Asymptotic Methods for the Atmosphere

By

OJESH KOUL
DISSERTATION

Submitted in partial satisfaction of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Applied Mathematics

in the

OFFICE OF GRADUATE STUDIES

of the

UNIVERSITY OF CALIFORNIA

DAVIS

Approved:

Joseph Biello, Chair

Paul Ullrich

Terry Nathan

Committee in Charge

2024

Contents

Abstract

In this thesis, we have looked at the application of asymptotic methods in studying atmospheric phenomenon. Asymptotic techniques are very useful to obtain simplified systems which approximate a very complex underlying system. In this document we primarily use the simplified models to model the Hadley circulation. The Hadley circulation is an atmospheric circulation with rising air at the tropics and descending air in the subtropics($\sim 30^\circ$ latitude). The dynamics of the Hadley cell have profound implications on the global atmosphere. It transports angular momentum, heat and moisture from the equator to the midlatitudes. The rising branch of the Hadley cell experiences increased rainfall and thunderstorms while the descending branch is marked by an increased aridity. So, understanding and accurately modeling the features of the Hadley cell is of great importance.

The thesis has been divided into four chapters. The first three chapter deal with applying the method of matched asymptotics in order to get an approximate solution valid for the entirety of the troposphere. Depending on the latitude, the system can be divided into three different layers. The innermost layer is the tropical region with dynamics on the mesoscales(500 km). The middle layer lies in the subtropics and modulates on the synoptic scales(1500 km). This is followed by a planetary scale(5000 km) outer layer in the midlatitudes. The aim of the study is to derive these systems using the formalism of matched asymptotics and obtain matching conditions between the solutions.

The full system of primitive equations and the non-dimensionalisation valid for large scale atmospheric flows have been described in chapter 1. Using these equations, the system valid for the tropics has been derived and its solutions have been described. The latitudinal extent of the tropical layer has been derived using scaling arguments for the tropical solutions. In chapter 2, we have looked at the shallow water system with non-dimensionalisation similar to those used in the first chapter. Shallow water formulation introduces a major simplification into the system by reducing the number of spatial dimensions by one. The tropical, subtropical and planetary layer models have been derived and their respective matching condition has been described . In chapter 3, we go back to the 3D system and derive the equations valid in the subtropics. The solution of the subtropical system yields an equation known as the Sawyer-Eliassen equation which is a second order partial differential equation. In the 3D system, the Sawyer-Eliassen equation is in 2 dimensions while in the shallow water system it is a 1 dimensional ODE. This makes the 3D system much more difficult

to solve since the PDE can be hyperbolic, parabolic or elliptic while no such complications arise in the shallow water system. A numerical solution scheme has been described for the subtropical system which solves the system when the Sawyer-Eliassen equation remains elliptic. The matching condition with the tropical boundary layer arises as the potential temperature restratification at the equator.

Chapter 4 deals with the instabilities arising in the subtropical jet. Informed by the pre-existing models of baroclinic instability, a damping model has been prescribed which incorporates the effect of baroclinic instability in the weak temperature gradient tropical mode. In the second part of the chapter, we have used the subtropical system obtained in chapter 3 to study these instabilities instead of the quasi-geostrophic system which has traditionally been used due to its simplicity. The effect of the momentum and temperature fluxes generated due to the instabilities has been studied. Since the fluctuations in linear analysis are much weaker than the mean flow, there is no interaction between the fluctuation and the mean flow. Using the method of multiple scale asymptotics, new equations have been derived to incorporate the fluxes generated due to the instabilities into the system describing the mean flow.

Acknowledgments

I would like to express my sincere gratitude towards my advisor, Prof. Joseph Biello. This work would not have been possible without his guidance and support. My discussions with him have deepened my understanding of both mathematics and atmospheric science.

I would also like to thank Prof. Terry Nathan and Prof. Paul Ullrich for agreeing to be on my thesis committee. The input of Prof. Stamen Dolaptchiev and Prof. Rupert Klein has been valuable throughout the span of this project and I am grateful for them taking the time to talk with me despite the time difference.

My parents and sister have been a constant source of unwavering support throughout my life. I have been blessed with a wonderful family and I am extremely grateful to them for their encouragement. I would also like to thank my friends for their support throughout the ups and downs of my graduate life.

Finally, I would also like to thank the administrative staff of the maths department for making my life as a graduate student as smooth as possible.

CHAPTER 1

Introduction and the Tropics

Scale analysis and asymptotics have been widely used in meteorology and fluid mechanics in general. The solutions of the Navier Stokes equation can be obtained through numerical methods, but it is often difficult to extract the relevant physics from those solutions. The addition of a temperature equation for atmospheric flows adds another layer of complexity to the system. Asymptotic methods use the length and time scales present in the system to derive much simpler equations which are generally easier to interpret.

Prandtl's boundary layer theory(Kevorkian and Cole [16], Lagerstrom and Cole [19]) of flows at high Reynold's number is arguably one of the most famous use of asymptotic analysis in fluid mechanics. On the other hand Stokes's equation and Oseen's equations are examples of low Reynolds number flows derived using asymptotic methods. The wide variety of phenomenon present in the atmosphere, ranging from few kilometers for clouds and storm systems to thousands of kilometers for planetary circulations, provide an obvious use case for applying asymptotic methods. Various models appropriate for these temporal and spatial scales have been obtained using asymptotic methods. These methods offer a systematic way of getting these models and puts them on a firm mathematical footing. The scale separation in the atmosphere is discussed in works by Klein (2000 [17],2010 [18]) which also makes the case for using a unified asymptotically small parameter. The asymptotic equations are valid in the limit of this parameter going to zero. Generally, a number of small parameters are present in a model, but they need to be represented in terms of this single parameter as the path dependent limits don't give a unique solution. This process of first expressing every small parameter in terms of a single small parameter and then taking a single limit is called taking distinguished limits.

For models near the tropics, Weak Temperature Gradient(WTG) approximation (Held and Hoskins, 1985 [11], Sobel et al., 2001 [40]) is widely used. A consequence of this is a balance between the vertical advection of the temperature and the background heating. Multiple scale asymptotics have also been used to model interaction across different spatial and temporal scales in near equatorial flows (Majda and Klein, 2003, [24]). Similarly quasi geostrophic theory (Pedlosky [27]) has been very successful in explaining phenomenon near the mid latitudes.

