one choice $i \in \{1, \ldots, m\}$ and a sufficiently small $\epsilon > 0$, either $X_0'' + \epsilon B_i \in \mathrm{int}(F')$ or $X_0'' - \epsilon B_i \in \mathrm{int}(F')$. This point serves as a sample point of $F'$.

Knowing a sample point $X_0'$ of $F' = F_j \cap \partial H$, we obtain a sample point of $F = F_j \cap H$ as follows. We take a orthogonal basis $C_1, \ldots, C_{m'}$ of $T_{X_0'} span(F)$, which is given analogously to the previous case. For at least one choice $i \in \{1, \ldots, m'\}$ and a sufficiently small $\epsilon > 0$, either $X_0' + \epsilon C_i \in \mathrm{int}(F)$ or $X_0' - \epsilon C_i \in \mathrm{int}(F)$. This point serves as a sample point of $F$. $\square$

### 5.4. Description of the algorithm

In this section, we will describe the algorithm proposed in Theorem 5.0.2. The algorithm utilizes the lemmas established in Sections 5.1 through 5.3.

**Algorithm for computing the poset structure of polyhedra in $\mathcal{P}(n)$.**

Consider a point $X \in \mathcal{P}(n)$ and a list $\mathcal{A}'$ of matrices $A_i'$, where $i = 1, \ldots, k'$. Define the half-spaces $H_i = \{\mathrm{tr}(A_i' \cdot Y) \geq 0\}$, and denote $P_l = \bigcap_{i=1}^{l} H_i$. We will use induction to show how to compute the poset data of $P_l$ for $l = 1, \ldots, k'$.

**Step (1).** We begin with $l = 0$. Since the polyhedron $P_0$ is the entire space $\mathcal{P}(n)$, we initialize

$$L^{face} = \{\varnothing\},\ L^{pos} = \{\varnothing\},\ L^{samp} = \{X\},$$

and $\mathcal{A} = \varnothing$.

**Step (2).** We increase $l$ by 1. By the induction assumption, we possess the poset data of $P_{l-1}$. That is, we have a set $\mathcal{A}$ of $n \times n$ symmetric matrices, such that $P_{l-1} = \bigcap_{A \in \mathcal{A}} \{\mathrm{tr}(A \cdot Y) \geq 0\}$, as well as lists denoted by $L^{face}$, $L^{pos}$, and $L^{samp}$ as required in Theorem 5.0.2.

**Step (3).** We will describe the computation of the poset data of $P_l = P_{l-1} \cap H_l$ from that of $P_{l-1}$. To begin with, we remove the first element of the list $\mathcal{A}'$, denoted by $A_l$, and append it to $\mathcal{A}$.

**Step (4).** We create a temporary list $L^{temp} = \{0, \ldots, 0\}$ of length equal to $|\mathcal{F}(P_{l-1})|$.

**Step (5).** We will replace the element $L_j^{temp}$ with a number from $\{1, \ldots, 6\}$ indicating the relative position of $(F_j, H_l)$, where $F_j \in \mathcal{F}(P_{l-1})$. To achieve this, we first determine the relative positions of the minimal faces in $\mathcal{F}(P_{l-1})$, i.e., the faces $F_j$ such that $L_j^{pos} = \varnothing$. Such a face $F_j$ is a plane $\bigcap_{i \in L_j^{face}} A_i^{\perp}$ in $\mathcal{P}(n)$.

If $A_l \in span_{i \in L_j^{face}}(A_i)$, then $F_j \in \mathcal{F}_{H_l}^{(1)}(P_{l-1})$, and we set $L_j^{temp} = 1$. Otherwise, we apply the algorithm described by Lemma 5.1.0.1 to determine if $F_j \cap H_l = \varnothing$. If it is not empty,

$F_j \in \mathcal{F}^{(6)}_{H_l}(P_{l-1})$, and we set $L_j^{temp} = 6$. If $F_j \cap H_l$ is empty, we compute $\mathrm{tr}(A_l X_j)$, where $X_j = L_j^{samp}$ is the sample point of $F_j$. If $\mathrm{tr}(A_l X_j) > 0$, we set $L_j^{temp} = 2$; if $\mathrm{tr}(A_l X_j) < 0$, we set $L_j^{temp} = 4$.

**Step (6).** Based on Lemma 5.2.1.1, we determine the relative position of $F_j$ when the relative positions of all the proper faces of $F_j$ are determined. If $L_{j'}^{temp} = 6$ for any $j' \in L_j^{pos}$, we set $L_j^{temp} = 6$. If $L_{j'_1}^{temp} \in \{2, 3\}$ and $L_{j'_2}^{temp} \in \{4, 5\}$ for any $j'_1, j'_2 \in L_j^{pos}$, we also set $L_j^{temp} = 6$.

If neither of the cases above applies to $F_j$ and $L_{j'}^{temp} \in \{1, 3, 5\}$ for any $j' \in L_j^{pos}$, we check if $A_l \in span_{i \in L_j^{face}}(A_i)$. If so, we set $L_j^{temp} = 1$. Otherwise, if $\mathrm{tr}(A_l X_j) > 0$, we set $L_j^{temp} = 3$; if $\mathrm{tr}(A_l X_j) < 0$, we set $L_j^{temp} = 5$.

If none of the cases above apply to $F_j$, either $L_{j'}^{temp} = 2$ for all $j' \in L_j^{pos}$, or $L_{j'}^{temp} = 4$ for all $j' \in L_j^{pos}$. Suppose that the former holds. We check if $span(F) \cap \partial H_l = \varnothing$ by applying Lemma 5.1.0.1. If it is empty, set $L_j^{temp} = 2$. Otherwise, let $X_0 \in span(F) \cap \partial H_l$ be the sample point we derived from Lemma 5.1.0.1. If $\mathrm{tr}(X_0 A_i) < 0$ for any $1 \le i \le l-1$, set $L_j^{temp} = 2$; otherwise, set $L_j^{temp} = 6$.

**Step (7).** We derive the data $L^{face}$ for $P_l$ according to Lemma 5.3.0.1. For any number $j$, if $L_j^{temp} \in \{1, 2, 3\}$, $F_j$ remains in $\mathcal{F}(P_l)$, and no action is taken for such $j$.

If $L_j^{temp} \in \{4, 5\}$, $F_j$ no longer exists as a face of $P_l$, thus we remove the elements $L_j^{face}$, $L_j^{temp}$, $L_j^{samp}$, and $L_j^{pos}$. Moreover, we remove any number $j$ that occurs in $L^{pos}$ and decrease by 1 any number greater than $j$.

If $L_j^{temp} = 6$, both $F_j$ and $F_j \cap H$ are faces of $P_l$. Since $span(F_j \cap H) = span(F_j)$, we keep the element $L_j^{face}$ representing the new face $F_j \cap H$ instead of $F_j$. Furthermore, we append an element $L_j^{face} \cup \{l\}$ to $L^{face}$ representing $F_j \cap \partial H$. Let this be the $\hat{j}$-th element of $L^{face}$.

**Step (8).** We derive the data $L^{pos}$ for $P_l$ according to Lemma 5.3.0.2. The case $L_j^{temp} \in \{4, 5\}$ will not occur. If $L_j^{temp} \in \{1, 2, 3\}$, no action is taken since the proper faces of $F_j$ in $\mathcal{F}^{(4)}_H(P_{l-1}) \cup \mathcal{F}^{(5)}_H(P_{l-1})$ have been removed from $L_j^{pos}$ in step (6).

If $L_j^{temp} = 6$, then $F_j \cap \partial H$ is a proper face of $F_j \cap H$, and we append the element $\hat{j}$ to $L_j^{pos}$. Additionally, for each $l \in L_j^{pos}$, we check the value $L_l^{temp}$. The case $L_l^{temp} \in \{4, 5\}$ will not occur, and no action is taken if $L_l^{temp} \in \{1, 2, 3\}$. If $L_l^{temp} = 6$, $F_l \cap \partial H$ is a proper face of both $F_j \cap H$ and $F_j \cap \partial H$, thus we append $\hat{l}$ to both $L_j^{pos}$ and $L_{\hat{j}}^{pos}$.

**Step (9).** We derive the data $L^{samp}$ for $P_l$ as decribed in Lemma 5.3.0.3.

**Step (10).** We check if all numbers in $\{1, \ldots, l\}$ appear in $L^{face}$. If a number $i \in \{1, \ldots, l\}$ does not appear, we remove $A_i$ from the list $\{A_1, \ldots, A_l\}$, decrease by 1 any numbers greater than $i$ appearing in $L^{face}$, and decrease $l$ by 1.

**Step (11).** Repeat steps (2) through (10) if $\mathcal{A}'$ is non-empty. If $\mathcal{A}'$ is empty, the algorithm terminates, and the data $\mathcal{A}$, $L^{face}$, $L^{pos}$ and $L^{samp}$ are the required output of the theorem.

CHAPTER 6

# On the Finite-sidedness of Dirichlet-Selberg Domains

The objective of this chapter is to classify discrete subgroups $\Gamma < SL(3,\mathbb{R})$ into the following types:

- The subgroup $\Gamma$ belongs to the **finitely-sided type**, if for every $X \in \mathcal{P}(3)$, the Dirichlet-Selberg domain $DS(X,\Gamma)$ is finitely-sided.
- The subgroup $\Gamma$ belongs to the **infinitely-sided type**, if for a generic choice of $X \in \mathcal{P}(3)$, the Dirichlet-Selberg domain $DS(X,\Gamma)$ is infinitely-sided.

Our study will focus on discrete Abelian subgroups of $SL(3,\mathbb{R})$ that consist of matrices with exclusively positive eigenvalues. Such matrices have favorable properties within $SL(3,\mathbb{R})$. In particular, if all eigenvalues of $g \in SL(3,\mathbb{R})$ are positive, then $g^k \in SL(3,\mathbb{R})$ for any $k \in \mathbb{R}$.
For every $X \in \mathcal{P}(3)$, discrete subgroup $\Gamma < SL(3,\mathbb{R})$, and $g \in SL(3,\mathbb{R})$, the Dirichlet-Selberg domains $DS(X,\Gamma)$ and $DS(g.X, g^{-1}\Gamma g)$ are isometric. Therefore, our initial focus will be classifying the conjugacy classes of Abelian subgroups of $SL(3,\mathbb{R})$ with only positive eigenvalues.

Proposition 6.0.1. *Let $\Gamma$ be a discrete Abelian subgroup of $SL(3,\mathbb{R})$ where all eigenvalues of each $\gamma \in \Gamma$ are positive real numbers. Then, $\Gamma$ is conjugate to a subgroup of $SL(3,\mathbb{R})$ generated by either of the following:*
*(i) For cyclic $\Gamma$, the generators are displayed below:*

| *Type* | *(1)* | *(2)* | *(3)* | *(4)* |
|---|---|---|---|---|
| *Generator* | $\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}$ | $\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}$ | $\begin{pmatrix}e^r&0&0\\0&e^s&0\\0&0&e^t\end{pmatrix}$ *($r+s+t=0$; $(r,s,t)\neq(0,0,0)$)* | $\begin{pmatrix}e^t&1&0\\0&e^t&0\\0&0&e^{-2t}\end{pmatrix}$ *($t\neq 0$)* |

Table 6.1. Generators of cyclic discrete Abelian subgroups of $SL(3,\mathbb{R})$.

*(ii) For 2-generated $\Gamma$, the generators are displayed below:*

| Type | (1) | (1') | (2) | (3) | (4) |
|---|---|---|---|---|---|
| Generators | $\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}$ | $\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}$ | $\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}$ | $\begin{pmatrix}e^r&0&0\\0&e^s&0\\0&0&e^t\end{pmatrix}$ | $\begin{pmatrix}e^t&1&0\\0&e^t&0\\0&0&e^{-2t}\end{pmatrix}$ |
| | $\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}$ | $\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}$ | $\begin{pmatrix}1&a&b\\0&1&a\\0&0&1\end{pmatrix}$ $(b \neq a(a-1)/2)$ | $\begin{pmatrix}e^{r'}&0&0\\0&e^{s'}&0\\0&0&e^{t'}\end{pmatrix}$ $(r+s+t = r'+s'+t'=0)$ | $\begin{pmatrix}e^s&a&0\\0&e^s&0\\0&0&e^{-2s}\end{pmatrix}$ $((s,t)\neq(0,0))$ |

TABLE 6.2. Generators of 2-generated discrete Abelian subgroups of $SL(3,\mathbb{R})$.

PROOF. First, we suppose that $\Gamma$ is cyclic, generated by $\gamma$. As all eigenvalues of $\gamma$ are positive, the Jordan canonical form of $\gamma$ falls into one of the types listed in Table 6.1.

Next, we suppose that $\Gamma$ is 2-generated, which breaks into the following cases:

(1) The subgroup $\Gamma$ is unipotent. For every non-identity element in $\Gamma$, the algebraic and geometric multiplicities of the eigenvalue $\lambda = 1$ are 3 and 2, respectively.
(2) The subgroup $\Gamma$ is unipotent. For every non-identity element in $\Gamma$, the algebraic multiplicity of the eigenvalue $\lambda = 1$ is 3, while for at least one element in $\Gamma$, the geometric multiplicity of $\lambda = 1$ is 1.
(3) All elements in $\Gamma$ are diagonalizable by similarity.
(4) The group $\Gamma$ contains both a non-unipotent and a non-diagonalizable element.

**Case (1).** We assume without loss of generality that one of the generators of $\Gamma$ is

$$\gamma_1 = \begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix},$$

and denote the other generator by $\gamma_2$. The conditions $\gamma_1\gamma_2 = \gamma_2\gamma_1$ and $\det(\lambda I - \gamma_2) = (\lambda-1)^3$ imply that

$$\gamma_2 = \begin{pmatrix}1&a&b\\0&1&0\\0&c&1\end{pmatrix}, \; a,b,c \in \mathbb{R}.$$

Since the geometric multiplicity of the eigenvalue 1 of $\gamma_2$ is 2, we have $rank(\gamma_2 - I) = 1$, implying that $b = 0$ or $c = 0$. If $c = 0$, then $b \neq 0$, as $\Gamma$ is discrete and 2-generated. Additionally, we can assume that $b > 0$, after replacing $\gamma_2$ with $\gamma_2^{-1}$ if necessary. Let

$$g = \begin{pmatrix} b^{1/3} & 0 & 0 \\ 0 & b^{1/3} & 0 \\ 0 & -ab^{-2/3} & b^{-2/3} \end{pmatrix},$$

then

$$g^{-1}\gamma_1 g = \gamma_1,\ g^{-1}\gamma_2 g = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which correspond to type (1) shown in Table 6.2.

If $b = 0$, then $c \neq 0$. We can similarly take the generators $\gamma_1$ and $\gamma_2$ to generators of type (1') via a similarity transformation.

**Case (2).** Assuming that $\gamma_1 \in \Gamma$ is unipotent with its eigenvalue $\lambda = 1$ having geometric multiplicity 1. Without loss of generality, we have

$$\gamma_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $\gamma_2$ be another generator of $\Gamma$. Since $\gamma_1\gamma_2 = \gamma_2\gamma_1$ and $\det(\gamma_2) = 1$, we deduce that

$$\gamma_2 = \begin{pmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix},$$

where $a, b \in \mathbb{R}$. As $\Gamma$ is 2-generated and discrete, $\gamma_2 \neq \gamma_1^a$, implying that $b \neq a(a-1)/2$. Thus, the generators $\gamma_1$ and $\gamma_2$ correspond to type (2) shown in Table 6.2.

**Case (3).** If $\gamma_1\gamma_2 = \gamma_2\gamma_1$ while both $\gamma_1$ and $\gamma_2$ are diagonalizable, they are simultaneously diagonalizable. Hence, $\gamma_1$ and $\gamma_2$ correspond to type (3) shown in Table 6.2.

**Case (4).** Let $\gamma_1$ be a generator of $\Gamma$ that has eigenvalues other than 1. Without loss of generality, we can assume $\gamma_1$ to be a Jordan matrix after applying a similarity transformation.

If $\gamma_1$ is diagonal, its diagonal elements cannot be pairwise distinct; otherwise, the other generator $\gamma_2$ will be diagonal, leading to a contradiction. Hence, we assume that $\gamma_1 = diag(e^s, e^s, e^{-2s})$, where

$s \neq 0$. Since $\gamma_1\gamma_2 = \gamma_2\gamma_1$, we deduce that

$$\gamma_2 = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix},$$

and $\gamma_2$ is not diagonalizable. By a similarity transformation of the first two rows and columns, this becomes

$$\gamma_2 = \begin{pmatrix} e^t & 1 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix},$$

for a real number $t$.