WTG approximation can be used to get a reasonable solution which matches the behaviour of Hadley cell near the tropics. The Hadley cell is an atmospheric circulation featuring rising air near the tropics and a descending branch near 30 degree latitude. The solutions obtained from the WTG regime have been given in sections further ahead. An eastward propagating jet stream and westward propagating trade winds are obtained which are a signature property of the Hadley cell (Vallis [43]).

Multiple scale asymptotics have also been used to explain the Madden Julian oscillation(MJO) (Majda and Biello 2004 [23],2005 [1]), which utilises the IPESD model. MJO is a large scale pattern which travels eastwards at speeds of $4-8m/s$. Various explanations have been put forward in trying to explain MJO and the asymptotic attempt is one such. This is also a problem where the general circulation models don't represent the MJO properly while the asymptotic methods do provide more insight into the problem.

The models people have been working with thus far are applicable in disparate regions, for example WTG near the tropics or QG in the midlatitudes. The systems thus obtained are often not closed and require us to plug in artificial boundary conditions. Majda and Biello (2012 [3]) discuss a closure of the MEWTG theory, which is a multiple scale modification of the WTG theory. In addition to this, these models are closed and don't feature any interaction with the winds outside their region of validity. In this study, we aim to look at the full system where the equatorial theory arises as a boundary layer and the mid latitude theory is the outer solution which can solve the boundary condition and non interaction problem.

In this chapter we will start with the non-dimensionalisation of the full set of primitive equations which are appropriate for large scale atmospheric dynamics. Using the dimensionless equations, we will look at two asymptotic regimes of the equations, one valid near the equator and the other valid in the midlatitudes. The tropical theory, as stated earlier is the so called weak temperature gradient(WTG) approximation. Solutions of this WTG system have been looked at for various different model parameters to model the Hadley cell. We will also look at how the same non dimensionalisation leads to the quasi geostrophic equation in the mid-latitudes.

1.1. The primitive equations

We consider hydrostatic, incompressible fluid with a cartesian geometry. Cartesian geometry has been chosen to simplify the model but the full functional form of the coriolis parameter has been chosen to emulate the spherical geometry of earth's atmosphere. Troposphere height $H_T = \pi H$ where $H = 5$ km with the buoyancy frequency $N = 10^{-2}s^{-1}$. This gives us a gravity wave speed of $c = NH = 50\ m/s$. The dimensional equations are as follows

$$
\begin{aligned}
\frac{Du}{Dt} - 2\Omega \sin\left(\frac{y}{R_E}\right) v + p_x &= S_u \\
\frac{Dv}{Dt} + 2\Omega \sin\left(\frac{y}{R_E}\right) u + p_y &= S_v \\
\frac{D\theta}{Dt} + \Gamma w &= S_\theta \\
p_z &= \frac{g}{\theta_*}\theta \\
u_x + v_y + w_z &= 0 \qquad (1.1) \\
\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w&\frac{\partial}{\partial z}
\end{aligned}
$$

where Γ is the dry adiabatic lapse rate. θ is the deviation of temperature from the assumed background temperature of Γz and thus the total potential temperature, Θ is given by $\Theta = \theta + \Gamma z$. The hydrostatic balance is essentially the vertical momentum equation with the vertical advection and coriolis terms omitted. The vertical velocity is much smaller than horizontal velocities and hence these terms can be safely assumed to be much smaller than the pressure gradient and the buoyancy term. S_u and S_v are the momentum forcing terms. For this work, we will choose a Rayleigh damping parametrisation for the forcing terms. We non dimensionalise the horizontal length scales by L, the time by T and the vertical length scale by H. The horizontal velocity is thus scaled by $u_0 = L/T$. Considering the system's time scale to be the diurnal time scale of 1 day, the non-dimensionalised zonal momentum equation becomes

$$\frac{Du}{Dt} - 2\Omega T \sin\left(\frac{L}{R_E} y\right) v + u_0^2 p_x = \frac{T^2}{L^2} S_u$$

For T=1 day, the non-dimensionalised coriolis parameter can be written as

$$2\Omega T = 4\pi = \frac{1}{\epsilon}$$

To have the coriolis term participate in the leading order equation, we will need the full coriolis term to be $O(1)$. This will only happen if $\frac{L}{R_E} = O(\epsilon)$. Let us say that $L/R_E = \epsilon \approx 0.1$, then the non-dimensionalised coriolis term becomes

$$\begin{aligned} 2\Omega T \sin\left(\frac{L}{R_E} y\right) &= \frac{\sin(\epsilon y)}{\epsilon} \\ &\sim y = O(1) \end{aligned}$$

This gives us the scale of L=500km and the scale of the horizontal velocities as $u_0 = L/T = 5m/s$. The vertical velocity scales as $H/T = 5cm/s$. This is the reason for neglecting the vertical velocity terms in the vertical momentum equation as it is two orders of magnitude less than the horizontal velocity terms. The ratio of the velocity scale, u_0 and the gravity wave speed is $u_0/NH = 0.1 = \epsilon$. These scalings are consistent with those used in Biello and Majda(2010) [3] which discusses the multi scale modification of weak temperature gradient equatorial flows. The pressure scales as $p_0 = u_0^2 \epsilon^{-1}$ and the temperature scales as $\theta_0 = \epsilon^1 \Gamma H = 3.3K$. With these scales, the non-dimensionalised primitive equations are as follows

$$\begin{aligned} \frac{Du}{Dt} - \frac{\sin(\epsilon y)}{\epsilon} v + \frac{p_x}{\epsilon} &= S_u \\ \frac{Dv}{Dt} + \frac{\sin(\epsilon y)}{\epsilon} u + \frac{p_y}{\epsilon} &= S_v \\ \epsilon \frac{D\theta}{Dt} + w &= S_\theta \\ p_z &= \Theta \\ u_x + v_y + w_z &= 0 \end{aligned} \tag{1.2}$$

These scalings will work as long as the momentum forcing is at most 5m/s per day and the heating is at most 33K per day. As we will see in later sections, this non-dimensionalisation gives us the WTG approximation for equatorial flows and the quasi geostrophic approximation for mid-latitude flows. A non-dimensionalisation like this is also helpful in clearly delineating the latitudes where the different regimes work i.e where does the tropics or the midlatitudes begin and end.

1.2. Potential vorticity

In this section, we will derive the vorticity and the potential vorticity for the hydrostatic, rotating and incompressible system. The potential voriticity is an important conserved quantity in the system and plays an important role in all the asymptotic regimes throughout this work.

Taking the z and y derivative of zonal momentum and x,z derivative of meridional momentum gives us

$$
\begin{aligned}
&\frac{\partial u_z}{\partial t} + (\vec{u}.\nabla)(u_z) + \frac{\partial \vec{u}}{\partial z}.\vec{\nabla}(u) - \tilde{f} v_z + \tilde{p}_{xz} = S_{u,z} \\
&\frac{\partial u_y}{\partial t} + (\vec{u}.\nabla)(u_y) + \frac{\partial \vec{u}}{\partial y}.\vec{\nabla}(u) - \tilde{f} v_y + \tilde{p}_{xy} - \beta v = S_{u,y} \\
&\frac{\partial v_z}{\partial t} + (\vec{u}.\nabla)(v_z) + \frac{\partial \vec{u}}{\partial z}.\vec{\nabla}(v) + \tilde{f} u_z + \tilde{p}_{yz} = S_{v,z} \\
&\frac{\partial v_x}{\partial t} + (\vec{u}.\nabla)(v_x) + \frac{\partial \vec{u}}{\partial x}.\vec{\nabla}(v) + \tilde{f} u_x + \tilde{p}_{yx} = S_{v,x} \\
&\frac{\theta_y}{\epsilon} = \tilde{p}_{zy} \\
&\frac{\theta_x}{\epsilon} = \tilde{p}_{zx}
\end{aligned} \tag{1.3}
$$

where $\tilde{f} = \frac{\sin(\epsilon y)}{\epsilon}$, $\beta = \frac{d\tilde{f}}{dy} = \cos(\epsilon y)$ and $\tilde{p} = \frac{p}{\epsilon}$. Eliminating the pressure terms gives us the vorticity equation

$$
\begin{aligned}
\frac{D\omega^x}{Dt} &= (\vec{\omega}.\nabla)u - u_z u_y + (\nabla \times F)^x \\
\frac{D\omega^y}{Dt} &= (\vec{\omega}.\nabla)v - v_z v_x + (\nabla \times F)^y \\
\frac{D\omega^z}{Dt} &= (\vec{\omega}.\nabla)w + (\nabla \times F)^z \\
\vec{\omega} &= -v_z \hat{i} + u_z \hat{j} + (v_x - u_y + \tilde{f})\hat{k} \\
\vec{F} &= (S_u, S_v, \frac{\theta}{\epsilon})
\end{aligned} \tag{1.4}
$$

From the temperature equation we get

$$\vec{\omega}.\frac{D\nabla.\tilde{\theta}}{Dt} = (\vec{\omega}.\nabla).\frac{D\tilde{\theta}}{Dt} - (\vec{\omega}.\nabla\vec{u})\nabla(\tilde{\theta})$$
$$= (\vec{\omega}.\nabla)S_\theta - (\vec{\omega}.\nabla\vec{u})\nabla(\tilde{\theta})$$
$$\tilde{\theta} = \epsilon\theta + z$$

(1.5)

To get the PV equation we add eq. 1.4 and eq. 3.12 to get

$$\frac{D(\vec{\omega}.\nabla\tilde{\theta})}{Dt} = (\vec{\omega}.\nabla S_\theta) + \nabla\tilde{\theta}.(\nabla \times \vec{F})$$

(1.6)

where the potential vorticity is given by $\vec{\omega}\nabla\tilde{\theta}$. In the absence of any forcing, this quantity is conserved along stream lines. The linear part of PV is

$$Q = (v_x - u_y) + f\theta_z + \frac{f}{\epsilon}$$

1.3. Quasi Geostrophy

As alluded to earlier, one of the singular limits of the non-dimensionalised primitive equations is the quasi geostrophic equations(Pedlosky [27], Vallis, [43]). This regime is applicable for latitudes where the Coriolis parameter is $O(1)$ i.e at latitudes $y = O(\epsilon^{-1})$. The dominant balance is the geostrophic balance due to the rotation being strong enough to overshadow the advective derivatives. The geostrophic balance leads to a much weaker vertical flow and the leading order flow is two dimensional. To derive the QG equations, we write the latitude y as

$$y \to \epsilon^{-1}Y_0 + y$$

where Y_0 and y are $O(1)$ i.e of the order of 500km. The coriolis parameter can be expanded using Taylor series as

$$\sin(\epsilon(\epsilon^{-1}Y_0 + y)) = \sin(Y_0) + \cos(Y_0)y + \dots$$
$$= f + \beta y + \dots$$

Plugging this into the primitive equations we get the following

$$-fv + p_x = \epsilon\Big(- \frac{Du}{Dt} + \beta yv + S_u + \dots \Big)$$
$$fu + p_y = \epsilon\Big(- \frac{Dv}{Dt} + S_v - \beta yu + \dots \Big)$$
$$p_z - \theta = 0$$
$$u_x + v_y + w_z = 0$$
$$w = S_\theta + \epsilon\Big(\frac{D\theta}{Dt}\Big) \tag{1.7}$$

Plugging in a regular asymptotic expansion of the form

$$\mathcal{U}^\epsilon = \mathcal{U} + \epsilon\mathcal{U}^{(1)} + \dots$$

where $\mathcal{U}$ is a stand-in for the solution $(u^\epsilon, v^\epsilon, w^\epsilon, p^\epsilon, \theta^\epsilon)$. At the leading order i.e $O(1)$ we get the following

$$u = -f^{-1}p_y$$
$$v = f^{-1}p_x$$
$$\theta = p_z$$
$$w_z = 0$$
$$w = S_\theta \tag{1.8}$$

With the rigid walls boundary condition at the bottom, the incompressibility equation implies that $w = 0$ throughout the domain. Hence the last equation above implies that the solution is consistent only if the $O(1)$ heating is identically zero. The heating not being zero implies that the scales of the variables in the asymptotic expansion is incorrect and we will have to increase all the terms by an order of epsilon. For this section we rewrite the heating as

$$S_\theta \to \epsilon S_\theta + \epsilon^2 S_\theta^1 + \dots$$

The solution of the leading order variables, as written in eq.1.8 is not closed since we have no way yet of figuring out the pressure. There are many equivalent ways of obtaining an equation for the leading order pressure. The first method requires us to go up an order in ϵ and look at the higher order balances.