If $\gamma_1$ is not diagonalizable, then

$$\gamma_1 = \begin{pmatrix} e^t & 1 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix},$$

where $t \neq 0$. The condition $\gamma_1\gamma_2 = \gamma_2\gamma_1$ implies that

$$\gamma_2 = \begin{pmatrix} e^s & a & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^{-2s} \end{pmatrix}$$

for some real numbers $s$ and $a$. In both cases, $\gamma_1$ and $\gamma_2$ correspond to type (4) shown in Table 6.2.

In any of the four cases, let $\gamma_3 \in SL(3,\mathbb{R})$ have only positive eigenvalues and commute with both $\gamma_1$ and $\gamma_2$. It is easy to show that $\gamma_3 = \gamma_1^{k_1}\gamma_2^{k_2}$ for $k_1, k_2 \in \mathbb{R}$, implying that $\langle\gamma_1, \gamma_2, \gamma_3\rangle$ is either 2-generated or non-discrete. Hence, Abelian discrete subgroups of $SL(3,\mathbb{R})$ with positive eigenvalues are at most 2-generated. $\square$

For clarity and organization, we further categorize cyclic groups of type (3) into two subtypes:

(3) The generator $\gamma = diag(e^r, e^s, e^t)$, where all of $r, s, t$ are nonzero.

(3') The generator $\gamma = diag(e^s, e^{-s}, 1)$, with $s \neq 0$.

Using the classification of discrete Abelian subgroups of $SL(3,\mathbb{R})$ with positive eigenvalues, we present the main result of this chapter:

THEOREM 6.0.1. *Let $\Gamma$ be a discrete and free Abelian subgroup of $SL(3,\mathbb{R})$, generated by matrices with exclusively positive eigenvalues.*

- *If $\Gamma$ is a cyclic group of type (1), (3), or (4), or if it is a 2-generated group of type (1) or (3), the Dirichlet-Selberg domain $DS(X,\Gamma)$ is finitely-sided for all $X \in \mathcal{P}(3)$.*
- *If $\Gamma$ is a cyclic group of type (2) or (3'), or if it is a 2-generated group of type (1'), (2) or (4), the Dirichlet-Selberg domain $DS(X,\Gamma)$ is infinitely-sided for all $X$ in a dense and Zariski open subset of $\mathcal{P}(3)$.*

## 6.1. An equivalent condition

Let $\Gamma$ be a discrete subgroup of $SL(3,\mathbb{R})$ and let $X \in \mathcal{P}(3)$. A facet of the Dirichlet-Selberg domain $D = DS(X,\Gamma)$ lies in a bisector $Bis(X, \gamma.X)$, where $\gamma \in \Gamma$. We denote such a facet by $F_\gamma$. The following lemma characterizes the existence of such facets:

LEMMA 6.1.0.1. *Let $\Gamma$ be a discrete subgroup of $SL(n,\mathbb{R})$. Suppose that there exists a smooth function $g : \mathbb{R}^m \to SL(n,\mathbb{R})$ such that $\Gamma = g(\Lambda)$, where $\Lambda$ is a discrete subset of $\mathbb{R}^m$, $\mathbf{0} \in \Lambda$, and $g(\mathbf{0}) = e$. For $A, X \in \mathcal{P}(n)$, define a function $s^g_{X,A} : \mathbb{R}^m \to \mathbb{R}$, $s^g_{X,A}(\mathbf{k}) = s(g(\mathbf{k}).X, A)$.*
*Then for any $\mathbf{k}_0 \in \Lambda \backslash \{\mathbf{0}\}$, the facet $F_{g(\mathbf{k}_0)}$ of $DS(X,\Gamma)$ exists if and only if there exists a matrix $A \in \mathcal{P}(n)$ such that $\mathbf{0}$ and $\mathbf{k}_0$ are the only minimum points of $s^g_{X,A}|_\Lambda$.*

PROOF. The existence of the facet $F_{g(\mathbf{k}_0)}$ is equivalent to the non-emptiness of

$$\mathrm{int}(F_{g(\mathbf{k}_0)}) = Bis(X, g(\mathbf{k}_0).X) \cap \left( \bigcap_{\mathbf{k} \in \Lambda - \{\mathbf{0}, \mathbf{k}_0\}} \{Y | s(I,Y) < s(g(\mathbf{k}).I, Y)\} \right).$$

Moreover, a point $A \in \mathcal{P}(n)$ lies in this intersection if and only if

$$s^g_{X,A}(\mathbf{k}_0) = s^g_{X,A}(\mathbf{0}),\ s^g_{X,A}(\mathbf{k}) = s^g_{X,A}(\mathbf{0}),\ \forall \mathbf{k} \in \Lambda - \{\mathbf{0}, \mathbf{k}_0\},$$

meaning that $\mathbf{0}$ and $\mathbf{k}_0$ are the only points where $s^g_{X,A}|_\Lambda$ attains its minima. $\square$

REMARK 6.1.1. *Let $X \in \mathcal{P}(n)$, $\Lambda \subset \mathbb{R}^m$ be a discrete subset, $g : \mathbb{R}^m \to Sym_n(\mathbb{R})$ a smooth function, and $\Gamma = g(\Lambda)$ a discrete subgroup of $SL(n,\mathbb{R})$. Lemma 6.1.0.1 implies the following:*

- *If for all but finitely many points $\mathbf{k} \in \Lambda$ and for every $A \in \mathcal{P}(n)$, the function $s^g_{X,A}|_\Lambda$ attains the minimum at points other than $\mathbf{k}$ and $\mathbf{0}$, then $F_\gamma$ is a facet of $DS(X,\Gamma)$ for* *only finitely many elements $\gamma = g(\mathbf{k}) \in \Gamma$. Thus, the Dirichlet-Selberg domain $DS(X, \Gamma)$ is finitely-sided.*

- *If there are infinitely many points $\mathbf{k} \in \Lambda$ such that $\mathbf{k}$ and $\mathbf{0}$ are the only two minimum points of $s^g_{X,A}|_\Lambda$ for a certain $A \in \mathcal{P}(n)$, then $F_\gamma$ is a facet of $DS(X, \Gamma)$ for infinitely many elements $\gamma = g(\mathbf{k}) \in \Gamma$. Thus, the Dirichlet-Selberg domain $DS(X, \Gamma)$ is infinitely-sided.*

Below is a generalization of Lemma 6.1.0.1:

Corollary 6.1.1. *Let $\Gamma < SL(n, \mathbb{R})$ be a discrete subgroup. Suppose that there is a smooth function $g : \mathbb{R}^m \to SL(n, \mathbb{R})$ such that $\Gamma = g(\Lambda)$, where $\Lambda \subset \mathbb{R}^m$ is a discrete subset, $\mathbf{0} \in \Lambda$, and $g(\mathbf{0}) = e$. Define the functions $s^g_{X,A}$ analogously to Lemma 6.1.0.1.*
*Suppose that there exists a matrix $A \in \mathcal{P}(n)$ and a finite subset $\Lambda_0 \subset \Lambda$ satisfying the following conditions:*

*(1) The point $\mathbf{0} \in \Lambda_0$.*
*(2) There exists a nonzero point $\mathbf{k_0} \in \Lambda_0$ such that $s^g_{X,A}(\mathbf{k_0}) = s^g_{X,A}(\mathbf{0})$.*
*(3) For any $\mathbf{k} \in \Lambda_0$, $s^g_{X,A}(\mathbf{k}) \leq s^g_{X,A}(\mathbf{0})$; for any $\mathbf{k} \in \Lambda \backslash \Lambda_0$, $s^g_{X,A}(\mathbf{k}) > s^g_{X,A}(\mathbf{0})$.*

*Then the Dirichlet-Selberg domain $DS(X, \Gamma)$ has a facet $F_{g(\mathbf{k})}$ for at least one element $\mathbf{k} \in \Lambda_0 \backslash \{\mathbf{0}\}$.*

Proof. Assume that $\Lambda_0 = \{\mathbf{0}, \mathbf{k}_0, \mathbf{k}_1, \dots, \mathbf{k}_r\}$, where

$$s^g_{X,A}(\mathbf{k}_0) \geq s^g_{X,A}(\mathbf{k}_1) \geq \cdots \geq s^g_{X,A}(\mathbf{k}_r).$$

Define

$$\Lambda'_i := (\Lambda \backslash \Lambda_0) \cup \{\mathbf{0}, \mathbf{k}_0, \dots, \mathbf{k}_i\},\ i = 0, \dots, r,$$

thus $\Lambda'_r = \Lambda$. We will prove the following assertion by induction on $i$:

(*) The Dirichlet-Selberg domain $DS(X, g(\Lambda'_i))$ contains a facet $F_{g(\mathbf{k}_j)}$ for a certain $j \in \{0, \dots, i\}$.

By Lemma 6.1.0.1, $DS(X, g(\Lambda'_0))$ contains the facet $F_{g(\mathbf{k}_0)}$. This serves as the base case for the assertion (*).
Assume that the claim (*) holds for $(i-1)$. To prove the assertion for $i$, note that

$$DS(X, g(\Lambda'_i)) = DS(X, g(\Lambda'_{i-1})) \cap H_i,\ H_i = \{Y | s(I, Y) \leq s(g(\mathbf{k}_i).I, Y)\}.$$

Let $F_{g(\mathbf{k}_{j'})} \in \mathcal{F}(DS(X, g(\Lambda'_{i-1})))$ be the facet in the induction assumption, where $j' \in \{0, 1, \ldots, i-1\}$. If the position of $(DS(X, g(\Lambda'_{i-1})), H_i)$ is not case (6) in Lemma 5.2.0.1, then $F_{g(\mathbf{k}_{j'})}$ remains a facet of $DS(X, g(\Lambda'_i))$. If it is case (6) in Lemma 5.2.0.1, then $\partial H_i \cap DS(X, g(\Lambda'_{i-1}))$ is the facet $F_{g(\mathbf{k}_i)}$ of $DS(X, g(\Lambda'_i))$. This confirms the claim (*).

Particularly, when $i = r$, (*) is equivalent to the lemma's statement. □

The proof of Theorem 6.0.1 consists of a series of assertions that will be described in the subsequent sections. For clarity, we shall consistently denote the $(i, j)$ entry of $X^{-1}$ and $A$ by $x^{ij}$ and $a_{ij}$, respectively.

### 6.2. Subgroups of $SL(3, \mathbb{R})$ with finitely-sided Dirichlet-Selberg domains

In this section, we will examine the cases when the Dirichlet-Selberg domain $DS(X, \Gamma)$ is finitely-sided for all $X \in \mathcal{P}(3)$, as asserted in Theorem 6.0.1.

Proof of Theorem 6.0.1, cyclic group of type (1). We interpret the group $\Gamma$ generated by

$$\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

as the image of $\mathbb{Z}$ under the 1-variable function

$$g(k) = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \forall k \in \mathbb{R}.$$

In this case, the function $s^g_{X,A}$ in Lemma 6.1.0.1 becomes

$$\begin{aligned} s^g_{X,A}(k) &= s(g(k).X, A) = \operatorname{tr}(((\gamma^{\mathrm{T}})^k.X.\gamma^k)^{-1}.A) \\ &= x^{11}a_{22}k^2 + 2(x^{11}a_{12} + x^{12}a_{22} + x^{13}a_{23})k + s(X, A), \end{aligned}$$

which is a quadratic polynomial in $k$. Since both $X$ and $A$ are positive definite, the leading term of $s^g_{X,A}$ has a positive coefficient. If $s^g_{X,A}$ evaluates the same at 0 and $k_0$, then $s^g_{X,A}(k) < s^g_{X,A}(0)$ for any $k$ between 0 and $k_0$.

Therefore, if $s^g_{X,A}|_{\mathbb{Z}}$ attains its minimum at both 0 and $k_0$, the integer $k_0$ can only be 1 or $-1$. Lemma 6.1.0.1 with $\Lambda = \mathbb{Z}$ implies that the Dirichlet-Selberg domain $DS(X, \Gamma)$ is 2-sided for any $X \in \mathcal{P}(3)$. $\square$

Proof of Theorem 6.0.1, cyclic group of type (3). We interpret the cyclic subgroup $\Gamma$ of type (3) as a one-parameter family given by

$$g(k) = \gamma^k = diag(e^{rk}, e^{sk}, e^{tk}),\ k \in \mathbb{Z},$$

where $r + s + t = 0$ and $r, s, t \neq 0$. Without loss of generality, we assume that $r \geq s > 0 > t$. For a given $X$ and $A \in \mathcal{P}(3)$, the function $s^g_{X,A}$ becomes

$$s^g_{X,A}(k) = x^{11}a_{11}e^{-2rk} + x^{22}a_{22}e^{-2sk} + x^{33}a_{33}e^{-2tk} + 2x^{23}a_{23}e^{rk} + 2x^{13}a_{13}e^{sk} + 2x^{12}a_{12}e^{tk}.$$

Since $x^{ii}$, $a_{ii} > 0$ for $i = 1, 2, 3$, there is a unique $k_c \in \mathbb{R}$ such that

$$\sqrt{x^{11}a_{11}}e^{-rk_c} + \sqrt{x^{22}a_{22}}e^{-sk_c} = \sqrt{x^{33}a_{33}}e^{-tk_c}.$$

Therefore, $s^g_{X,A}(k) = c \cdot f(k - k_c)$, where

$$f(n) = e^{2(r+s)n} + 2p\alpha_{13}e^{sn} + 2(1-p)\alpha_{23}e^{rn} + p^2e^{-2rn} + (1-p)^2e^{-2sn} + 2p(1-p)\alpha_{12}e^{-(r+s)n},$$

and

$$c = x^{33}a_{33}e^{-2tk_c} > 0,\ p = \frac{\sqrt{x^{11}a_{11}}e^{-rk_c}}{\sqrt{x^{33}a_{33}}e^{-tk_c}} \in (0, 1),\ \alpha_{ij} = \frac{x^{ij}a_{ij}}{\sqrt{x^{ii}x^{jj}a_{ii}a_{jj}}}.$$

For any $i \neq j$, $|\alpha_{ij}| < \xi := \max_{i\neq j} \frac{|x^{ij}|}{\sqrt{x^{ii}x^{jj}}}$, $\xi < 1$ and depends only on $X$. For any $0 \leq p \leq 1$ and any $-\xi \leq \alpha_{ij} \leq \xi$, $\lim_{n\to\infty} f'(n) = \infty$ and $\lim_{n\to-\infty} f'(n) = -\infty$. Since $f'(n; p, \alpha_{ij})$ is continuous with respect to $p$ and $\alpha_{ij}$, there exists a number $N > 0$ determined by $r, s, t$, and $\xi$, such that

$$f'(n; p, \alpha_{ij}) > 0,\ \forall n > N;\ f'(n; p, \alpha_{ij}) < 0,\ \forall n < -N,$$

for every $(p, \alpha_{12}, \alpha_{13}, \alpha_{23})$ in the compact region $[0, 1] \times [-\xi, \xi]^3$. If $k = 0$ and $k = k_0$ are the only minimum points of $s^g_{X,A}|_{\mathbb{Z}}$, then $(-k_c)$ and $(k_0 - k_c)$ are the only minimum points of $f|_{\mathbb{Z}-k_c}$. Thus $|k_c|, |k_0 - k_c| < N + 1$, implying that $|k_0| < 2(N + 1)$. Since there are finitely many choices of such $k_0$, Lemma 6.1.0.1 with $\Lambda = \mathbb{Z}$ implies that the Dirichlet-Selberg domain $DS(X, \Gamma)$ is finitely-sided for any $X \in \mathcal{P}(3)$. $\square$

Proof of Theorem 6.0.1, cyclic group of type (4). We interpret the cyclic subgroup $\Gamma$ of type (4) as a one-parameter family given by

$$g(k) = \gamma^k = \begin{pmatrix} e^{sk} & ke^{s(k-1)} & 0 \\ 0 & e^{sk} & 0 \\ 0 & 0 & e^{-2sk} \end{pmatrix},\ k \in \mathbb{Z},$$

where $s \neq 0$. Without loss of generality, we assume that $s > 0$. The function $s^g_{X,A}$ becomes

$$s^g_{X,A}(k) = x^{33}a_{33}e^{4sk} + (2x^{13}a_{13} + 2x^{23}a_{23} - 2ke^{-s}x^{23}a_{13})e^{sk}$$
$$+ (x^{11}a_{11} + x^{22}a_{22} + 2x^{12}a_{12} - 2ke^{-s}(x^{12}a_{11} + x^{22}a_{12}) + k^2e^{-2s}x^{22}a_{11})e^{-2sk}.$$

A unique $k_c \in \mathbb{R}$ exists such that

$$(\sqrt{x^{11}a_{11}} + \sqrt{x^{22}a_{22}} + e^{-s}\sqrt{x^{22}a_{11}})e^{-sk_c} = \sqrt{x^{33}a_{33}}e^{2sk_c}.$$

Hence, $s^g_{X,A}(k) = c \cdot f(k - k_c)$, where

$$f(n) = e^{4sn} + (2\alpha_{13}p + 2\alpha_{23}q - 2\beta_3(1-p-q)n)e^{sn}$$
$$+ (p^2 + q^2 + 2\alpha_{12}pq - 2(\beta_1 p + \beta_2 q)(1-p-q)n + (1-p-q)^2n^2)e^{-2sn},$$

and

$$c = x^{33}a_{33}e^{4sk_c} > 0,\ p = \frac{\sqrt{x^{11}a_{11}}e^{-sk_c}}{\sqrt{x^{33}a_{33}}e^{2sk_c}},\ q = \frac{\sqrt{x^{22}a_{22}}e^{-sk_c}}{\sqrt{x^{33}a_{33}}e^{2sk_c}},$$
$$\alpha_{ij} = \frac{x^{ij}a_{ij}}{\sqrt{x^{ii}x^{jj}a_{ii}a_{jj}}},\ \beta_1 = \frac{x^{12}}{\sqrt{x^{11}x^{22}}},\ \beta_2 = \frac{a_{12}}{\sqrt{a_{11}a_{22}}},\ \beta_3 = \frac{x^{23}a_{13}}{\sqrt{x^{22}x^{33}a_{11}a_{33}}},$$

where $p, q > 0,\ p + q < 1,\ |\alpha_{ij}|, |\beta_1|, |\beta_3| < \xi := \max_{i \neq j} \frac{|x^{ij}|}{\sqrt{x^{ii}x^{jj}}}$, and $|\beta_2| < 1$. Analogously to the proof for cyclic groups of type (3), there exists a number $N > 0$ determined by $s$ and $\xi$, such that

$$f'(n; p, q, \alpha_{ij}, \beta_i) > 0,\ \forall n > N; f'(n; p, q, \alpha_{ij}, \beta_i) < 0,\ \forall n < -N,$$

and for every $(p, q, \alpha_{ij}, \beta_i)$ in the compact region $\{(p,q) | p, q \geq 0, p + q \leq 1\} \times [-\xi, \xi]^5 \times [-1, 1]$. Hence, if $k = 0$ and $k = k_0$ are the only minimum points of $s^g_{X,A}|_{\mathbb{Z}}$, then $|k_0| < 2(N+1)$ analogously. Lemma 6.1.0.1 with $\Lambda = \mathbb{Z}$ implies that $DS(X, \Gamma)$ is finitely-sided for any $X \in \mathcal{P}(3)$.