1.3.1. $O(\epsilon)$ system. Collecting the $O(\epsilon)$ terms, after plugging the regular asymptotic series, we get the following system

$$
\begin{aligned}
-fv^{(1)} + (p^{(1)})_x &= -\frac{Du}{Dt} + \beta y v + S_u \\
fu^{(1)} + (p^{(1)})_y &= -\frac{Dv}{Dt} - \beta y u + S_v \\
(p^{(1)})_z - \theta^{(1)} &= 0 \\
u_x^{(1)} + v_y^{(1)} + w_z^{(1)} &= 0 \\
w^{(1)} &= S_\theta - \frac{D\theta}{Dt}
\end{aligned}
\tag{1.9}
$$

Plugging $v^{(1)}$,$u^{(1)}$ and $w^{(1)}$ from the zonal momentum, meridional momentum and the heat equation into the incompressibility condition we get

$$
\begin{aligned}
\frac{Dq}{Dt} + \beta\frac{\partial p}{\partial x} &= f^2\frac{\partial S_\theta}{\partial z} + f(\partial_x S_v - \partial_y S_u) \\
q &= \partial_{xx}^2 p + \partial_{yy}^2 p + f^2\partial_{zz}^2 p
\end{aligned}
\tag{1.10}
$$

This is the closure equation for the leading order pressure which arises from applying incompressibility to the higher order solutions. The above equation is an advection equation for q which we will later see is just the potential vorticity for the system, and the pressure is obtained from q by inverting the elliptic operator. So, to solve the inversion operator, we need the some form of boundary conditions for the pressure(Dirichlet, Neumann or Robin).

Another way to directly obtain the closure condition of eq.1.10 is to write the PV equation for this system. Let us compute the terms upto $O(1)$ of eq.1.6

$$\begin{aligned}\vec{\omega}.\nabla\tilde{\theta} &= \frac{f}{\epsilon} + v_x - u_y + f\theta_z + \beta y \\ &= \frac{f}{\epsilon} + \frac{p_{xx}+p_{yy}}{f} + fp_{zz} + \beta y + O(\epsilon) \\ \vec{\omega}.\nabla(S_\theta) &= fS_{\theta,z} + O(\epsilon) \\ \nabla\tilde{\theta}.\nabla\times\vec{F} &= S_{v,x} - S_{u,y} + O(\epsilon)\end{aligned}$$

Since, the advective derivative of $\frac{f}{\epsilon}$ is zero, this term of the PV doesn't participate in the equation. Plugging these into the PV equation, at O(1), we get

$$\frac{D}{Dt}(p_{xx} + p_{yy} + f^2 p_{zz} + \beta y) = f^2 S_{\theta,z} + f(\partial_x S_v - \partial_y S_u) \tag{1.11}$$

The above equation is equivalent to eq.1.10. One of the drawbacks of the theory as written here is that it is only valid for large, $O(\epsilon^{-1})$ latitudes and in small $O(1)$ patches with no way of continuously extending it to lower latitudes. Another problem is that the velocity in the geostrophic balance blows up as $y \to 0$.

1.3.2. Extending QG to lower latitudes. A way to deal with the singularity in the leading order horizontal velocity is to transform the pressure term so that the coriolis term in the geostrophic balance gets canceled. This can be done by transforming the pressure and temperature as follows

$$p \to fp$$

$$\theta \to f\theta$$

With this transformation, the equations become

$$
\begin{aligned}
-v + p_x &= \frac{\epsilon}{f}\big(-\frac{Du}{Dt} + S_u\big) \\
u + p_y &= \frac{\epsilon}{f}\big(-\frac{Dv}{Dt} + S_v - \beta p\big) \\
p_z - \theta &= 0 \\
u_x + v_y + w_z &= 0 \\
w &= \frac{\epsilon}{f}S_\theta + \epsilon\frac{Df\theta}{Dt} \qquad (1.12)
\end{aligned}
$$

Let us fix the latitude at $Y_0 = \epsilon^{-\alpha}Y$ where $Y = O(1)$ and $\alpha \leq 1$. This makes the coriolis parameter $f = O(\epsilon^{1-\alpha})$ and the contribution of to the ageostrophic wind $O(\epsilon^\alpha)$. This is why we can have the heating to be $O(\epsilon^\alpha)$ or $O(\epsilon/f)$. In other words, this expansion works as long as the ratio $\frac{S_\theta}{y} = O(1)$ for $y >> 1$. As $\epsilon \to 0$, at leading order we get the following

$$
\begin{bmatrix} u \\ v \\ w \\ p \\ \theta \end{bmatrix} = \begin{bmatrix} -p_y \\ p_x \\ 0 \\ p \\ -p_z \end{bmatrix} \qquad (1.13)
$$

The leading order vertical velocity vanishes, just like in QG and the remaining perturbations are of the order $O(\frac{\epsilon}{f})$. So, we plug in the following expansion in eq.1.12

$$
\mathcal{U}^\epsilon = \mathcal{U} + \frac{\epsilon}{f}\mathcal{U}^{(1)} + \dots
$$

where $\mathcal{U}$ denotes the solution $(u^\epsilon, v^\epsilon, w^\epsilon, p^\epsilon, \theta^\epsilon)$. The above expansion is an asymptotic expansion as long as $\frac{\epsilon}{f} << 1$. This is true for $y >> 1$ i.e for $\alpha > 0$. Analogous to QG, the zeroth order equations are not closed since we do not know what the pressure is. To find an equation for the pressure, we go to the first order solution.

$$-v^{(1)} + p_x^{(1)} = -\frac{Du}{Dt} + S_u$$
$$u^{(1)} + p_y^{(1)} = -\frac{Dv}{Dt} + S_v - \beta p$$
$$p_z^{(1)} = \theta^{(1)}$$
$$u_x^{(1)} + v_y^{(1)} + w_z^{(1)} = 0$$
$$w^{(1)} = S_\theta$$

The above equations are valid for $0 < \alpha < 1$. For $\alpha = 1$, the temperature advection terms is the same order as the heating and cannot be ignored. For $\alpha < 1$, the vertical velocity is given by the heating directly. This is known as the weak temperature gradient approximation and will be talked about in greater detail in the equatorial theory section later on. We can now proceed in the same manner as we did for regular QG to get a closure for the leading order pressure by taking the curl of the momentum equations and the z derivative of the temperature equation to get

$$\frac{Dq}{Dt} + \beta p_x = \partial_z(S_\theta) + \partial_x(S_v) - \partial_y(S_u)$$
$$q = p_{xx} + p_{yy}$$

The above equation is basically the same as the QG potential vorticity equation derived above but instead of a 3D Poisson equation we only have to solve a 2D Poisson equation. The extension here only works as long as the leading order horizontal velocity remains $O(1)$. These assumptions, as we will see in later sections cannot be justified and hence this extension is not valid, at least for Earth-like atmospheres.

1.4. Equatorial theory

One of the major difference between the equatorial model and the quasi geostrophic, mid-latitude model, apart from the differences in latitude is the presence of strong heating near the equator. This increased heating results in the atmospheric circulation know as the Hadley cell. The Hadley cell is characterised by an equator to polewards circulation that has a pronounced affect on a wide variety of phenomenon on our planet. In the northern hemisphere, the surface winds coming from the north east, also called the Trade winds rise up as they converge towards the equator. This is called the inter tropical convergence zone (ITCZ) and results in thunderstorms and increased rainfall. As the rising air moves towards the poles, it drifts westwards due to the coriolis effect and eventually descends around 20 to 30 degrees latitude. The latitudes around this descending branch are marked for their aridity and the presence of many deserts.

In this section we will look at the asymptotic formulation of our tropical theory and study a simplified model of the Hadley cell. As we will see in the results section, this model does a pretty good job of modeling various features of the Hadley cell like the Trade winds and the subtropical jet. We begin by looking at $O(1)$ distances in the meridional direction, i.e $y = O(1)$. At this latitude the coriolis parameter can be expanded using the Taylor series as

$$\sin(\epsilon y) = \epsilon y - \epsilon^3 \frac{y^3}{6} + \dots$$

We use the following regular expansion for the solution $(u^\epsilon, v^\epsilon, w^\epsilon, p^\epsilon, \theta^\epsilon)$

$$\begin{aligned} u^\epsilon &= u + \epsilon u_1 + \dots \\ v^\epsilon &= v + \epsilon v_1 + \dots \\ w^\epsilon &= w + \epsilon w_1 + \dots \\ p^\epsilon &= \epsilon(p + \epsilon p_1 + \dots) \\ \theta^\epsilon &= \epsilon(\theta + \epsilon \theta_1 + \dots) \end{aligned} \tag{1.14}$$

Due to the coriolis parameter being an order of ϵ lower than the latitude where QG is valid, the magnitude of the pressure perturbation is also lower by an order of magnitude. There is a caveat in this argument which we will discuss a bit further. With the above expansion, we look at the resulting simplified systems at $O(1)$ and $O(\epsilon)$

1.4.1. O(1) solution. Collecting all the terms at $O(1)$ after we plug in the regular expansion in eq.2.2, we get

$$\begin{aligned}\frac{Du}{Dt} - yv + p_x &= S_u \\ \frac{Dv}{Dt} + yu + p_y &= S_v \\ w &= S_\theta \\ p_z &= \theta \\ u_x + v_y + w_z &= 0\end{aligned} \tag{1.15}$$

The horizontal momentum equations only contain the simplification of the coriolis parameter from its full form to its linearised form. This is called the equatorial beta plane approximation. The temperature equation on the other hand undergoes an enormous simplification. The advection of the temperature completely disappears and the only remaining term on the left hand side of the equation is the vertical velocity. These terms do not participate in the leading order balance because the gradients of the perturbation temperature are much weaker than the gradient of the background potential temperature which is why this approximation regime is also called the weak temperature gradient approximation(Neelin and Held [26], Sobel and Bretherton [39], Sobel et al. [40]). The remaining w term is the vertical advection of the background potential temperature. The equation now acts as a prognostic equation for the vertical velocity rather than the potential temperature.

Since the vertical velocity is directly obtained from the temperature equation, we only need to solve the two momentum equations along with the incompressibility constraint. To solve the momentum equation, we eliminate the pressure by taking the curl of the momentum equations. This gives us the vorticity equation

$$\begin{aligned}\frac{D\omega}{Dt} &= (\vec{\omega}.\nabla S_\theta) + S_{v,x} - S_{u,y} \\ \omega &= v_x - u_y\end{aligned} \tag{1.16}$$

We can time step this equation to obtain the vorticity as a function of time. To obtain the velocity from the time stepped vorticity, we can use the Helmholtz decomposition

$$(u, v) = -\nabla\phi - \nabla \times (\psi\hat{k}) \tag{1.17}$$

where $\nabla\phi$ and $\nabla \times \psi$ are the curl free and divergence free components of the velocity field respectively. With the above decomposition, the vorticity and incompressibility equation become

$$\begin{aligned} \Delta\psi &= \omega \\ \Delta\phi &= w_z \end{aligned} \tag{1.18}$$

To obtain the horizontal velocity, we have to solve the above two Poisson equation at each time step.

1.4.1.1. *Potential vorticity.* We can use the scaling in the equatorial region and the PV equation (1.6) to find the leading order PV equation near the equator

$$\begin{aligned} \omega &= -v_z\hat{i} + u_z\hat{j} + (v_x - u_y + y)\hat{k} \\ \tilde{\theta} &= \epsilon^2\theta + z \\ Q &= \epsilon^2(v_z\theta - x + u_z\theta_y) + (v_x - u_y + y) \\ &= \epsilon(v_z\theta - x + u_z\theta_y) + \omega^z \end{aligned}$$

So,the leading order Potential vorticity(Q) is simply the z component of the vorticity. Hence, the PV equation becomes

$$\frac{D\omega^z}{Dt} = (\vec{\omega}.\nabla S_\theta) + S_{v,x} - S_{u,y} \tag{1.19}$$

This equation is the same as the vorticity equation we obtained at the end of the previous section. So, unlike the mid-latitude QG theory, the PV equation in the tropical theory doesn't give us any new information.

1.4.2. Zonally symmetric case. The zonally symmetric case provides an excellent simplified model of the Hadley cell(Schneider and Lindzen [36], Schneider [33], Held and Hou [12]). The removal of the x derivative from the equation set also simplifies the solution of the system. We no longer need to invert two Laplacian operators in eq.1.18 at each time step. The vertical velocity, as in the full system is still given by the WTG approximation. The meridional velocity on the other hand is directly obtained from the incompressibility equation.

$$\begin{aligned} w &= S_\theta \\ v_y &= -w_z = -S_{\theta,z} \end{aligned} \tag{1.20}$$