$\square$

We now consider 2-generated subgroups. To utilize Lemma 6.1.0.1, we investigate the level curves of the function $s^g_{X,A} : \mathbb{R}^2 \to \mathbb{R}$.

PROOF OF THEOREM 6.0.1, 2-GENERATED GROUP OF TYPE (1). We interpret the group $\Gamma$ as a two-parameter family

$$g(k,l) = \gamma_1^k \gamma_2^l = \begin{pmatrix} 1 & k & l \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \forall (k,l) \in \mathbb{Z}^2.$$

Thus, the function $s^g_{X,A}$ is expressed as

$$s^g_{X,A}(k,l) = a_{11}(x^{22}(k-k_c)^2 + 2x^{23}(k-k_c)(l-l_c) + x^{33}(l-l_c)^2) + const,$$

where

$$k_c = \frac{a_{12}}{a_{11}} + \frac{x^{12}x^{33} - x^{13}x^{23}}{x^{22}x^{33} - (x^{23})^2},\ l_c = \frac{a_{13}}{a_{11}} + \frac{x^{13}x^{22} - x^{12}x^{23}}{x^{22}x^{33} - (x^{23})^2}.$$

As $x^{22}x^{33} > (x^{23})^2$, the level curves of $s^g_{X,A}$ are ellipses centered at $(k_c, l_c)$. These ellipses share the same eccentricity, depending solely on $X$.

If $(0,0)$ and $(k_0, l_0)$ are the only minimum points of $s^g_{X,A}|_{\mathbb{Z}^2}$, then there exists a level curve of $s^g_{X,A}$ that surrounds these two points and excludes all other points in $\mathbb{Z}^2$. Since any circle of diameter greater than $\sqrt{5}$ encompasses at least 3 points in $\mathbb{Z}^2$, the minor axis length of the mentioned level curve is at most $\sqrt{5}$. Thus, the major axis length of the level curve is bounded by a constant depending solely on $X$. Consequently, there are only finitely many choices of $(k_0, l_0)$ such that $(0,0)$ and $(k_0, l_0)$ are the only minimum points of $s^g_{X,A}$. By Lemma 6.1.0.1 with $\Lambda = \mathbb{Z}^2$, the Dirichlet-Selberg domain $DS(X, \Gamma)$ is finitely-sided for any $X \in \mathcal{P}(3)$. $\square$

PROOF OF THEOREM 6.0.1, 2-GENERATED GROUP OF TYPE (3). We interpret the group $\Gamma$ as a two-parameter family

$$g(k,l,m) = diag(e^k, e^l, e^m),\ \forall (k,l,m) \in \Lambda,$$

where the domain of $g$ is the 2-plane

$$\{(k,l,m) \in \mathbb{R}^3 | k+l+m = 0\},$$

and $\Lambda = \mathbb{Z}(r,s,t) \oplus \mathbb{Z}(r',s',t')$ is contained in this 2-plane.

The function $s^g_{X,A}$ is given by

$$\begin{aligned}
&s^g_{X,A}(k,l,m) \\
&= (x^{11}a_{11})e^{2k} + (x^{22}a_{22})e^{2l} + (x^{33}a_{33})e^{2m} + (2x^{23}a_{23})e^{-k} + (2x^{13}a_{13})e^{-l} + (2x^{12}a_{12})e^{-m} \\
&= c(e^{2(k-k_c)} + e^{2(l-l_c)} + e^{2(m-m_c)} + 2\alpha_{23}e^{-(k-k_c)} + 2\alpha_{13}e^{-(l-l_c)} + 2\alpha_{12}e^{-(m-m_c)}),
\end{aligned}$$

where

$$c = \sqrt[3]{x^{11}x^{22}x^{33}a_{11}a_{22}a_{33}},\ \alpha_{12} = \frac{x^{12}a_{12}}{\sqrt{x^{11}x^{22}a_{11}a_{22}}},\ \alpha_{13} = \frac{x^{13}a_{13}}{\sqrt{x^{11}x^{33}a_{11}a_{33}}},\ \alpha_{23} = \frac{x^{23}a_{23}}{\sqrt{x^{22}x^{33}a_{22}a_{33}}},$$

and

$$k_c = -\frac{1}{3}\log\frac{x^{11}a_{11}}{\sqrt{x^{22}x^{33}a_{22}a_{33}}},\ l_c = -\frac{1}{3}\log\frac{x^{22}a_{22}}{\sqrt{x^{11}x^{33}a_{11}a_{33}}},\ m_c = -\frac{1}{3}\log\frac{x^{33}a_{33}}{\sqrt{x^{11}x^{22}a_{11}a_{22}}}.$$

Here, the point $(k_c,l_c,m_c)$ lies on the plane $\{k+l+m=0\}$. For any $i \neq j$, the coefficient $|\alpha_{ij}| < \xi := \max_{i\neq j}\frac{|x^{ij}|}{\sqrt{x^{ii}x^{jj}}}$, where $\xi < 1$ and depends only on $X$. Thus,

$$f_-(k-k_c, l-l_c, m-m_c) \le s^g_{X,A}(k,l,m)/c \le f_+(k-k_c, l-l_c, m-m_c),\ \forall k+l+m=0,$$

where

$$f_\pm(k-k_c,l-l_c,m-m_c) := e^{2(k-k_c)} + e^{2(l-l_c)} + e^{2(m-m_c)} \pm 2\xi(e^{-(k-k_c)} + e^{-(l-l_c)} + e^{-(m-m_c)}).$$

Let $d = d(k,l,m)$ represents the Euclidean distance between $(k_c,l_c,m_c)$ and $(k,l,m)$. When $d$ is fixed, it is evident that $f_+(k-k_c,l-l_c,m-m_c)$ reaches its maximum at $(k-k_c,l-l_c,m-m_c) = (\frac{2d}{\sqrt{6}}, -\frac{d}{\sqrt{6}}, -\frac{d}{\sqrt{6}})$, and $f_-(k-k_c,l-l_c,m-m_c)$ reaches its minimum at $(k-k_c,l-l_c,m-m_c) = (-\frac{2d}{\sqrt{6}}, \frac{d}{\sqrt{6}}, \frac{d}{\sqrt{6}})$. Therefore,

$$2(1-\xi)e^{d} - 4\xi e^{-d/2} + e^{-2d} := f_-(d) \le s^g_{X,A}(k,l,m)/c \le f_+(d) := 2e^{2d} + 4\xi e^{d/2} + 2(1+\xi)e^{-d}.$$

If a level curve of $s^g_{X,A}(k,l,m)$ surrounds only two points $(0,0,0)$ and $(k_0,l_0,m_0)$ among the points in $\Lambda$, the inscribed radius of the level curve is less than a certain constant $\rho > 0$ determined by $\Lambda$. Since $\xi < 1$, it follows that $\lim_{d\to\infty} f_-(d) = \infty$. Hence, there exists a constant $R < \infty$ depending solely on $\xi$ and $\rho$, such that $f_+(\rho) = f_-(R)$. Consequently, the value taken on this level curve is less than $f_+(\rho) = f_-(R)$, implying that the diameter of the level curve is less than $2R$. Therefore, there are only finitely many choices of $(k_0, l_0, m_0) \in \Lambda$, such that $(0,0,0)$ and $(k_0, l_0, m_0)$ are the only minimum points of $s^g_{X,A}$. By Lemma 6.1.0.1, the Dirichlet-Selberg domain $DS(X,\Gamma)$ is finitely-sided for any $X \in \mathcal{P}(3)$. $\square$

### 6.3. Subgroups of $SL(3,\mathbb{R})$ with infinitely-sided Dirichlet-Selberg domains

We proceed to the cases when the Dirichlet-Selberg domain $DS(X,\Gamma)$ is infinitely-sided for a generic choice of $X \in \mathcal{P}(3)$ as asserted in Theorem 6.0.1.

Proof of Theorem 6.0.1, cyclic group of type (2). We interpret the cyclic group $\Gamma$ of type (2) as a one-parameter family

$$g(k) := \gamma^k = \begin{pmatrix} 1 & k & k(k+1)/2 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix},\ k \in \mathbb{Z}.$$

Thus for any $A = (a_{ij})$ and $X = (x^{ij})^{-1} \in \mathcal{P}(3)$, the function $s^g_{X,A}$ is expressed as

$$\begin{aligned}
&s^g_{X,A}(k) = s(\gamma^k.X, A) = \operatorname{tr}(((\gamma^k)^{\mathrm{T}} X \gamma^k)^{-1} A) \\
&= (x^{33}a_{11}/4)k^4 + \left(-x^{33}a_{12} + \left(x^{33}/2 - x^{23}\right)a_{11}\right)k^3 \\
&+ \left(x^{33}a_{13} + x^{33}a_{22} + (3x^{23} - x^{33})a_{12} + \left(x^{33}/4 - x^{23} + x^{13} + x^{22}\right)a_{11}\right)k^2 \\
&+ \left(-2x^{33}a_{23} + (x^{33} - 2x^{23})a_{13} - 2x^{23}a_{22} + (x^{23} - 2x^{13} - 2x^{22})a_{12} + (x^{13} - 2x^{12})a_{11}\right)k \\
&+ (x^{33}a_{33} + x^{22}a_{22} + x^{11}a_{11} + 2x^{23}a_{23} + 2x^{13}a_{13} + 2x^{12}a_{12}),
\end{aligned}$$

a quartic polynomial in $k$. We will demonstrate that for any $X \in \mathcal{P}(3)$ and any $k_0 \in \mathbb{Z}$, there exists a positive definite matrix $A$ such that

$$s^g_{X,A}(k) = k^2(k - k_0)^2 + const,$$

representing a quartic function whose global minimum points are $k = 0$ and $k = k_0$. By comparing coefficients of the $k^4$ and $k^3$ terms, we derive

$$x^{33}a_{11}/4 = 1, \quad -x^{33}a_{12} + \left(x^{33}/2 - x^{23}\right)a_{11} = -2k_0,$$

implying that

$$a_{11} = 4/x^{33} > 0,\ a_{12} = (2(k_0+1)x^{33} - 4x^{23})/((x^{33})^2).$$

Furthermore, we let $a_{22}$ be sufficiently large, so that $a_{11}a_{22} > a_{12}^2$. By comparing the coefficients of the $k^2$ and $k^1$ terms, we deduce:

$$\begin{aligned}
&x^{33}a_{13} + x^{33}a_{22} + (3x^{23} - x^{33})a_{12} + \left(x^{33}/4 - x^{23} + x^{13} + x^{22}\right)a_{11} = {k_0}^2,\\
&-2x^{33}a_{23} + (x^{33} - 2x^{23})a_{13} - 2x^{23}a_{22} + (x^{23} - 2x^{13} - 2x^{22})a_{12} + (x^{13} - 2x^{12})a_{11} = 0,
\end{aligned}$$

which is a linear equation system in the unknowns $a_{13}$ and $a_{23}$. This linear equation system has an invertible coefficient matrix, thus has a unique solution for $a_{13}$ and $a_{23}$.
Lastly, we determine $a_{33}$ by setting $\det(A) = 1$. These steps yield a matrix $A \in F_{\gamma^k}$. By Lemma 6.1.0.1 with $\Lambda = \mathbb{Z}$, the Dirichlet-Selberg domain $DS(X, \Gamma)$ is infinitely sided for any $X \in \mathcal{P}(3)$. $\square$

Proof of Theorem 6.0.1, cyclic group of type (3'). We interpret the cyclic group $\Gamma$ of type (3') as a one-parameter family

$$g(k) := \gamma^k = diag(e^{sk}, e^{-sk}, 1),\ k \in \mathbb{Z}.$$

Thus, the function $s^g_{X,A}$ is expressed as

$$s^g_{X,A}(k) = s(\gamma^k.X, A) = a_{22}x^{22}e^{2sk} + 2a_{23}x^{23}e^{sk} + 2a_{13}x^{13}e^{-sk} + a_{11}x^{11}e^{-2sk} + const.$$

Additionally, we assume that $x^{23}, x^{13} \neq 0$. We will demonstrate that for any $k_0 \in \mathbb{Z}$, there exists a positive definite matrix $A$ such that

$$s^g_{X,A}(k) = e^{2sk} - 2(e^{sk_0}+1)e^{sk} - 2e^{sk_0}(e^{sk_0}+1)e^{-sk} + e^{2sk_0}e^{-2sk} + const,$$

which represents a function with minimum points at $k = 0$ and $k = k_0$. A suitable solution is given by:

$$a_{11} = e^{2tk_0}/x^{11}, \quad a_{12} = 0, \quad a_{22} = 1/x^{22}, \quad a_{23} = -(e^{tk_0}+1)/x^{23}, \quad a_{13} = -e^{tk_0}(e^{tk_0}+1)/x^{13},$$

and $a_{33}$ is determined by $\det(A) = 1$. Analogously to the preceding case, the existence of such a solution for $A$ implies that $DS(X, \Gamma)$ is infinitely sided whenever the center $X$ does not belong to the proper Zariski closed subset $\{X = (x^{ij})^{-1} \in \mathcal{P}(3) | x^{13}x^{23} = 0\}$. $\square$

We proceed to consider the cases of 2-generated subgroups:

Proof of Theorem 6.0.1, 2-generated group of type (1'). We interpret the group $\Gamma$ as a two-parameter family, given by

$$g(k,l) = \gamma_1^k \gamma_2^l = \begin{pmatrix} 1 & 0 & l \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}, \ \forall (k,l) \in \mathbb{Z}^2,$$

and the function $s^g_{X,A}$ is expressed as

$$s^g_{X,A}(k,l) = x^{33}(a_{22}(k-k_c)^2 + 2a_{12}(k-k_c)(l-l_c) + a_{11}(l-l_c)^2) + const,$$

where

$$k_c = \frac{x^{23}}{x^{33}} + \frac{a_{11}a_{23} - a_{12}a_{13}}{a_{11}a_{22} - a_{12}^2}, \ l_c = \frac{x^{13}}{x^{33}} + \frac{a_{22}a_{13} - a_{12}a_{23}}{a_{11}a_{22} - a_{12}^2}.$$

We claim that for any coprime pair $(k_0, l_0) \in \mathbb{Z}^2$, there exists a matrix $A \in \mathcal{P}(3)$ such that $s^g_{X,A}|_{\mathbb{Z}^2}$ achieves its global minimum at $(k,l) = (0,0)$ and $(k,l) = (k_0, l_0)$. Specifically, we claim that the matrix $A$ can be chosen so that

$$s^g_{X,A}(k,l) = \epsilon^2(k_0(k-k_0/2) + l_0(l-l_0/2))^2 + (l_0(k-k_0/2) - k_0(l-l_0/2))^2 + const \tag{6.1}$$

for arbitrarily small $\epsilon > 0$. In other words, a particular level set of $s^g_{X,A}$ is an ellipse, with its major axis being the line segment between $(0,0)$ and $(k_0, l_0)$, and its minor axis length being $\epsilon$ times the length of the major axis.