The first equation above is the WTG approximation and the second is the incompressibility constraint. To get a closed Hadley cell, we have to impose a restriction on the heating S_θ. In the case of the heating being a symmetric function about the equator, we will have a vanishing meridional velocity at the equator. For a closed Hadley cell, we need the meridional velocity to vanish at some latitude $Y*$ away from the equator. Integrating the incompressibility equation for $y = 0$ to $y = Y^*$, we get a constraint for the heating function

$$v(Y^*, z, t) - v(0, z, t) = -\int_0^{Y^*} S_{\theta,z}(s, z, t)ds$$

Since the left side vanishes, so too does the integral of $S_{\theta,z}$. In the example solutions discussed here, heating functions satisfying the above constraint have been chosen. The specific form of the heating function is

$$S_\theta = S_0 \sin(z)[\cos(\pi Y/Y^*) + \cos(2\pi Y/Y^*)] \tag{1.21}$$

The vertical structure is the first baroclinic mode while the meridional behaviour ensures a Hadley cell closing at $y = Y^*$. The forcing terms in the momentum equation is chosen to be the Rayleigh damping parametrization,

$$\begin{aligned} S_u &= -d(z)u \\ d(z) &= d_0 \exp(-z/\lambda) \end{aligned} \tag{1.22}$$

The vertical decaying behaviour of the damping ensures a higher damping at the surface due to surface drag while the winds at the top of the troposphere experience little to no drag. For the time independent solution, we solve the zonal momentum equation along stream lines which converts the equation from a 2-D PDE to a 1-D ODE. As seen from Fig.1.1, the stream lines form closed loops. This means that we need to impose a perdiodic boundary condition while solving the ODE. The ODE has been solved by MATLAB's inbuilt ode45 function which uses a Runge Kutta solver. To plot the solution we have interpolated the solution obtained on points along the stream lines to a regular Cartesian grid using the *scatteredinterpolant* function.

1.4.2.1. *Note on periodic solutions.* We solve the steady state equations on closed streamlines and thus need to impose periodic boundary conditions. Since solving on stream lines corresponds to converting the 2D PDE into an ODE on stream lines, let us look at how to impose periodic boundary condition numerically. Let us consider the following ODE with a period of 1.

$$\frac{dy}{dx} + a(x)y = f(x)$$

Since $a(x)$ and $f(x)$ depend upon the heating and stream function, these are supposed to be periodic. For $a(x) = 0$, the condition for existence of solution is $\int_0^1 f(x)dx = 0$. Otherwise, the solution is given by

$$\frac{d}{dx}(\mu(x)y) = f(x)\mu(x)$$
$$\mu(x) = \exp\left(\int_0^x a(s)ds\right)$$

The solution comes out to be

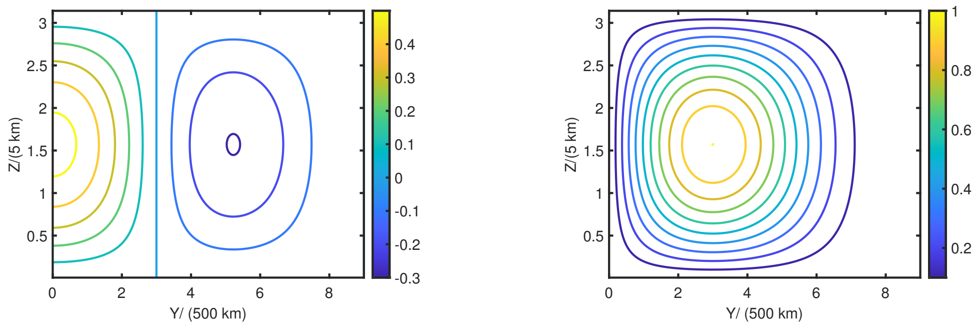
$$y(x)\mu(x) - y(0) = \int_0^x f(s)\mu(s)ds$$

Imposing periodicity of 1, we get

$$y(0) = \frac{\int_0^1 f(s)\mu(s)ds}{\mu(1) - 1}$$

If $\mu(1) \neq 1$, we have a periodic solution with the initial value given by the above equation. If $\mu(1) = 1$, then a periodic solution exists if $\int_0^1 f(s)\mu(s)ds = 0$ and the initial condition is arbitrary.

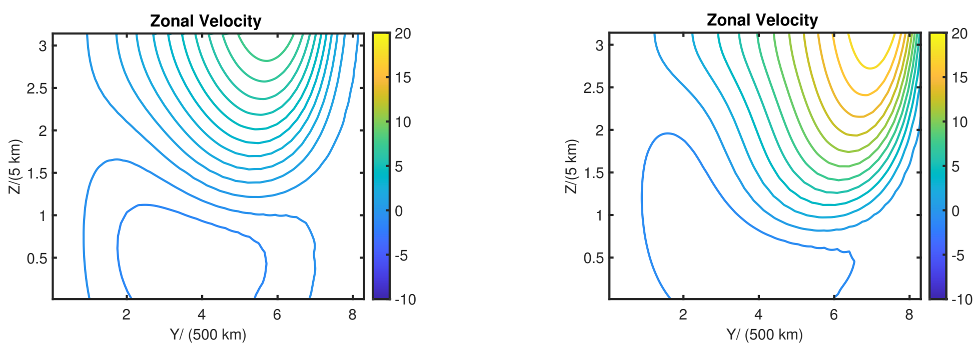
1.4.2.2. *Results.* Let us discuss the properties of the solution obtained when using a zonally symmetric heating, as given in eq.1.21. The contour plots of the heating profile and the corresponding stream function have been plotted in Fig.1.1 for a heating maximum $S_0 = 1$ or 33 K/day in dimensional terms . The maximum extent of the circulation has been set to $Y^* = 4500$km. This



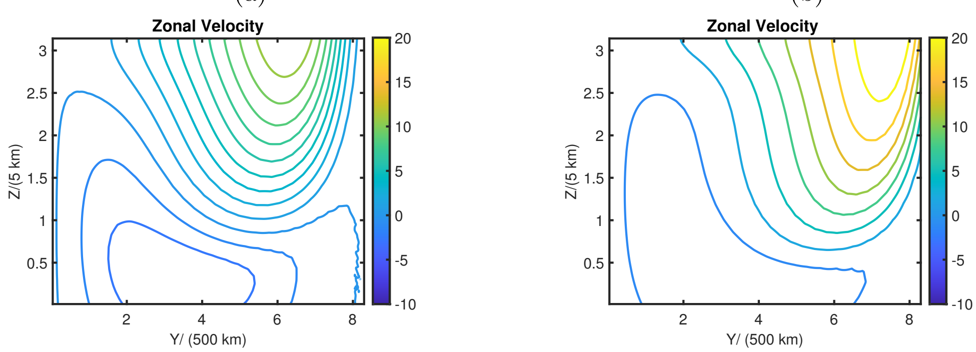
(a) (b)

FIGURE 1.1. The heating profile, S_θ corresponding to a maximum meridional velocity of 5m/s is plotted in a). b) is the corresponding stream function, Ψ.

corresponds to a latitude of 35°. As stated before, the closure of the cell requires us to chose a heating with vanishing meridional integral. If we have heating near the equator, the zero integral condition necessitates a cooling region in the function S_θ. This can be seen from the plot of the heating where the ratio of the heating to cooling region has been set to 2:1.