A comparison of the coefficients of the $k^2$, $kl$ and $l^2$ terms yields that:

$$a_{11} = \epsilon^2 l_0^2 + k_0^2, \ a_{12} = -(1-\epsilon^2)k_0 l_0, \ a_{22} = \epsilon^2 k_0^2 + l_0^2.$$

Therefore, $a_{11}a_{22} - a_{12}^2 = 2\epsilon^2(k_0^2 + l_0^2)^2 > 0$. A comparison of the coefficients of the $k^1$ and $l^1$ terms implies that:

$$k_0/2 = k_c = \frac{x^{23}}{x^{33}} + \frac{a_{11}a_{23} - a_{12}a_{13}}{a_{11}a_{22} - a_{12}^2}, \ l_0/2 = l_c = \frac{x^{13}}{x^{33}} + \frac{a_{22}a_{13} - a_{12}a_{23}}{a_{11}a_{22} - a_{12}^2}.$$

This can be interpreted as a 2-variable linear equation system in the unknowns $a_{13}$ and $a_{23}$. The coefficient matrix of the equation system is

$$\begin{pmatrix} a_{11} & -a_{12} \\ -a_{12} & a_{22} \end{pmatrix},$$

which is invertible. Thus, a unique solution for $a_{13}$ and $a_{23}$ is determined by $k_0$, $l_0$, $\epsilon$ and $X$. Finally, we determine $a_{33}$ by setting $\det(A) = 1$.

The matrix $A$ we constructed is positive definite. Furthermore, a certain level curve of $s^g_{X,A}$ is an ellipse whose major axis is the line segment between $(0,0)$ and $(k_0, l_0)$. Assuming that $k_0$ and $l_0$ are coprime, the ellipse will exclude all other points in $\mathbb{Z}^2$ if the minor axis is sufficiently short, i.e., the number $\epsilon$ is sufficiently small. By Lemma 6.1.0.1, the Dirichlet-Selberg domain $DS(X, \Gamma)$ is infinitely-sided for any $X \in \mathcal{P}(3)$. □

Proof of Theorem 6.0.1, 2-generated group of type (2). We interpret the 2-generated subgroup $\Gamma$ of type (2) for given constants $a$ and $b$ as a two-parameter family:

$$g(k,l) = \begin{pmatrix} 1 & -k & k^2 - l \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{pmatrix}, \ \forall (k,l) \in \Lambda = \Lambda(a,b),$$

where

$$\Lambda(a,b) = \left\{ (k,l) \,\middle|\, k = x + ay, l = \frac{1}{2}\left(a^2(y^2 - y) + 2axy + 2by + x^2 - x\right),\ (x,y) \in \mathbb{Z} \right\}$$

is a discrete subset of $\mathbb{R}^2$. The function $s^g_{X,A}$ is expressed as

$$s^g_{X,A}(k,l) = (a_{11}x^{22} + 2a_{12}x^{23} + a_{22}x^{33})(k-k_c)^2 + 2(a_{11}x^{23} + a_{12}x^{33})(k-k_c)(l-l_c) + (a_{11}x^{33})(l-l_c)^2 + const,$$

where $k_c$ and $l_c$ are given by:

$$k_c = -\frac{\begin{gathered} a_{11}^2(x^{12}x^{33} - x^{13}x^{23}) + a_{11}a_{12}(x^{22}x^{33} - (x^{23})^2) \\ + (a_{11}a_{22} - a_{12}^2)x^{23}x^{33} + (a_{11}a_{23} - a_{12}a_{13})(x^{33})^2 \end{gathered}}{a_{11}^2(x^{22}x^{33} - (x^{23})^2) + (a_{11}a_{22} - a_{12}^2)(x^{33})^2},$$

$$l_c = -\frac{\begin{gathered} a_{11}^2(x^{13}x^{22} - x^{12}x^{23}) - a_{11}a_{12}(x^{12}x^{33} - x^{13}x^{23}) + (a_{11}a_{13} - a_{12}^2)(x^{22}x^{33} - (x^{23})^2) \\ + (a_{13}a_{22} - a_{12}a_{23})(x^{33})^2 - (a_{11}a_{23} - a_{12}a_{13})x^{23}x^{33} + (a_{11}a_{22} - a_{12}^2)(x^{13}x^{33} - (x^{23})^2) \end{gathered}}{a_{11}^2(x^{22}x^{33} - (x^{23})^2) + (a_{11}a_{22} - a_{12}^2)(x^{33})^2}.$$

We assert that for any sufficiently small $\delta > 0$, there exists $\epsilon = \epsilon(X, \delta) > 0$, such that $\epsilon = O(\delta^2)$ as $\delta \to 0$. Furthermore, for any $(k_0, l_0) \in \Lambda$ with $|k_0/l_0| = \delta$, there exists a positive definite matrix $A$ such that the equation (6.1) holds. In other words, the following properties hold:

- A certain level curve of $s^g_{X,A}$ is an ellipse whose major axis is between $(0, 0)$ and $(k_0, l_0)$.
- The minor axis length is $\epsilon$ times the length of the major axis.

By comparing the coefficients of the $k^2$, $kl$ and $l^2$ terms, we obtain:

$$a_{11}x^{22} + 2a_{12}x^{23} + a_{22}x^{33} = \epsilon^2 k_0^2 + l_0^2,\ a_{11}x^{33} = \epsilon^2 l_0^2 + k_0^2,\ a_{11}x^{23} + a_{12}x^{33} = (\epsilon^2 - 1)k_0 l_0.$$

This equation system has a unique solution, namely

$$\begin{aligned} a_{11} &= \frac{k_0^2 + \epsilon^2 l_0^2}{x^{33}}, a_{12} = -\frac{(k_0^2 + \epsilon^2 l_0^2)x^{23} + k_0 l_0 (1 - \epsilon^2)x^{33}}{{x^{33}}^2}, \\ a_{22} &= \frac{(l_0^2 + \epsilon^2 k_0^2){x^{33}}^2 + 2k_0 l_0 (1 - \epsilon^2)x^{23}x^{33} - (k_0^2 + \epsilon^2 l_0^2)(x^{22}x^{33} - 2{x^{23}}^2)}{{x^{33}}^3}. \end{aligned}$$

In order to satisfy the positive definite condition $a_{11}a_{22} > a_{12}^2$, the following inequality must hold:

$$-\frac{l_0^4(x^{22}x^{33} - {x^{23}}^2)}{{x^{33}}^4}\epsilon^4 + \frac{(k_0^2 + l_0^2)^2{x^{33}}^2 - 2k_0^2 l_0^2(x^{22}x^{33} - {x^{23}}^2)}{{x^{33}}^4}\epsilon^2 - \frac{k_0^4(x^{22}x^{33} - {x^{23}}^2)}{{x^{33}}^4} > 0.$$

As $k_0/l_0 \to 0$, the roots of the quartic function on the left-hand side are $\epsilon = \pm\epsilon_+$ and $\epsilon = \pm\epsilon_-$, where $\epsilon_\pm = \epsilon_\pm(k_0/l_0)$ have the following series expansions:

$$\epsilon_+ = \frac{x^{33}}{\sqrt{x^{22}x^{33} - {x^{23}}^2}} + O\left((k_0/l_0)^2\right),\ \epsilon_- = \frac{\sqrt{x^{22}x^{33} - {x^{23}}^2}}{x^{33}}(k_0/l_0)^2 + O\left((k_0/l_0)^4\right),$$

with all coefficients determined by $X$. We let $\delta = |k_0/l_0|$ and set $\epsilon = \epsilon(\delta)$ such that $\epsilon(\delta) > \epsilon_-(\delta)$ and $\epsilon(\delta) \sim \epsilon_-(\delta)$ as $\delta \to 0$. This choice ensures the positive definite condition $a_{11}a_{22} > a_{12}^2$.
By comparing the coefficients of the $k^1$ and $l^1$ terms, we deduce that $k_c = k_0/2$ and $l_c = l_0/2$. Substituting the solution for $a_{11}$, $a_{12}$ and $a_{22}$ above, the denominators of the expressions for both $k_c$ and $l_c$ are

$$a_{11}^2(x^{22}x^{33} - (x^{23})^2) + (a_{11}a_{22} - a_{12}^2)(x^{33})^2 = -\epsilon^2(k_0^2 + l_0^2)^2 \neq 0.$$

Thus, the equations $k_c = k_0/2$ and $l_c = l_0/2$ form a linear equation system with unknowns $a_{13}$ and $a_{23}$, with an invertible coefficient matrix. Consequently, the equation system has a unique solution for $a_{13}$ and $a_{23}$. Finally, we determine $a_{33}$ by setting $\det(A) = 1$.

We continue the proof by utilizing Corollary 6.1.1. Our proof consists of two cases, depending on whether the entry $a$ of the generator $\gamma_2$ is rational.

**Case (1):** $a \in \mathbb{Q}$**.** Assume that $a = p/q$, where $(p, q)$ are coprime. The first components of points in $\Lambda$ take values in $(1/q)\mathbb{Z}$, and we have

$$\Lambda \cap \{(k,l)|k = 1/q\} = \{(1/q, l_n)|l_n = (a(a-1) - 2b)qn + l_0,\ n \in \mathbb{Z}\},$$

where $l_0$ is a constant depending on $a$ and $b$. Let $\delta_n = (1/q)/l_n$, then $\delta_n = O(n^{-1})$. Our construction of the matrix $A$ yields a level curve of $s^g_{X,A}$, whose major axis is the line segment between $(0, 0)$ and $(1/q, l_n)$, and the minor axis length is $\epsilon_n$ times the length of the major axis. Here, $\epsilon_n = \epsilon(\delta_n) = O(n^{-2})$. Consequently, the level curve intersects with the line $\{l = 0\}$ at $k = 0$ and

$$k = \frac{\epsilon_n^2/q(1/q^2 + l_n^2)}{\epsilon_n^2/q^2 + l_n^2} = O(n^{-4}).$$

For sufficiently large $n$, the level curve we constructed does not surround any points other than $(0, 0)$ and $(1/q, l_n)$ in $\Lambda$. By Lemma 6.1.0.1, the Dirichlet-Selberg domain $DS(X, \Gamma)$ is infinitely-sided.

**Case (2):** $a \notin \mathbb{Q}$**.** Choose any $(k_1, l_1)$. Our construction yields a point $A_1 \in \mathcal{P}(3)$ and a level curve of $s^g_{X,A_1}$ surrounding $(0, 0)$ and $(k_1, l_1)$. We choose the points $(k_i, l_i)$ inductively as follows: suppose we have chosen points $A_j \in \mathcal{P}(3)$ and $(k_j, l_j) \in \Lambda$, where $j = 1, \dots, (i-1)$. Let $\Lambda_j$ be the set of points in $\Lambda$ surrounded by the level curve of $s^g_{X,A_j}$ through $(0, 0)$ and $(k_j, l_j)$, then the union $\bigcup_{j=1}^{i-1} \Lambda_j$ is a finite set. Since $a \notin \mathbb{Q}$, we can choose $(k_i, l_i) \in \Lambda$ such that $k_i$ is sufficiently small and $l_i$ is sufficiently large, ensuring that the level curve of $s^g_{X,A_i}$ in our construction excludes all points in $\bigcup_{j=1}^{i-1} \Lambda_j \backslash \{(0, 0)\}$. By Corollary 6.1.1, $DS(X, \Gamma)$ has a facet $F_{g(k'_i, l'_i)}$, where $(k'_i, l'_i) \in \Lambda_i \backslash \{(0, 0)\}$ for all $i \in \mathbb{N}$. Our construction implies that these points are pairwise distinct. Thus the Dirichlet-Selberg domain $DS(X, \Gamma)$ is infinitely-sided. $\square$

Proof of Theorem 6.0.1, 2-generated group of type (4). A 2-generated subgroup $\Gamma$ of type (4) is interpreted as a two-parameter family

$$g(k,l) = \begin{pmatrix} e^{-k} & -le^{-k} & 0 \\ 0 & e^{-k} & 0 \\ 0 & 0 & e^{2k} \end{pmatrix}, \ \forall (k,l) \in \Lambda = (t,1)\mathbb{Z} \oplus (s,a)\mathbb{Z} \subset \mathbb{R}^2,$$

where $(s,t) \neq (0,0)$ and $a \in \mathbb{R}$. The function $s^g_{X,A}$ is expressed as:

$$\begin{aligned} s^g_{X,A}(k,l) &= e^{2k}(a_{11}x^{11} + 2a_{12}x^{12} + a_{22}x^{22} + 2l(a_{11}x^{12} + a_{12}x^{22}) + l^2 a_{11}x^{22}) \\ &+ 2e^{-k}(a_{13}x^{13} + a_{23}x^{23} + la_{13}x^{23}) + e^{-4k}a_{33}x^{33}. \end{aligned}$$

We assert that if $X = (x^{ij})^{-1}$ satisfies $x^{23} \neq 0$, then for any $(k_0, l_0) \in \Lambda$ where $k_0 \neq 0$, there exists a point $A \in \mathcal{P}(3)$, such that:

- A level curve of $s^g_{X,A}$ is connected and passes through $(0,0)$ and $(k_0,l_0)$.
- The level curve lies between the lines $k = 0$ and $k = k_0$, and is tangent to these lines at $(0,0)$ and $(k_0,l_0)$, respectively.

Indeed, the level curve $s^g_{X,A} = c$ is the union of graphs of the following functions:

$$l = l_\pm(e^{-k}; c) = l_0(e^{-k}; c) \pm \sqrt{l_1(e^{-k}; c)},$$

where

$$\begin{aligned} l_0(e^{-k}; c) &= -\left(\frac{x^{12}}{x^{22}} + \frac{a_{12}}{a_{11}}\right) - \frac{a_{13}x^{23}}{a_{11}x^{22}}e^{-3k}, \\ l_1(e^{-k}; c) &= 2\frac{a_{11}a_{13}(x^{12}x^{23} - x^{13}x^{22}) + x^{22}x^{23}(a_{12}a_{13} - a_{11}a_{23})}{a_{11}^2 {x^{22}}^2}e^{-3k} \\ &- \frac{a_{11}a_{33}x^{22}x^{33} - a_{13}^2 {x^{23}}^2}{a_{11}^2 {x^{22}}^2}e^{-6k} + \frac{c}{a_{11}x_{22}}e^{-2k} - \left(\frac{x^{11}x^{22} - {x^{12}}^2}{{x^{22}}^2} + \frac{a_{11}a_{22} - a_{12}^2}{a_{11}^2}\right). \end{aligned}$$

The function $l_1$ is a polynomial in $t = e^{-k}$ of degree 6, with a negative leading coefficient. If $t = e^{-k} = 1$ and $t = e^{-k} = e^{-k_0}$ are the only positive zeroes of $l_1(t)$, then $l_1(e^{-k}) \geq 0$ if and only if $0 \leq k \leq k_0$, implying the connectedness of the level curve. Thus, it is sufficient to select numbers $a_{ij}$, such that the matrix $A = (a_{ij}) \in \mathcal{P}(3)$, and

- The values of the function $l = l_0(e^{-k}; c)$ at $k = 0$ and $k = k_0$ are $0$ and $l_0$, respectively.

- The only positive zeroes of the function $l_1(t;c)$ are $t=1$ and $t=e^{-k_0}$.