(a) (b)



(c) (d)

FIGURE 1.2. The contours of the zonal velocity have been plotted for different maximum meridional velocity and the damping length scale λ. a) $v_{max} = 0.5$, $\lambda = \pi/2$. b) $v_{max} = 0.5$, $\lambda = \pi/4$. c) $v_{max} = 1$, $\lambda = \pi/2$. d) $v_{max} = 1$, $\lambda = \pi/4$

The meridional slices of the zonal velocity for different maximum meridional velocities and damping length scales have been plotted in Fig.1.2. We get a region of very fast winds at the top of the troposphere at the polar end of the circulation. This is the region describing the subtropical jet. The winds as they move towards the pole conserve angular momentum and gain speed. When the meridional velocity starts to go down, the damping effect kicks in and reduces the zonal velocity which eventually goes to zero at the poleward end of the Hadley cell. Analogous to the poleward moving wind at the tropopause, the equatorward moving wind parcels at the surface gain westward zonal velocity as they move towards lower latitudes but due to surface damping, the winds are not as fast as those in the jet region. These easterly winds near the equator are known as trade winds which have been used by sailors for navigation for centuries. At, the top of the troposphere, since the vertical velocity is zero, the steady zonal momentum equation can be written as

$$vu_y - yv = -d(H_T)u$$

$$v(u - y^2/2)_y = -d(H_T)u$$

where H_T is the height of the troposphere. Without any damping, the solution of the above equation would be $u = u_0 + y^2/2$. With a damping component added in, the y^2 behaviour will only be valid for low latitudes till the damping term kicks in. The zonal velocity will start decreasing when $y < \frac{d(H_T)}{v}u$. As we can see from the plots, the lower the value of the damping at the top, the further away this latitude lies. A direct result of this is an increase in the region where the zonal velocity is an increasing function, hence increasing the maximum velocity of the winds in the jet. The zonal velocity at the top of the troposphere for different value of the damping parameter have been plotted in Fig.1.4. As the length scale of the damping decreases, the maximum of the jet moves towards the poleward end of the jet and the subsequent drop in the velocity becomes sharper.

Once we get the zonal velocity, we can use the meridional momentum equation to find the pressure and then the potential temperature using the hydrostatic balance. These, along with the total temperature have been plotted in Fig.1.3. The potential temperature at the tropopause decreases as we move polewards. At the poleward end of the cell, the temperature contours flip, with cooler air at the top and hot air at the bottom. This type of a profile is gravitationally unstable which means that the hydrostatic approximation no longer holds. As we will see in later sections, the

WTG theory itself is not applicable at higher latitudes where this flip of the potential temperature contours happens.

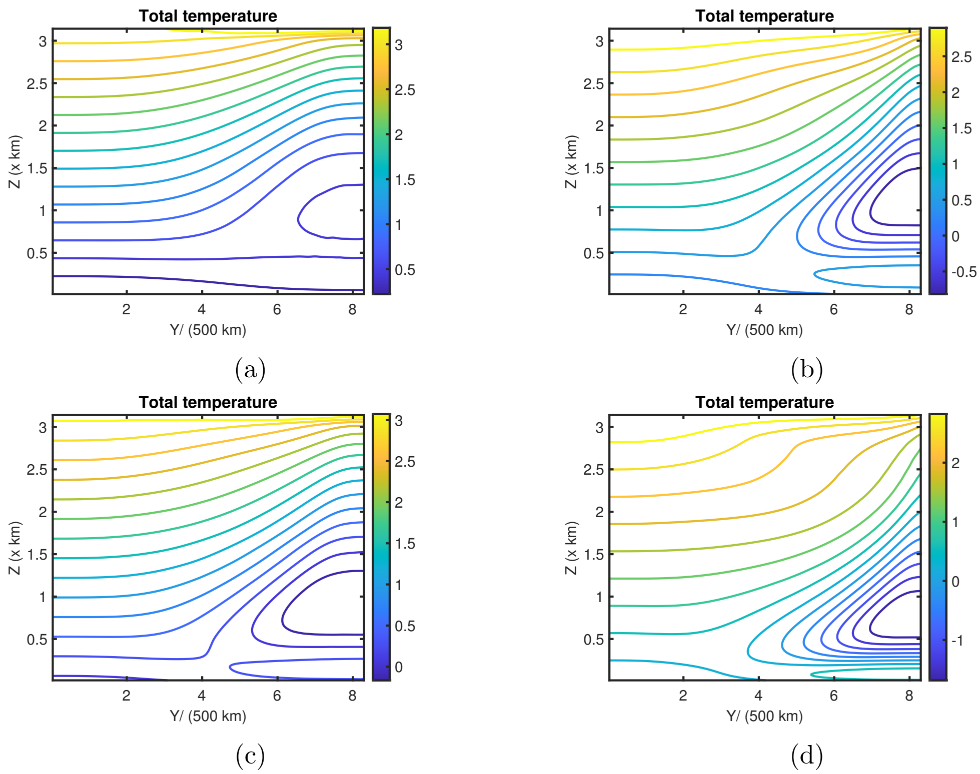


FIGURE 1.3. The contours of the total temperature have been plotted for varying maximum meridional velocity and the damping length scale λ. a) $v_{max} = 0.5$, $\lambda = \pi/2$. b) $v_{max} = 0.5$, $\lambda = \pi/4$. c) $v_{max} = 1$, $\lambda = \pi/2$. d) $v_{max} = 1$, $\lambda = \pi/4$

1.4.2.3. *Zonal velocity at the tropopause.* The system we have chosen here to model zonal velocity has a fixed tropopause height throughout. A natural consequence of this is to have vanishing vertical velocity at the tropopause. This makes the steady state zonal momentum equation at the tropopause an ODE and we can understand the behaviour of the zonal velocity by a simple model of the meridional velocity. Let us assume that the horizontal domain is from $y = 0$ to $y = 1$ and the meridional velocity is given by

$$v(y) = Ay(1 - y)$$

The velocity is zero at both the ends to represent a closed circulation. The zonal momentum equation becomes

$$v\frac{du}{dy} - yv = -d(H)u$$
$$d(z) = d_0 e^{-z/\lambda}$$

For our chosen meridional velocity, this can be written as

$$\frac{du}{dy} + \gamma\frac{u}{y(1-y)} = y$$
$$\gamma = d(H)/A$$

For a small length scale, the parameter γ will be very small. The meridional velocity chosen lets us compute the solution of the above equation exactly. The solution is given by