We set $a_{11}=1$. The first requirement yields a linear equation system in the unknowns $a_{12}$ and $a_{13}$, with a unique set of solutions

$$a_{12}=-\frac{x^{12}}{x^{22}}-\frac{l_0e^{3k_0}}{e^{3k_0}-1},\quad a_{13}=-\frac{l_0e^{3k_0}x^{22}}{(e^{3k_0}-1)x^{23}}.$$

With $a_{11}$, $a_{12}$ and $a_{13}$ given above, the second requirement yields a linear equation system in the unknowns $a_{23}$ and $a_{33}$, resulting in a set of solutions in terms of $k_0$, $l_0$, $X$, $c$ and $a_{22}$:

$$\begin{aligned}
a_{23} &= \Big(-a_{22}e^{6k_0}{x^{22}}^2x^{23}+a_{22}{x^{22}}^2x^{23}+ce^{4k_0}x^{22}x^{23}-cx^{22}x^{23}+e^{6k_0}l_0^2{x^{22}}^2x^{23}\\
&+2e^{3k_0}l_0x^{12}x^{22}x^{23}+2e^{6k_0}l_0x^{12}x^{22}x^{23}-2e^{3k_0}l_0x^{13}{x^{22}}^2-e^{6k_0}x^{11}x^{22}x^{23}+2e^{6k_0}{x^{12}}^2x^{23}+\\
&x^{11}x^{22}x^{23}-2{x^{12}}^2x^{23}\Big)/\left(2\left(e^{3k_0}-1\right)x^{22}{x^{23}}^2\right),\\
a_{33} &= -\left(e^{3k_0}\left(-a_{22}e^{3k_0}{x^{22}}^2+a_{22}{x^{22}}^2+ce^{k_0}x^{22}-cx^{22}+e^{3k_0}l_0^2{x^{22}}^2+2e^{3k_0}l_0x^{12}x^{22}\right.\right.\\
&\left.\left.-e^{3k_0}x^{11}x^{22}+2e^{3k_0}{x^{12}}^2+x^{11}x^{22}-2{x^{12}}^2\right)\right)/\left(\left(e^{3k_0}-1\right)x^{22}x^{33}\right).
\end{aligned}$$

With $a_{23}$ and $a_{33}$ given above, the determinant $\det(a_{ij})$ forms a quadratic polynomial in $c$, with coefficients depending on $k_0, l_0, X$, and $a_{22}$. The coefficient of the $c^2$ term is

$$-\frac{\left(1+e^{k_0}\right)^2\left(1+e^{2k_0}\right)^2}{4\left(e^{k_0}+e^{2k_0}+1\right)^2{x^{23}}^2}<0.$$

We select $c=c(k_0,l_0,X,a_{22})$ to be the maximum point of this quadratic function. Assuming this, $\det(a_{ij})$ becomes a quadratic polynomial in the variable $a_{22}$, whose coefficients are in terms of $k_0, l_0$, and $X$. The coefficient of the $a_{22}^2$ term is

$$\frac{e^{4k_0}\left(e^{2k_0}{x^{23}}^2+\left(1+e^{k_0}\right)^2\left(1+e^{2k_0}\right)x^{22}x^{33}\right)}{\left(1+e^{k_0}\right)^2\left(1+e^{2k_0}\right)^2{x^{33}}^2}>0.$$

Hence, when $a_{22}$ is sufficiently large, both $\det(A)>0$ and $a_{11}a_{22}-a_{12}^2>0$ hold. This results in a positive definite matrix $A=(a_{ij})$; up to a positive scaling, it yields $\det(A)=1$.

To show that $t=1$ and $t=e^{-k_0}$ are the only positive zeroes of $l_1$, we note that the derivative $l_1'(t)$ contains terms of $t^5$, $t^2$, and $t^1$ only, implying that $l_1'(t)$ has a zero $t=0$. The other zeroes of $l_1'(t)$

satisfy $3\alpha t^4 = 3\beta t + c_0$, where

$$\alpha = \frac{a_{11}a_{33}x^{22}x^{33} - a_{13}^2 {x^{23}}^2}{a_{11}^2 {x^{22}}^2},\ \beta = \frac{a_{11}a_{13}(x^{12}x^{23} - x^{13}x^{22}) + x^{22}x^{23}(a_{12}a_{13} - a_{11}a_{23})}{a_{11}^2 {x^{22}}^2},\ c_0 = \frac{c}{a_{11}x^{22}}.$$

We have $\alpha > 0$. Moreover, $(0,0)$ lies on the level curve, thus $c = \mathrm{tr}(X^{-1} \cdot A) > 0$, implying that $c_0 > 0$. Consequently, $l_1'(t)$ has at most one positive zero, and $l_1(t)$ has at most two positive zeroes, which must be $t = 1$ and $t = e^{-k_0}$.

The remainder of the proof is divided in two cases, based on whether $a := t/s$ is rational. If $a = p/q \in \mathbb{Q}$, then the first components of points in $\Lambda = (t,1)\mathbb{Z} \oplus (s,a)\mathbb{Z}$ take discrete values $(s/q)\mathbb{Z}$. If $a \notin \mathbb{Q}$, there exists a point $(k_n, l_n) \in \Lambda$ where $k_n$ is arbitrarily large and $l_n$ is arbitrarily close to 0. Analogously to the previous case, the Dirichlet-Selberg domain $DS(X, \Gamma)$ is infinitely-sided whenever the center $X$ does not belong to the proper Zariski closed subset $\{X = (x^{ij})^{-1} \in \mathcal{P}(3) | x^{23} = 0\}$. $\square$

CHAPTER 7

# Schottky Groups in $SL(n,\mathbb{R})$

Using Dirichlet-Selberg domains in $\mathcal{P}(n)$, we extend the notion of **Schottky groups** [Mas67] to subgroups of $SL(n,\mathbb{R})$:

Definition 7.0.1. *A discrete subgroup $\Gamma < SL(n,\mathbb{R})$ is called a Schottky group of rank $k$ if there exists a point $X \in \mathcal{P}(n)$, such that the Dirichlet-Selberg domain $DS(X,\Gamma)$ is $2k$-sided and is ridge free.*

Similarly to the case of $SO^+(n,1)$, we establish the following property for Schottky groups in $SL(n,\mathbb{R})$:

Proposition 7.0.1. *Suppose that $\Gamma < SL(n,\mathbb{R})$ is a Schottky group, and the facets of $DS(X,\Gamma)$ are $Bis(X, g_i.X)$ and $Bis(X, g_i^{-1}.X)$, $i = 1,\dots,k$ for a point $X \in \mathcal{P}(n)$. Then $\Gamma$ is generated by $g_1,\dots,g_k$ and is free over those generators.*

Proof. Since $DS(X,\Gamma)$ is a fundamental domain of the discrete subgroup $\Gamma$, $\Gamma$ is generated by the facet pairing transformations of $DS(X,\Gamma)$, namely $g_1,\dots,g_k$.
To show that $\Gamma$ is free over $g_1,\dots,g_k$, we assume that

$$w = g_{i_1}^{\epsilon_1}\dots g_{i_r}^{\epsilon_r}$$

is an arbitrary reduced word in letters of $g_1,\dots,g_k$, where $i_1,\dots,i_r \in \{1,\dots,k\}$ and $\epsilon_1,\dots,\epsilon_r \in \{1,-1\}$. Since $DS(X,\Gamma)$ is bounded by the $2k$ bisectors $Bis(X, g_i^{\pm}.X)$ and is ridge-free, the complement of $\mathrm{int}(DS(X,\Gamma))$ consists of $2k$ disjoint half-spaces, namely

$$H_i^{\pm} = \{Y | s(g_i^{\pm}.X, Y) \geq s(X,Y)\},\ i = 1,\dots,k.$$

Furthermore,

$$g_i^{\pm}.(H_i^{\mp})^{\mathsf{c}} = \mathrm{int}(H_i^{\pm}),\ i = 1,\dots,k.$$

As $w$ is a reduced word, it follows that

$$g_{i_1}^{\epsilon_1}.X \in \mathrm{int}(H_{i_1}^{\epsilon_1}) \subset (H_{i_2}^{-\epsilon_2})^{\mathsf{c}},\ (g_{i_1}^{\epsilon_1} g_{i_2}^{\epsilon_2}).X \in \mathrm{int}(H_{i_2}^{\epsilon_2}) \subset (H_{i_3}^{-\epsilon_3})^{\mathsf{c}},\ \ldots, w.X \in \mathrm{int}(H_{i_r}^{\epsilon_r}).$$

Therefore, $w.X \notin DS(X, \Gamma)$, implying that $w \neq Id$. Hence, $\Gamma$ is free over the generators $g_1, \ldots, g_k$.

□

In this Chapter, we show that Schottky groups in $SL(n, \mathbb{R})$ exist in the generic case when $n$ is even and only exist in a degenerated case when $n$ is odd.

### 7.1. Schottky groups in $SL(n, \mathbb{R})$: $n$ is even

DEFINITION 7.1.1. *For any $A \in SL(n, \mathbb{R})$ with only positive eigenvalues, one defines the* ***attracting and repulsing subspaces*** *of $\mathbb{R}\mathbf{P}^{n-1}$ as follows:*

$$C_A^+ = span_{\lambda_i > 1}(\mathbf{v}_i)/\mathbb{R}^\times, \quad C_A^- = span_{0<\lambda_j<1}(\mathbf{v}_j)/\mathbb{R}^\times,$$

*where $\mathbf{v}_i$ denotes the eigenvector of $A^{\mathrm{T}}$ associated with the eigenvalue $\lambda_i$, $i = 1, \ldots, n$.*

THEOREM 7.1.2. *(cf. [**Tit72**]) Suppose that $A_1, \ldots, A_k \in SL(2n, \mathbb{R})$ are such that the attracting and repulsing spaces $C_{A_i}^\pm$, $i = 1, \ldots, k$, are all $(n-1)$-dimensional and pairwise disjoint. Then there exists an integer $M > 0$ such that the group $\Gamma = \langle A_1^M, \ldots, A_k^M \rangle$ is a Schottky group of rank $k$.*

PROOF. Denote the eigenvalues of $A_i$ by

$$\lambda_{i,1} \geq \cdots \geq \lambda_{i,n} > 1 > \lambda_{i,n+1} \geq \cdots \geq \lambda_{i,2n} > 0,\ i = 1, \ldots, k.$$

We claim that there exists an integer $M$ satisfying the following conditions:

- For any real numbers $m_i^\pm \geq M$, $i = 1, \ldots, k$, the $2k$ bisectors $Bis(I, A_i^{m_i^+}.I)$, $Bis(I, A_i^{-m_i^-}.I)$ are pairwise disjoint.
- For each bisector $\sigma$ among the $2k$ ones, the center $I$ of the Dirichlet-Selberg domain and the other $(2k-1)$ bisectors lie in the same connected component of $\sigma^{\mathsf{c}} = \mathcal{P}(2n) \backslash \sigma$.

To prove our first claim, we define

$$f_{m,i}^+(\mathbf{x}) = \lambda_{i,n+1}^m \left( \frac{||(A_i^{-m})^{\mathrm{T}} \mathbf{x}||^2}{||\mathbf{x}||^2} - 1 \right),\ f_{m,i}^-(\mathbf{x}) = \lambda_{i,n}^{-m} \left( \frac{||(A_i^m)^{\mathrm{T}} \mathbf{x}||^2}{||\mathbf{x}||^2} - 1 \right),\ \forall \mathbf{x} \in \mathbb{R}\mathbf{P}^{2n-1},$$

which are smooth functions on $\mathbb{R}\mathbf{P}^{2n-1}$. For any $\mathbf{x} \in C^{\pm}_{A_i}$,

$$\lim_{m\to\infty} f^{\pm}_{m,i}(\mathbf{x}) = 0,$$

while for any $\mathbf{x} \notin C^{\pm}_{A_i}$,

$$\lim_{m\to\infty} f^{\pm}_{m,i}(\mathbf{x}) = \infty.$$

Since $\mathbb{R}\mathbf{P}^{2n-1}$ is compact and the $(n-1)$-dimensional submanifolds $C^{\pm}_{A_i}$ are pairwise disjoint, there exists a positive number $M$, such that for any $m^{\pm}_i \geq M$, the sum of any two among the $2k$ functions $f^{\pm}_{m^{\pm}_i,i}$, $i = 1, \ldots, k$ is positive. That is, the sum of any two among the $2k$ symmetric matrices

$$\lambda^{m^+_i}_{i,n+1}((A^{m^+_i}_i.I)^{-1} - I),\ \lambda^{-m^-_i}_{i,n}((A^{-m^-_i}_i.I)^{-1} - I),\ i = 1, \ldots, k$$

is a positive definite matrix. By Theorem 4.1.3, the bisectors $Bis(A^{\pm m^{\pm}_i}_i.I, I)$ are pairwise disjoint for any numbers $m^{\pm}_i \geq M$.

To prove our second claim, we assume the opposite: there are bisectors $\sigma_1$ and $\sigma_2$ among the $2k$ bisectors $Bis(A^{\pm m^{\pm}_i}_i.I, I)$, $m^{\pm}_i \geq M$, such that $\sigma_2$ and the center $I$ lie in different components of $\sigma^{\mathsf{c}}_1$. Without loss of generality, we suppose that $\sigma_1 = Bis(A^{m_1}_1.I, I)$ and $\sigma_2 = Bis(A^{m_2}_2.I, I)$. Let $X$ be an arbitrary point in $Bis(A^{m_2}_2.I, I)$; the assumption implies that $I$ and $X$ lie in different components of $\sigma^{\mathsf{c}}_1$. Since

$$\lim_{m\to\infty} s(A^m_1.I, I) = \lim_{m\to\infty} s(A^m_1.I, X) = \infty,$$

the points $I$ and $X$ lie in the same component of $Bis(A^m_1.I, I)^{\mathsf{c}}$ for $m$ large enough. Thus, there exists a real number $m'_1 > m_1 \geq M$ such that

$$X \in Bis(A^{m'_1}_1.I, I).$$

However, $X \in Bis(A^{m_2}_2.I, I)$, implying that $Bis(A^{m_2}_2.I, I)$ and $Bis(A^{m'_1}_1.I, I)$ intersect at $X$, which contradicts our first claim. This completes the proof of our second claim.

In conclusion, there exists a number $M > 0$ such that the Dirichlet-Selberg domain $DS(I, \Gamma)$ of $\Gamma = \langle A^M_1, \ldots, A^M_k\rangle$ is bounded by the $2k$ bisectors $Bis(A^{\pm M}_i.I, I)$ and is ridge-free. Consequently, $\Gamma$ is a Schottky group of rank $k$. $\square$

## 7.2. Schottky groups in $SL(n,\mathbb{R})$: $n$ is odd

Note that for odd $n$, Schottky groups in $SL(n,\mathbb{R})$ can be obtained from these in $SL(n-1,\mathbb{R})$ via the inclusion map $SL(n-1,\mathbb{R}) \hookrightarrow SL(n,\mathbb{R})$, $g \mapsto diag(g,1)$:

EXAMPLE 7.2.1. *Consider the matrices*

$$A = diag(A_0,1) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B = diag(B_0,1) = \begin{pmatrix} 5/3 & 4/3 & 0 \\ 4/3 & 5/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

*It is evident that each pair from the four matrices $((A^{\pm}.I)^{-1} - I) = diag((A_0^{\pm}.I)^{-1} - I, 0)$ and $((B^{\pm}.I)^{-1} - I) = diag((B_0^{\pm}.I)^{-1} - I, 0)$ has a positive semi-definite linear combination. By Theorem 4.1.3, the bisectors $Bis(A^{\pm}.I, I)$ and $Bis(B^{\pm}.I, I)$ are pairwise disjoint. Moreover, for any bisector $\sigma$ among these four, the point $I$ and the other three bisectors lie in the same connected component of $\sigma^{\mathsf{c}}$. Thus, $\Gamma$ is a Schottky group. In fact, $\Gamma$ projects to the Schottky group $\langle A_0, B_0\rangle$ in $SL(2,\mathbb{R})$.*

In contrast, Bobb and Riestenberg proved that $SL(3,\mathbb{R})$ does not contain Schottky groups in the generic case. We show that for odd number $n$, $SL(n,\mathbb{R})$ does not contain Schottky groups under the following non-degeneracy assumption.

DEFINITION 7.2.2. *We call the discrete subgroup $\Gamma < SL(n,\mathbb{R})$ a non-degenerate Schottky group of rank $k$, if there is a point $X \in \mathcal{P}(n)$ satisfying the following criteria:*

- *The Dirichlet-Selberg domain $DS(X,\Gamma)$ is $2k$-sided and ridge free.*
- *For any facet $Bis(A_i.X, X)$ of $DS(X,\Gamma)$ where $A_i \in \Gamma$ and $i = 1,\dots,k$, and for any eigenvalue $\lambda_{i,j}$ of $A_i$ where $j = 1,\dots,n$, the absolute value $|\lambda_{i,j}| \neq 1$.*

THEOREM 7.2.3. *For any odd number $n$, there are no non-degenerate Schottky groups in $SL(n,\mathbb{R})$.*

PROOF. Assume to the contrary that $\Gamma = \langle A_1,\dots,A_k\rangle \subset SL(n,\mathbb{R})$ is a non-degenerate Schottky group of rank $k$, i.e., there exists a point $X \in \mathcal{P}(n)$ such that the Dirichlet-Selberg domain $DS(X,\Gamma)$ is ridge-free, with $2k$ facets $Bis(A_i.X, X)$ and $Bis(A_i^{-1}.X, X)$, $i = 1,\dots,k$. Without loss of generality, we can assume that $X = I$ after an isometry of the Dirichlet-Selberg domain $DS(X,\Gamma)$; the Dirichlet-Selberg domain after the isometry corresponds to a subgroup conjugate to $\Gamma$.

We extend the notions of attracting and repulsing subspaces:

$$C^+_{A_i,\mathbb{C}} = span_{\mathbb{C},|\lambda_j|>1}(\mathbf{v}_j)/\mathbb{C}^\times,\ C^-_{A_i,\mathbb{C}} = span_{\mathbb{C},|\lambda_j|<1}(\mathbf{v}_j)/\mathbb{C}^\times,$$

where $\mathbf{v}_j \in \mathbb{C}^n$ is the eigenvector of $A_i^{\mathrm{T}}$ associated with the eigenvalue $\lambda_j$. Then, $C^+_{A_i,\mathbb{C}}$ and $C^-_{A_i,\mathbb{C}}$ are proper subspaces of $\mathbb{C}\mathbf{P}^{n-1}$. Since $\Gamma$ is assumed to be non-degenerate, $\dim_{\mathbb{C}}(C^+_{A_i,\mathbb{C}})+\dim_{\mathbb{C}}(C^-_{A_i,\mathbb{C}}) = n-2$, $i = 1,\dots,k$. That is, either $\dim_{\mathbb{C}}(C^+_{A_i,\mathbb{C}}) \geq (n-1)/2$ or $\dim_{\mathbb{C}}(C^-_{A_i,\mathbb{C}}) \geq (n-1)/2$. Without loss of generality, we can assume that $\dim_{\mathbb{C}}(C^+_{A_1,\mathbb{C}}), \dim_{\mathbb{C}}(C^+_{A_2,\mathbb{C}}) \geq (n-1)/2$. Consequently, the intersection

$$C^+_{A_1,\mathbb{C}} \cap C^+_{A_2,\mathbb{C}} \neq \varnothing.$$

On the one hand, for any $m \in \mathbb{N}$, the bisectors $Bis(A_1^m.I, I)$ and $Bis(A_2^m.I, I)$ are disjoint. For $m = 1$, this follows from the definition of Schottky groups. For $m \geq 2$, we note that the bisectors $Bis(A_i^m.I, I)$, $i = 1,\dots,k$ do not intersect the Dirichlet-Selberg domain $DS(I,\Gamma)$. In other words, $Bis(A_i^m.I, I)$ lies in $DS(I,\Gamma)^{\mathsf{c}}$. Therefore, the bisector $Bis(A_i^m.I, I)$ and the point $A_i^m.I$ lie in the same connected component of $DS(I,\Gamma)^{\mathsf{c}}$. Since $A_i^m.I$ lies in the component

$$\mathrm{int}(H_i^+) = \{Y | s(A_i.I, Y) > s(I, Y)\},$$

the bisectors $Bis(A_1^m.I, I)$ and $Bis(A_2^m.I, I)$ belong to different components of $DS(I,\Gamma)^{\mathsf{c}}$. Thus, they are disjoint.

On the other hand, we will derive a contradiction by showing that the bisectors $Bis(A_1^m.I, I)$ and $Bis(A_2^m.I, I)$ intersect for sufficiently large $m \in \mathbb{N}$. Take vectors

$$\mathbf{v} \in C^+_{A_1,\mathbb{C}} \cap C^+_{A_2,\mathbb{C}},\ \mathbf{w} \in (C^+_{A_1,\mathbb{C}} \cup C^+_{A_2,\mathbb{C}})^{\mathsf{c}}.$$

Similarly to the proof of Theorem 7.1.2, we establish that

$$\mathbf{w}^*((A_1^m.I)^{-1} - I)\mathbf{w} > 0$$

and

$$\mathbf{w}^*((A_2^m.I)^{-1} - I)\mathbf{w} > 0,$$

for sufficiently large $m$. Furthermore,

$$\mathbf{v}^*(A_1^m.I)^{-1}\mathbf{v} = ||(A_1^{-m})^{\mathrm{T}}\mathbf{v}||^2 = ||\varphi^m(\mathbf{v})||^2 \leq ||\varphi^m||^2 \cdot ||\mathbf{v}||^2,$$

where $\varphi$ represents the restriction of the linear transformation $(A_1^{-1})^{\mathrm{T}}$ to the $A_1^{\mathrm{T}}$-invariant subspace $\mathbb{C} \cdot C^+_{A_1,\mathbb{C}}$ of $\mathbb{C}^n$. Gelfand's theorem implies that

$$\lim_{m\to\infty} ||\varphi^m||^{1/m} = \rho(\varphi) = \max_{|\lambda_j|>1} |\lambda_j|^{-1} < 1,$$

where $\rho(\varphi)$ denotes the spectral radius of $\varphi$. It follows that $\lim_{m\to\infty} ||\varphi^m|| = 0$, and consequently

$$\lim_{m\to\infty} \mathbf{v}^*(A_1^m)^{-1}\mathbf{v} = 0.$$

Similarly,

$$\lim_{m\to\infty} \mathbf{v}^*(A_2^m)^{-1}\mathbf{v} = 0.$$

Thus, inequalities

$$\mathbf{v}^*((A_1^m.I)^{-1} - I)\mathbf{v} < 0$$

and

$$\mathbf{v}^*((A_2^m.I)^{-1} - I)\mathbf{v} < 0$$

hold for sufficiently large $m$.

The inequalities above imply that the pencil

$$((A_1^m.I)^{-1} - I, (A_2^m.I)^{-1} - I)$$

is indefinite for sufficiently large $m$. According to Theorem 4.1.3, the bisectors $Bis(A_1^m.I, I)$ and $Bis(A_2^m.I, I)$ intersect for sufficiently large $m$, leading to a contradiction. □

APPENDIX A

# Motivation for the Angle-like Function

This appendix describes how we discovered the equation (3.2), particularly the methodology employed to identify a suitable function:

$$\theta:\ \Sigma_n^{(2)} \sqcup \Sigma_n^{(0)} \to [0,\pi],\ (A,B) \mapsto \theta(A^\perp, B^\perp)$$

which fulfills the properties (1) through (4) in Definition 3.1.1. Here, $\Sigma_n^{(2)}$ represents the set consisting of pairs $(A,B) \in Sym_n(\mathbb{R})^2$ for which the corresponding co-oriented hyperplane pair $(A^\perp, B^\perp)$ is of type (2), and that the generalized eigenvalues of the pencil $(A,B)$ are pairwise distinct. Furthermore, $\Sigma_n^{(0)}$ represents the set consisting of pairs $(A,B)$ in $Sym_n(\mathbb{R})^2$, such that $A, B \neq O$ and $B = c \cdot A$ for some $c \in \mathbb{R}$; that is, the corresponding co-oriented hyperplanes $A^\perp$ and $B^\perp$ are either identical or oppositely co-oriented.

We begin by observing that a restriction of $\theta$ factors through $\mathbf{S}^1 \times \mathbf{S}^1$. Here and after, we consistently regard $\mathbf{S}^1$ as $\mathbb{R}/2\pi\mathbb{Z}$.

Lemma A.0.0.1. *For any pair $(A,B) \in \Sigma_n^{(2)}$, $\theta|_{(span(A,B)-\{O\})^2}$ factors through $\mathbf{S}^1 \times \mathbf{S}^1$ via*

$$(span(A,B) - \{O\})^2 \xrightarrow{\varphi\times\varphi} \mathbf{S}^1 \times \mathbf{S}^1 \xrightarrow{\angle} [0,\pi],$$

*where $\varphi$ is a map $\varphi_{(A,B)} : span(A,B) - \{O\} \to \mathbf{S}^1$, and $\angle : \mathbf{S}^1 \times \mathbf{S}^1 \to [0,\pi]$ is the Euclidean angle.*

Proof. Consider an arbitrary pair $(A,B) \in \Sigma_n^{(2)}$. We define the map $\varphi_{(A,B)}$ as follows:

- For any $c > 0$, we have $\varphi(c \cdot A) = 0$; for any $c < 0$, $\varphi(c \cdot A) = \pi$.
- If $C$ is a positive linear combination of $B$ and $A$, or $B$ and $-A$, then $\varphi(C) = \theta(A^\perp, C^\perp)$.
- If $C$ is a positive linear combination of $-B$ and $A$ or $-B$ and $-A$, then $\varphi(C) = -\theta(A^\perp, C^\perp)$.

The definition of $\varphi$ ensures that $\angle(\varphi(C_1), \varphi(C_2)) = \theta(C_1^\perp, C_2^\perp)$ whenever $C_1 \in span(A)$ or $C_2 \in span(A)$.

Next, we will show that $\angle(\varphi(C_1), \varphi(C_2)) = \theta(C_1^\perp, C_2^\perp)$ holds for any $C_1, C_2 \in span(A, B) - span(A)$. After a positive scaling of $C_1$ and $C_2$, we can assume that $C_1 = \epsilon_1 B + c_1 A$ and $C_2 = \epsilon_2 B + c_2 A$, where $c_1, c_2 \in \mathbb{R}$, and $\epsilon_1, \epsilon_2 \in \{1, -1\}$. The proof is divided in three cases:

**Case (1).** Assume that $\epsilon_1 = \epsilon_2 = 1$ and $c_1 > c_2$. In this case, $C_1$ is a positive linear combination of $A$ and $C_2$. The property (4) of $\theta$ in Definition 3.1.1 implies that

$$\theta(A^\perp, C_2^\perp) - \theta(A^\perp, C_1^\perp) = \theta(C_1^\perp, C_2^\perp) > 0,$$

which leads to $0 < \varphi(C_1) < \varphi(C_2) < \pi$. Hence, the angle is computed as

$$\angle(\varphi(C_1), \varphi(C_2)) = \varphi(C_2) - \varphi(C_1) = \theta(A^\perp, C_2^\perp) - \theta(A^\perp, C_1^\perp) = \theta(C_1^\perp, C_2^\perp).$$

**Case (2).** If $\epsilon_1 = \epsilon_2 = -1$, we can show that $\angle(\varphi(C_1), \varphi(C_2)) = \theta(C_1^\perp, C_2^\perp)$, following a reasoning analogous to the Case (1).

**Case (3).** Assume that $\epsilon_1 = 1$ and $\epsilon_2 = -1$. When $c_1 + c_2 = 0$, it follows that $C_2 = -C_1$. The property (3) of $\theta$ in Definition 3.1.1 implies that

$$\varphi(C_2) = -\theta(A^\perp, -C_1^\perp) = -\pi + \theta(A^\perp, C_1^\perp) = \varphi(C_2) - \pi,$$

leading to the conclusion that $\angle(\varphi(C_1), \varphi(C_2)) = \pi = \theta(C_1^\perp, C_2^\perp)$.

When $c_1 + c_2 > 0$, $A$ is a positive linear combination of $C_1$ and $C_2$. The property (4) of $\theta$ in Definition 3.1.1 implies that

$$\theta(C_1^\perp, C_2^\perp) = \theta(A^\perp, C_1^\perp) + \theta(A^\perp, C_2^\perp) = \varphi(C_1) - \varphi(C_2).$$

Therefore, $\varphi(C_1)$ and $\varphi(C_2) \in \mathbf{S}^1$ satisfy that $0 < \varphi(C_1) - \varphi(C_2) < \pi$, leading to the conclusion:

$$\theta(C_1^\perp, C_2^\perp) = \varphi(C_1) - \varphi(C_2) = \angle(\varphi(C_1), \varphi(C_2)).$$

When $c_1 + c_2 < 0$, we can show that $\angle(\varphi(C_1), \varphi(C_2)) = \theta(C_1^\perp, C_2^\perp)$, analogously to the preceding argument. $\square$

To construct an invariant angle function $\theta$ on $\Sigma_n^{(2)} \sqcup \Sigma_n^{(0)}$, we aim at constructing a family of maps $\varphi_{(A,B)} : span(A, B) - \{O\} \to \mathbf{S}^1$ for $(A, B) \in \Sigma_n^{(2)}$ that satisfies particular properties. We note that the definition of invariant angle function requires a connection between pairs in $\Sigma_n^{(2)}$.

Specifically, the property (2) in Definition 3.1.1 suggests a connection between the pairs $(A, B)$ and $(g.A, g.B)$ for any $g \in SL(n, \mathbb{R})$. Moreover, the pairs $(A, B)$ and $(A', B')$ are inherently related when $span(A', B') = span(A, B)$. This observation motivates us to introduce an equivalence relation on $\Sigma_n^{(2)}$, formalized as follows:

DEFINITION A.0.1. *Two pairs $(A, B)$, $(A', B') \in \Sigma_n^{(2)}$ are equivalent if*

$$(A', B') = (g.(pA + qB), g.(rA + sB)),$$

*for an element $g \in SL(n, \mathbb{R})$ and numbers $p, q, r, s \in \mathbb{R}$ with $ps - qr \neq 0$. We denote this equivalence relation by $(A, B) \sim (A', B')$.*

The concept of the cross-ratio plays an important role in characterizing the equivalence classes on $\Sigma_n^{(2)}$, as will be introduced in Section A.1.

## A.1. Further insights to the family of functions $\varphi_{(A,B)}$

We refer to [Har67] for the following concepts and propositions related to the cross-ratio:

DEFINITION A.1.1. *The **cross-ratio** of four distinct points $p_i = [x_i : y_i] \in \mathbb{R}\mathbf{P}^1$, $i = 1, 2, 3, 4$, is defined as*

$$R_\times(p_1, p_2; p_3, p_4) = \frac{(x_3 y_1 - y_3 x_1)(x_4 y_2 - y_4 x_2)}{(x_3 y_2 - y_3 x_2)(x_4 y_1 - y_4 x_1)}.$$

*For points $\mathbf{x}_i \in \mathbb{R}^2 - \{(0, 0)\}$, the cross ratio $R_\times(\mathbf{x}_1, \mathbf{x}_2; \mathbf{x}_3, \mathbf{x}_4)$ denotes $R_\times([\mathbf{x}_1], [\mathbf{x}_2]; [\mathbf{x}_3], [\mathbf{x}_4])$, where $[\mathbf{x}_i] \in \mathbb{R}\mathbf{P}^1$, for $i = 1, 2, 3, 4$.*

PROPOSITION A.1.1. *Consider points $p_i \in \mathbb{R}\mathbf{P}^1$ for $i = 1, 2, 3, 4$. The cross-ratio $R_\times(p_1, p_2; p_3, p_4)$ is greater than $1$ if and only if the points $p_1$, $p_2$, $p_3$, and $p_4$ are arranged on $\mathbb{R}\mathbf{P}^1$ according to their cyclic order.*

PROPOSITION A.1.2. *Identify points $p_i \in \mathbb{R}\mathbf{P}^1$ as one-dimensional linear subspaces of $\mathbb{R}^2$, where $i = 1, 2, 3, 4$. Denote by $\theta_i$ the angle between $p_i$ and $p_{i+1}$ for $i = 1, 2, 3$. The cross-ratio $R_\times(p_1, p_2; p_3, p_4)$ is expressed as a function of $\theta_i$:*

$$R_\times(p_1, p_2; p_3, p_4) = R_\times(\theta_1, \theta_2, \theta_3) := \frac{\sin(\theta_1 + \theta_2)\sin(\theta_2 + \theta_3)}{\sin\theta_2 \sin(\theta_1 + \theta_2 + \theta_3)}.$$

Proposition A.1.3. *Consider two-dimensional vector spaces $V$ and $W$ and an invertible linear map $\varphi : V \to W$. Let $p_i \in V$ and $q_i \in W$ represent distinct points for $i = 1, 2, 3, 4$. Assuming that $q_i = \varphi(p_i)$ for $i = 1, 2, 3$, then the equality*

$$R_\times(p_1, p_2; p_3, p_4) = R_\times(q_1, q_2; q_3, q_4)$$

*holds if and only if $q_4$ is a non-zero multiple of $\varphi(p_4)$.*

Remark A.1.1. *Invertible linear maps on $\mathbb{R}^2$ correspond to Möbius transformations on $\mathbb{R}\mathbf{P}^1$. Thus, Proposition A.1.3 remains valid when the parts "two-dimensional spaces" and "invertible linear map" are replaced by "projective lines" and "Möbius transformation" respectively.*

The cross-ratios of consecutive points among $n \geq 4$ points in $\mathbb{R}\mathbf{P}^1$ depend on each other:

Lemma A.1.1.1. *Consider $n \geq 4$ distinct points $p_1, \ldots, p_n \in \mathbb{R}\mathbf{P}^1$, denote $R_i = R_\times(p_i, p_{i+1}; p_{i+2}, p_{i+3})$ for $i = 1, \ldots, n$, with indices being taken modulo $n$. It follows that $R_{n-2}$, $R_{n-1}$, and $R_n$ are determined by $R_1$ through $R_{n-3}$.*

Proof. A unique Möbius transformation exists that takes $p_1$, $p_2$ and $p_3$ to 0, 1 and $\infty$, respectively. As noted in Remark A.1.1, Möbius transformations preserve the cross-ratios among points in $\mathbb{R}\mathbf{P}^1$. Consequently, we may assume that $(p_1, p_2, p_3) = (0, 1, \infty)$ without loss of generality. Under this assumption, equations $R_i = R_\times(p_i, p_{i+1}; p_{i+2}, p_{i+3})$, $i = 1, \ldots, n-3$ uniquely determine the points $p_4$ through $p_n$. Hence, the cross-ratios $R_{n-2}$, $R_{n-1}$ and $R_n$ are also uniquely determined. $\square$

We return to the equivalence classes in $\Sigma_n^{(2)}$.

Definition A.1.2. *Define $M'_n$ as the $(n-3)$-dimensional space given by*

$$M'_n = \{(R_1, \ldots, R_n) \in \mathbb{R}^n_{>1} | R_i = R_i(R_1, \ldots, R_{n-3}), i = n-2, n-1, n\},$$

*where $R_i(R_1, \ldots, R_{n-3})$, for $i = n-2, n-1, n$, are as described in Lemma A.1.1.1. Introduce the following equivalence relations on $M'_n$:*

$$(R_1, \ldots, R_n) \sim (R_n, \ldots, R_1),\ (R_1, \ldots, R_n) \sim (R_{i+1}, \ldots, R_{i+n}),\ i = 1, \ldots, n-1,$$

*with indices taken modulo $n$. Define $M_n = M'_n / \sim$ as the quotient space by these equivalence relations.*

*For a point in $M_n$ represented by $(R_1, \ldots, R_n) \in M'_n$, denote by $\Sigma_n^{R_1,\ldots,R_n}$ the set consisting of pairs $(A, B) \in \Sigma_n^{(2)}$, such that the generalized eigenvalues of $(A, B)$ are ordered as $\lambda_n > \cdots > \lambda_1$, and their cross-ratios*

$$R'_i := R_\times(\lambda_i, \lambda_{i+1}; \lambda_{i+2}, \lambda_{i+3}),\ i = 1, \ldots, n,$$

*satisfy that $(R_1, \ldots, R_n) \sim (R'_1, \ldots, R'_n)$. For conciseness, we omit the last three components from the superscript and denote the set above by $\Sigma_n^{R_1,\ldots,R_{n-3}}$.*

Lemma A.1.2.1. *Each equivalence class in $\Sigma_n^{(2)}$ is contained in $\Sigma_n^{R_1,\ldots,R_{n-3}}$ for a specific element $(R_1, \ldots, R_n) \in M_n$.*

Proof. For each $(A, B) \in \Sigma_n^{(2)}$ with eigenvalues $\lambda_n > \cdots > \lambda_1$, Proposition A.1.1 implies that the cross-ratios $R_i = R_\times(\lambda_i, \lambda_{i+1}; \lambda_{i+2}, \lambda_{i+3}) > 1$ for $i = 1, \ldots, n$.
Consider a pair $(A, B) \in \Sigma_n^{R_1,\ldots,R_{n-3}}$, where $(R_1, \ldots, R_n) \in M_n$. According to Proposition 2.2.4, the generalized eigenvalues of $(g.A, g.B)$ are equal to those of $(A, B)$ for any $g \in SL(n, \mathbb{R})$. Thus, $(g.A, g.B) \in \Sigma_n^{R_1,\ldots,R_{n-3}}$. Moreover, for any $p, q, r, s \in \mathbb{R}$ with $ps - qr \neq 0$, Lemma 2.3.3.1 demonstrates that the generalized eigenvalues of $(pA + qB, rA + sB)$ are equal to $(p\lambda_i + q)/(r\lambda_i + s)$, where $i = 1, \ldots, n$, which are the images of $\lambda_1, \ldots, \lambda_n$ under a Möbius transformation. According to Remark A.1.1, Möbius transformations preserve the cross-ratios. Additionally, such transformations alter the order of $\lambda_i$, $i = 1, \ldots, n$ by a cyclic permutation with/or a reversal, thus they alter the order of $R_\times(\lambda_i, \lambda_{i+1}; \lambda_{i+2}, \lambda_{i+3})$, $i = 1, \ldots, n$ in the same manner. Such permutations preserve the equivalence classes in $M'_n$. Hence, $(pA + qB, rA + sB) \in \Sigma_n^{R_1,\ldots,R_{n-3}}$ as well. $\square$

Remark A.1.2. *For $(A, B) \in \Sigma_n^{R_1,\ldots,R_{n-3}}$, the set of singular matrices in span$(A, B)$ consists of $n$ lines, specifically span$(A - \lambda_i B)$, where $\lambda_i$, $i = 1, \ldots, n$ are the generalized eigenvalues of $(A, B)$. Additionally, $R_i = R_\times(A - \lambda_i B, A - \lambda_{i+1} B; A - \lambda_{i+2} B, A - \lambda_{i+3} B)$ for $i = 1, \ldots, n$.*

We make a further assumption on the family of maps $\varphi_{(A,B)}$. For any $(R_1, \ldots, R_n) \in M_n$, we assume that the angles $\theta_i = \angle(\varphi_{(A,B)}(C_i), \varphi_{(A,B)}(C_{i+1}))$ remain constant despite different selections of $(A, B) \in \Sigma_n^{R_1,\ldots,R_{n-3}}$. Here, $C_1, \ldots, C_n$ are the generators of the $n$ lines of singular matrices in *span*$(A, B)$, with $R_i = R_\times(C_i, C_{i+1}; C_{i+2}, C_{i+3})$, $i = 1, \ldots, n$. According to Proposition A.1.2, the angles $\theta_i$, $i = 1, \ldots, n$, satisfy the following condition:

$$\begin{aligned} &\sum_{i=1}^{n} \theta_i = \pi, \\ &R_\times(\theta_i, \theta_{i+1}, \theta_{i+2}) = R_i, \ \forall i = 1, \ldots, n. \end{aligned} \tag{A.1}$$

Denote by $\Theta$ a proposed function that takes $(R_1, \ldots, R_n) \in M'_n$ to $(\theta_1, \ldots, \theta_n) \in (\mathbf{S}^1)^n$ satisfying (A.1). Note that a cyclic permutation of the elements $C_1, \ldots, C_n$ results in identical permutations of both $(R_1, \ldots, R_n)$ and $(\theta_1, \ldots, \theta_n)$. Consequently, we establish another condition:

$$\begin{aligned} &\Theta(R_{1+j}, \ldots, R_{n+j}) = (\theta_{1+j}, \ldots, \theta_{n+j}), \ \forall j = 0, \ldots, n-1, \\ &\Theta(R_n, \ldots, R_1) = (\theta_2, \theta_1, \theta_n, \ldots, \theta_3). \end{aligned} \tag{A.2}$$

We describe the requirements for the family of maps $\varphi_{(A,B)}$ as below.

Proposition A.1.4. *Suppose that there exists a map $\Theta : M'_n \to (\mathbf{S}^1)^n$ satisfying conditions (A.1) and (A.2). Moreover, assume that for each tuple $(R_1, \ldots, R_n) \in M'_n$ and and the corresponding output $(\theta_1, \ldots, \theta_n) = \Theta(R_1, \ldots, R_n)$, there is a family of maps $\varphi_{(A,B)} : span(A, B) - \{O\} \to \mathbf{S}^1$ for $(A, B) \in \Sigma_n^{R_1,\ldots,R_{n-3}}$ satisfying the criteria:*

1. *The map $\varphi_{(A,B)}$ is the composition of a linear map $\psi_{(A,B)} : span(A, B) \to \mathbb{R}^2$ and the canonical quotient map $\mathbb{R}^2 - \{O\} \to (\mathbb{R}^2 - \{O\})/\mathbb{R}^+ = \mathbf{S}^1$.*
2. *Suppose that $C_1, \ldots, C_n$ is any set of pairwise linearly independent singular elements in $span(A, B)$ satisfying that*

$$R_\times(C_i, C_{i+1}; C_{i+2}, C_{i+3}) = R_i,$$

*for $i = 1, \ldots, n$, and $C_2, \ldots, C_{n-1}$ represent positive linear combinations of $C_1$ and $C_n$, where the indices are taken modulo $n$. Then the relationship*

$$\varphi_{(A,B)}(C_i) = \eta + \sum_{j=1}^{i-1} \theta_j, \ i = 1, \ldots, n, \tag{A.3}$$

*holds for a specific $\eta = \eta(C_1, \ldots, C_n) \in \mathbf{S}^1$.*

*Under these criteria, the function*

$$\theta(A^\perp, B^\perp) := \angle(\varphi_{(A,B)}(A), \varphi_{(A,B)}(B))$$

*is well-defined and serves as an invariant angle function.*

PROOF. We begin by proving the uniqueness, up to an additive constant, of the map $\varphi_{(A,B)}$. Assuming that $\varphi_{(A,B)}(C_1) = 0$, the criteria (1) and (2) imply the expressions

$$\psi_{(A,B)}(C_1) = k_1(1,0),\ \psi_{(A,B)}(C_2) = k_2(\cos\theta_1, \sin\theta_1),\ \psi_{(A,B)}(C_n) = k_n(-\cos\theta_n, \sin\theta_n),$$

where $k_1$, $k_2$, and $k_n$ are positive scalars. This determines a unique linear map $\psi_{(A,B)}$ up to a positive multiple, implying the uniqueness of $\varphi_{(A,B)}$ under the assumption $\varphi_{(A,B)}(C_1) = 0$. Consequently, $\varphi_{(A,B)}$ is unique up to a constant without this assumption. Thus, the function $\theta$ is uniquely determined and is independent of $\eta$.

Properties (1), (3) and (4) in Definition 3.1.1 are inherently fulfilled by the angular nature of $\theta$. Regarding the property (2), suppose that a choice of singular elements $C_1, \ldots, C_n$ in $span(A, B)$ satisfies the criteria in the proposition. Thus, the elements $g.C_1, \ldots, g.C_n$ satisfy these criteria for $span(g.A, g.B)$. The uniqueness of $\varphi_{(g.A,g.B)}$ implies that

$$\varphi_{(g.A,g.B)}(g.C) = \varphi_{(A,B)}(C),$$

for any $C \in span(A, B) - \{O\}$. Hence,

$$\theta((g.A)^{\perp}, (g.B)^{\perp}) = \angle(\varphi_{(g.A,g.B)}(g.A), \varphi_{(g.A,g.B)}(g.B)) = \angle(\varphi_{(A,B)}(A), \varphi_{(A,B)}(B)) = \theta(A^{\perp}, B^{\perp}).$$

□

REMARK A.1.3. *The proof of the existence of the map $\varphi_{(A,B)}$, guaranteed by the equation (A.1), is omitted here for conciseness.*

The objective now shifts to identifying an appropriate map $\Theta : M'_n \to (\mathbf{S}^1)^n$. We will deal with this in the next section, focusing on cases with small values of $n$.

## A.2. Examples: cases $n = 3$, $4$, and $5$

In this section, we demonstrate the derivation of the formula (3.2) when $n$ takes the values 3, 4, and 5.

EXAMPLE A.2.1. *Suppose that $n = 3$. In this case, $M_3$ is reduced to a point. Consider any pair $(A, B) \in \Sigma_3^{(2)}$ whose generalized eigenvalues are arranged as $\lambda_1 < \lambda_2 < \lambda_3$. Both choices of singular* *elements $C_i$, $i = 1,2,3$, satisfy the criterion (2) in Proposition A.1.4:*

$$C_1 = A - \lambda_1 B,\ C_2 = A - \lambda_2 B,\ C_3 = A - \lambda_3 B,$$

*or*

$$C_1 = A - \lambda_2 B,\ C_2 = A - \lambda_3 B,\ C_3 = -(A - \lambda_1 B).$$

*Thus the function $\varphi_{(A,B)}$ satisfies (A.3) for either choice of $C_i$, yielding that*

$$(\eta, \eta + \theta_1, \eta + \theta_1 + \theta_2) = (\eta' + \theta_1, \eta' + \theta_1 + \theta_2, \eta' + \pi),$$

*where $\eta, \eta'$ are specific elements in $\mathbf{S}^1$. This condition implies that $\theta_1 = \theta_2 = \theta_3 = \pi/3$, a choice that also satisfies the equation A.2. Consequently, the criterion (1) in Proposition A.1.4 implies that*

$$\psi_{(A,B)}(A - \lambda_i B) = k_i \left( \cos\left(\frac{(i-1)\pi}{3}\right), \sin\left(\frac{(i-1)\pi}{3}\right)\right), \quad i = 1,2,3,$$

*where $k_i$, $i = 1,2,3$ are positive numbers. To determine these numbers, we note that the matrices $C_i = A - \lambda_i B$, $i = 1,2,3$, are linearly dependent, expressed as:*

$$\frac{1}{\lambda_3 - \lambda_1}(C_3 - C_1) = \frac{1}{\lambda_2 - \lambda_1}(C_2 - C_1).$$

*Therefore,*

$$\frac{1}{\lambda_3 - \lambda_1}(\psi_{(A,B)}(C_3) - \psi_{(A,B)}(C_1)) = \frac{1}{\lambda_2 - \lambda_1}(\psi_{(A,B)}(C_2) - \psi_{(A,B)}(C_1)).$$

*Combining these equations, we derive a solution, which is unique up to a positive multiple:*

$$k_1 = \frac{1}{\lambda_3 - \lambda_2}, k_2 = \frac{1}{\lambda_3 - \lambda_1}, k_3 = \frac{1}{\lambda_2 - \lambda_1}.$$

*Therefore, we determine $\psi_{(A,B)}(A)$ and $\psi_{(A,B)}(B)$ up to a positive multiple, particularly determining the angle between them. Straightforward computations show that*

$$\begin{aligned}
&\cos\theta(A^\perp, B^\perp) = \cos\angle(\psi_{(A,B)}(A), \psi_{(A,B)}(B)) \\
\text{(A.4)}\qquad &= \frac{\lambda_1^2\lambda_2 + \lambda_2^2\lambda_3 + \lambda_3^2\lambda_1 + \lambda_1\lambda_2^2 + \lambda_2\lambda_3^2 + \lambda_3\lambda_1^2 - 6\lambda_1\lambda_2\lambda_3}{2\sqrt{\begin{array}{l}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - \lambda_1\lambda_2 - \lambda_2\lambda_3 - \lambda_3\lambda_1) \\ \quad (\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 - \lambda_1^2\lambda_2\lambda_3 - \lambda_2^2\lambda_3\lambda_1 - \lambda_3^2\lambda_1\lambda_2)\end{array}}}.
\end{aligned}$$

Example A.2.2. *Consider the case where* $n = 4$. *Let* $p_1$, $p_2$, $p_3$, *and* $p_4 \in \mathbb{R}\mathbf{P}^1$ *and denote their cross-ratio by* $R = R_\times(p_1, p_2; p_3, p_4)$. *The cross-ratios of these points in different orders are as follows:*

$$R_\times(p_2, p_3; p_4, p_1) = \frac{R}{R-1},\ R_\times(p_3, p_4; p_1, p_2) = R,\ R_\times(p_4, p_1; p_2, p_3) = \frac{R}{R-1}.$$

*Therefore,* $M_4 = \mathbb{R}_{>1}/\{R \sim \frac{R}{R-1}\}$. *For any* $R \in M_4$ *and* $(A, B) \in \Sigma_4^R$ *with generalized eigenvalues* $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$, *both choices of the singular elements* $C_i$, $i = 1, 2, 3, 4$, *satisfy the criterion (2) in Proposition A.1.4 with* $R_\times(C_1, C_2; C_3, C_4) = R$*:*

$$C_1 = A - \lambda_1 B,\ C_2 = A - \lambda_2 B,\ C_3 = A - \lambda_3 B,\ C_4 = A - \lambda_4 B,$$

*or*

$$C_1 = A - \lambda_3 B,\ C_2 = A - \lambda_4 B,\ C_3 = -(A - \lambda_1 B),\ C_4 = -(A - \lambda_2 B).$$

*Thus, the function* $\varphi_{(A,B)}$ *satisfies* (A.3) *for both choices of* $C_i$. *Similarly to Example A.2.1, this condition implies that*

$$\theta_1 = \theta_3,\ \theta_2 = \theta_4.$$

*This condition also satisfies equation* (A.2). *Moreover, Proposition A.1.2 implies that*

$$R = R_\times(\theta_1, \theta_2, \theta_3) = \frac{\sin \pi/2 \cdot \sin \pi/2}{\sin(\pi/2 - \theta_1)\sin(\pi/2 + \theta_1)} = \frac{1}{\cos^2 \theta_1},$$

*hence we have:*

$$\theta_1 = \theta_3 = \arccos \sqrt{1/R},\ \theta_2 = \theta_4 = \arcsin \sqrt{1/R}.$$

*Consequently, the criterion (1) in Proposition A.1.4 implies that* $\psi_{(A,B)} : span(A, B) \to \mathbb{R}^2$ *is characterized by:*

$$\psi_{(A,B)}(A - \lambda_1 B) = k_1(1, 0),\ \psi_{(A,B)}(A - \lambda_2 B) = k_2(1, \sqrt{R-1}),$$

$$\psi_{(A,B)}(A - \lambda_3 B) = k_3(0, 1),\ \psi_{(A,B)}(A - \lambda_4 B) = k_4(-\sqrt{R-1}, 1),$$

*for some positive numbers $k_i$, $i = 1, 2, 3, 4$. By a straightforward computation similar to Example A.2.1, we determine these numbers and derive the following formula:*

$$\begin{aligned} \cos\theta(A^\perp, B^\perp) &= \cos\angle(\psi_{(A,B)}(A), \psi_{(A,B)}(B)) \\ &= \frac{\lambda_4\lambda_2 - \lambda_3\lambda_1}{\sqrt{(\lambda_4 - \lambda_3 + \lambda_2 - \lambda_1)(\lambda_4\lambda_3\lambda_2 - \lambda_4\lambda_3\lambda_1 + \lambda_4\lambda_2\lambda_1 - \lambda_3\lambda_2\lambda_1)}}. \end{aligned} \tag{A.5}$$

Example A.2.3. *We consider the case where $n = 5$. For points $p_i \in \mathbb{R}\mathbf{P}^1$, $i = 1, \dots, 5$, we define the cross-ratios as usual:*

$$R_i = R_\times(p_i, p_{i+1}; p_{i+2}, p_{i+3}).$$

*As asserted by Lemma A.1.1.1, cross-ratios $R_3$, $R_4$ and $R_5$ are dependent on $R_1$ and $R_2$. This is clear from the following relationships:*

$$R_3 = \frac{R_2}{R_1(R_2 - 1)}, \ R_4 = \frac{1}{R_1 + R_2 - R_1R_2}, \ R_5 = \frac{R_1}{R_2(R_1 - 1)}.$$

*For any $(A, B) \in \Sigma_5^{R_1,R_2}$ with generalized eigenvalues $\lambda_1 < \cdots < \lambda_5$, the following choice of singular elements $C_i$, $i = 1, \dots, 5$, satisfies the assumption in criterion (2) with cross-ratios $(R_{1+j}, \dots, R_{5+j}) \in M_5'$:*

$$C_1 = A - \lambda_{j+1}B, \ \dots, \ C_{5-j} = A - \lambda_5 B, \ C_{6-j} = -(A - \lambda_1 B), \ \dots, \ C_5 = -(A - \lambda_j B),$$

*where $j = 0, \dots, 4$, and the indices are taken modulo 5.*

*To find a function $\Theta$ satisfying equations (A.1) and (A.2), we note that $\theta_3$, $\theta_4$ and $\theta_5$ are dependent on $\theta_1$ and $\theta_2$ according to Proposition A.1.2. Specifically, if we set $c_i = \cot\theta_i$, $i = 1, \dots, 5$, then the second equation in (A.1) implies that*

$$R_i = \frac{(c_{i+1} + c_i)(c_{i+1} + c_{i+2})}{c_ic_{i+1} + c_ic_{i+2} + c_{i+1}c_{i+2} - 1}, \quad i = 1, \dots, 5.$$

*Consequently, we regard $c_3$, $c_4$ and $c_5$ as rational functions of $c_1$ and $c_2$, with parameters $R_1$ and $R_2$. After straightforward computation, the condition $\sum\theta_i = \pi$ appears superfluous. We define*

$$F(c_1, c_2; R_1, R_2) = c_1 + c_2 + \sum_{j=3}^{5} c_j(c_1, c_2; R_1, R_2),$$

*which is a rational function of* $(c_1, c_2) \in \mathbb{R}^2$ *with parameters* $R_1$ *and* $R_2$*. Further computation reveals that* $F(c_1, c_2)$ *has a unique minimum point at:*

$$\begin{aligned}
c_1 &= \frac{(R_1R_2 - R_1 - R_2)(R_1R_2 - R_1 + R_2 + 1)}{\sqrt{\begin{aligned}&(R_1-1)(R_2-1)(R_1R_2 - R_1 - R_2)\\ &\cdot (3R_1^2R_2^2 - 6R_1R_2^2 - 6R_1^2R_2 + 3R_1^2 + 3R_2^2 + 5R_1R_2 - 3R_1 - 3R_2)\end{aligned}}},\\
c_2 &= \frac{(R_1-1)(R_2^2 + 2R_1R_2 + R_2 - R_1R_2^2 - R_1)}{\sqrt{\begin{aligned}&(R_1-1)(R_2-1)(R_1R_2 - R_1 - R_2)\\ &\cdot (3R_1^2R_2^2 - 6R_1R_2^2 - 6R_1^2R_2 + 3R_1^2 + 3R_2^2 + 5R_1R_2 - 3R_1 - 3R_2)\end{aligned}}}.
\end{aligned} \tag{A.6}$$

PROPOSITION A.2.1. *Let* $\Theta : M_5' \to (\mathbf{S}^1)^5$, $\Theta(R_1, \dots, R_5) = (\theta_1, \dots, \theta_5) := (\mathrm{arccot} c_1, \dots, \mathrm{arccot} c_5)$, *where* $c_1 = c_1(R_1, R_2)$ *and* $c_2 = c_2(R_1, R_2)$ *are given by* (A.6)*, and* $c_j = c_j(c_1, c_2; R_1, R_2)$ *for* $j = 3, 4, 5$ *are described above. Then,* $\Theta$ *satisfies the conditions* (A.1) *and* (A.2)*.*

PROOF. It suffices to show that $\Theta(R_{1+j}, R_{2+j}) = (\theta_{1+j}, \dots, \theta_{5+j})$ for $j = 1, \dots, 4$, where the indices are taken modulo 5. Suppose that $\Theta(R_{1+j}, R_{2+j}) = (\theta'_{1+j}, \dots, \theta'_{5+j})$. By definition, both $(\theta_1, \dots, \theta_5)$ and $(\theta'_1, \dots, \theta'_5)$ are points on the variety

$$\{(\theta_1, \dots, \theta_5) | R_\times(\theta_i, \theta_{i+1}, \theta_{i+2}) = R_i, i = 1, \dots, 5\},$$

that minimize $\sum_{i=1}^5 \cot\theta_i$. We have verified that such a minimum point is unique. Therefore, $\theta'_i = \theta_i$ for $i = 1, \dots, 5$. $\square$

*From the function* $\Theta : M_5 \to (\mathbf{S}^1)^5$ *constructed above, we derive the following formula by a straightforward computation, analogously to Examples A.2.1 and A.2.2:*

(A.7)

$$\begin{aligned}
&\cos\theta(A^\perp, B^\perp) = \cos\angle(\psi_{(A,B)}(A), \psi_{(A,B)}(B))\\
&= \frac{\sum(3\lambda_1^2\lambda_2\lambda_3\lambda_4 + 3\lambda_1\lambda_2\lambda_3\lambda_4^2 - \lambda_1^2\lambda_3^2\lambda_4 - \lambda_1^2\lambda_3\lambda_4^2 - \lambda_1\lambda_2^2\lambda_3\lambda_4 - \lambda_1\lambda_2\lambda_3^2\lambda_4) - 10\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5}{2\sqrt{\begin{aligned}&(\sum(\lambda_1^2\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_3^2 + \lambda_1^2\lambda_3\lambda_4 - \lambda_1^2\lambda_3^2 - \lambda_1\lambda_2\lambda_3\lambda_4 - \lambda_1\lambda_2^2\lambda_3))\\ &(\sum(\lambda_1^2\lambda_2^2\lambda_3\lambda_4 + \lambda_1^2\lambda_2\lambda_3\lambda_4^2 + \lambda_1\lambda_2\lambda_3^2\lambda_4^2 - \lambda_1\lambda_2^2\lambda_3^2\lambda_4 - \lambda_1^2\lambda_2^2\lambda_4^2 - \lambda_1^2\lambda_2\lambda_3\lambda_4\lambda_5))\end{aligned}}}.
\end{aligned}$$

We can further simplify Equation (A.7). Noticeably, the cyclic sums appearing in equation (A.7) represent cyclic sums of products of linear terms in $\lambda_1, \dots, \lambda_5$, namely,

$$\begin{aligned}
&\sum_{cyc}(\lambda_1^2\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_3^2 + \lambda_1^2\lambda_3\lambda_4 - \lambda_1^2\lambda_3^2 - \lambda_1\lambda_2\lambda_3\lambda_4 - \lambda_1\lambda_2^2\lambda_3)\\
&= -\sum_{cyc}(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_4)(\lambda_4 - \lambda_5),
\end{aligned}$$

$$\sum_{cyc}(3\lambda_1^2\lambda_2\lambda_3\lambda_4+3\lambda_1\lambda_2\lambda_3\lambda_4^2-\lambda_1^2\lambda_3^2\lambda_4-\lambda_1^2\lambda_3\lambda_4^2-\lambda_1\lambda_2^2\lambda_3\lambda_4-\lambda_1\lambda_2\lambda_3^2\lambda_4)-10\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5$$
$$=-\sum_{cyc}(\lambda_1+\lambda_5)(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)(\lambda_4-\lambda_5),$$

and

$$4\sum_{cyc}(\lambda_1^2\lambda_2^2\lambda_3\lambda_4+\lambda_1^2\lambda_2\lambda_3\lambda_4^2+\lambda_1\lambda_2\lambda_3^2\lambda_4^2-\lambda_1\lambda_2^2\lambda_3^2\lambda_4-\lambda_1^2\lambda_2^2\lambda_4^2-\lambda_1^2\lambda_2\lambda_3\lambda_4\lambda_5)$$
$$=-\sum_{cyc}(\lambda_1+\lambda_5)^2(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)(\lambda_4-\lambda_5),$$

implying that

$$\cos\theta(A,B)=\frac{\sum_{i=1}^{5}\frac{\lambda_{i+1}+\lambda_i}{\lambda_{i+1}-\lambda_i}}{\sqrt{\left(\sum_{i=1}^{5}\frac{1}{\lambda_{i+1}-\lambda_i}\right)\left(\sum_{i=1}^{5}\frac{(\lambda_{i+1}+\lambda_i)^2}{\lambda_{i+1}-\lambda_i}\right)}},$$

which is a special case of equation (3.2) when $k=5$. After obtaining such an equation for general $k$, it is notable that equations (A.4) and (A.5) are special cases of the equation (3.2) when $k=3$ and $k=4$, respectively. This observation motivates us to prove that the function given by (3.2) is an invariant angle function for every $k\geq 3$.

APPENDIX B

# Infinite-sidedness for the Integer Heisenberg Group

In this appendix, we consider the Heisenberg group:

DEFINITION B.0.1. *The **Heisenberg group** over a ring $R$ is defined as*

$$H_3(R) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a, b, c \in R \right\}.$$

The Heisenberg group over $\mathbb{R}$ is a 3-dimensional non-Abelian nilpotent Lie group. The geometric structure of the Heisenberg group, known as the **Nil geometry**, is one of Thurston's eight model geometries [Sco83].

The **real Heisenberg group** $H_3(\mathbb{R})$ is a 3-dimensional Lie subgroup of $SL(3, \mathbb{R})$, while the **integer Heisenberg group** $H_3(\mathbb{Z})$ is a discrete subgroup of $H_3(\mathbb{R})$ and thus of $SL(3, \mathbb{R})$. The latter group is interesting among non-Abelian discrete subgroups of $SL(3, \mathbb{R})$. It is worth exploring a question analogous to the question raised in Chapter 6, namely whether the Dirichlet-Selberg domain for $H_3(\mathbb{Z})$ in $\mathcal{P}(3)$ is infinitely-sided when centered at a given point $X \in \mathcal{P}(3)$. At least for certain choices of $X$, the answer is affirmative:

PROPOSITION B.0.1. *Suppose that $X \in \mathcal{P}(3)$, with $X^{-1} = (x^{ij})_{i,j=1}^3$, and $x^{23} = 0$. Then the Dirichlet-Selberg domain $DS(X, H_3(\mathbb{Z}))$ is infinitely-sided.*

PROOF. We view $H_3(\mathbb{Z})$ as a three-parameter family:

$$g(k, l, m) = \begin{pmatrix} 1 & m & km + l \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -m & -l \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{pmatrix}^{-1}, \ \forall (k, l, m) \in \mathbb{Z}^3,$$

and express the function $s^g_{X,A}$ as:

$$s^g_{X,A}(k, l, m) = x^{33}(a_{22}(k - k_c)^2 + 2a_{12}(k - k_c)(l - l_c) + a_{11}(l - l_c)^2) + x^{22}a_{11}(m - m_c)^2 + const,$$

where $A = (a_{ij})$ as usual, and

$$k_c = \frac{a_{11}a_{23} - a_{12}a_{13}}{a_{11}a_{22} - a_{12}^2},\ l_c = \frac{x^{13}}{x^{33}} + \frac{a_{22}a_{13} - a_{12}a_{23}}{a_{11}a_{22} - a_{12}^2},\ m_c = \frac{a_{12}x^{22} + a_{11}x^{12}}{a_{11}x^{22}}.$$

We separate the variables as $s^g_{X,A}(k,l,m) = s_1(k,l) + s_2(m) + const$, where

$$s_1(k,l) = x^{33}(a_{22}(k-k_c)^2 + 2a_{12}(k-k_c)(l-l_c) + a_{11}(l-l_c)^2),\ s_2(m) = x^{22}a_{11}(m-m_c)^2.$$

The function $s_1$ coincides with $s^g_{X,A}$ for two generated groups of type (1') that appeared in Chapter 6. As we have proven, for any coprime pair $(k_0, l_0) \in \mathbb{Z}^2$, there is a family of matrices $A = A(\epsilon) \in \mathcal{P}(3)$, $0 < \epsilon < \epsilon_0$, such that $(0,0)$ and $(k_0, l_0)$ are the only minimal points of $s_1|_{\mathbb{Z}^2}$. Here, $\epsilon_0$ is a positive number dependent on $k_0$ and $l_0$, and the matrix $A(\epsilon)$ is dependent on $k_0$, $l_0$ as well as $x^{ij}$. Among these coprime pairs, infinitely many pairs $(k_0, l_0) \in \mathbb{Z}^2$ exist for which the matrix $A(\epsilon)$ constructed above ensures that $s_2|_{\mathbb{Z}}$ has a unique minimum point at $m = 0$ when $\epsilon$ approaches 0. Indeed, $s_2(m)$ is a quadratic polynomial with a positive leading coefficient. Consequently, $m = 0$ is the sole minimum point of $s_2|_{\mathbb{Z}}$ if and only if $|m_c| < 1/2$; the latter condition yields that

$$\frac{-x^{22} - 2x^{12}}{2x^{22}}(\epsilon^2 l_0^2/k_0 + k_0) < -(1-\epsilon^2)l_0 < \frac{x^{22} - 2x^{12}}{2x^{22}}(\epsilon^2 l_0^2/k_0 + k_0).$$

As $\epsilon$ tends to 0, the above inequality simplifies to:

$$\left(\frac{x^{12}}{x^{22}} - \frac{1}{2}\right)k_0 < l_0 < \left(\frac{x^{12}}{x^{22}} + \frac{1}{2}\right)k_0,$$

which holds for infinitely many coprime pairs of integers $(k_0, l_0)$. For each of these pairs, there exists a sufficiently small $\epsilon$, dependent on $k_0$ and $l_0$, such that a level curve of $s^g_{X,A(\epsilon)}$ solely encloses integer points $(0,0)$ and $(k_0, l_0)$. Thus, these are the only minimum points of $s_1|_{\mathbb{Z}^2}$, while $m = 0$ serves as the only minimum point of $s_2|_{\mathbb{Z}}$. Consequently, $(k,l,m) = (0,0,0)$ and $(k,l,m) = (k_0, l_0, 0)$ are the only minimum points of $s^g_{X,A}(k,l,m)$. Lemma 6.1.0.1 with $\Lambda = \mathbb{Z}^3$ implies that the Dirichlet-Selberg domain $DS(X, H_3(\mathbb{Z}))$ is infinitely sided for any $X \in \{X = (x^{ij})^{-1} \in \mathcal{P}(3) | x^{23} = 0\}$. $\square$

The assumption $x^{23} = 0$ ensures that the variables of the function $s^g_{X,A}$ can be separated, allowing a concise proof of the proposition. We ask if the proposition still holds without this assumption:

Conjecture B.0.2. *Is there any $X \in \mathcal{P}(3)$ such that the Dirichlet-Selberg domain $DS(X, H_3(\mathbb{Z}))$ is finitely sided?*

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