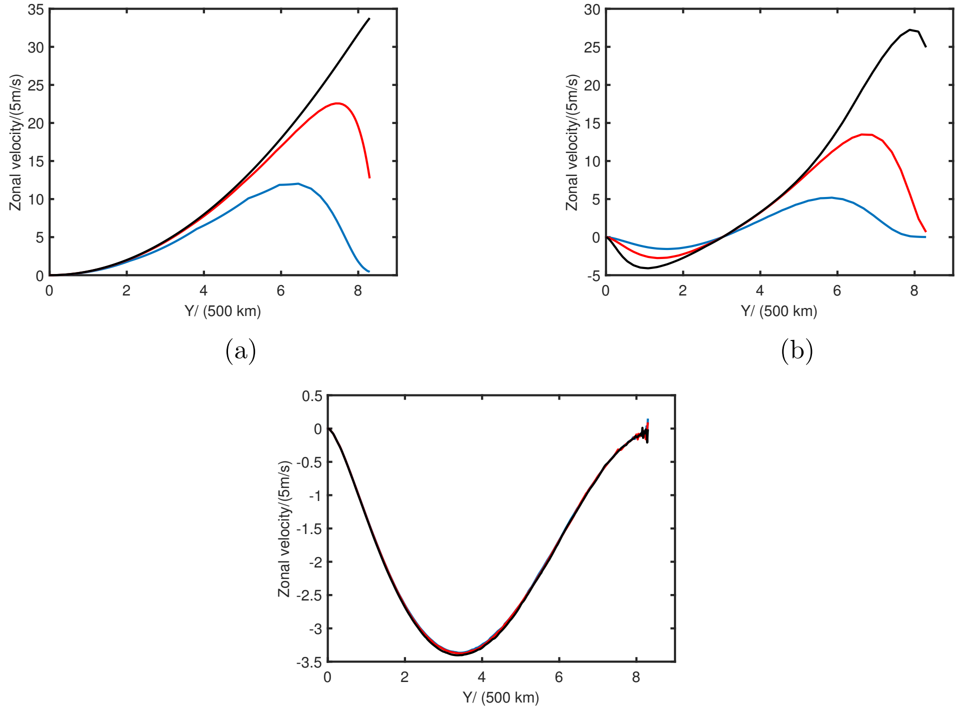
$$\left(\frac{y}{1-y}\right)^{\gamma} u = \int_0^y s^{1+\gamma}(1-s)^{-\gamma} ds$$

Since γ is small we can expand the integrand as a Taylor series and integrate to get an asymptotic series

$$u(y) = (1-y)^{\gamma}\left[\frac{y^2}{2+\gamma} + \gamma\frac{y^3}{3+\gamma} + \dots\right]$$

This shows the initial y^2 behaviour of the zonal velocity and its eventual drop to zero at the end of the circulation($y = 1$).

1.4.3. Multiple Scales. For the WTG theory, we have used a background temperature given by $\Theta = z$ which is not decided by the theory but needs to be put in as a parameter. The perturbations in temperature were two orders of ϵ separated from the leading order temperature. In this section we will use multiple scales asymptotics to show that the theory can support lower order temperature perturbations. This section follows the multiple scale separation of the MEWTG(Multiscale equatorial WTG) theory in Majda and Klein([24]), Biello and Majda([3]) but goes one step further by having the leading order temperature perturbation to be of the same scale as the background temperature perturbation. We define a planetary length scale $X = \epsilon x$ where a unit of X corresponds to 5000 km in the zonal direction. With this, the zonal derivative



(a) (b) (c)

FIGURE 1.4. The variation of zonal velocity with latitude for $v_{max} = 1$ for multiple λ has been plotted for different heights. a), b) and c) have been plotted at the tropopause, middle of the troposphere and the surface respectively. Blue, red and black correspond to $\lambda = \pi/2$, $\pi/4$ and $\pi/8$ respectively.

becomes

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x} + \epsilon \frac{\partial}{\partial X}$$

For the solution we again use a regular asymptotic expansion

$$u^\epsilon = u + \epsilon u_1 + \dots$$

$$v^\epsilon = v + \epsilon v_1 + \dots$$

$$w^\epsilon = w + \epsilon w_1 + \dots$$

$$p^\epsilon = \epsilon^{-1}\Pi + p_0 + \epsilon(p + \epsilon p_1 + \dots)$$

$$\theta^\epsilon = \epsilon^{-1}\Theta + \theta_0 + \epsilon(\theta + \epsilon\theta_1 + \dots) \tag{1.23}$$

where all the terms are a function of (t, X, x, y, z). The leading order terms in the zonal and meridional momentum equation are at $O(\epsilon^{-2})$. This gives us

$$\Pi_x = \Pi_y = 0$$

$$\Pi_z = \Theta \tag{1.24}$$

The leading order pressure and temperature cannot depend on the meridional or short zonal distance. Collecting $O(\epsilon^{-1})$ terms gives us

$$\Pi_X + p_{0,x} = 0$$

$$p_{0,y} = 0$$

$$p_{0,z} = \theta_0 \tag{1.25}$$

The first equation above can be leveraged to gain even more information about the variables Π and p_0. We define the zonal average as

$$\overline{f}(X, y, z, t) = \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} f(x, X, y, z, t)dx$$

If we take the zonal mean of the first equation in eq.1.25, the zonal mean of p_{0x} vanishes(due to periodicity) and all we are left with is the following

$$\Pi_X = 0$$

The above equation and the first equation in eq.1.25 gives us $p_{0,x} = 0$. Collecting the $O(1)$ terms gives us the following system

$$\frac{Du}{Dt} - yv + p_x = S_u$$

$$\frac{Dv}{Dt} + yu + p_y = S_v$$

$$\Theta_t + w(1 + \Theta_z) = S_\theta$$

$$p_z = \theta$$

$$u_x + v_y + w_z = 0 \tag{1.26}$$

The above set is not closed since the number of variables is more than the number of equations. In the WTG approximation of the previous section, we needed a cooling part in the forcing term S_θ to close the Hadley cell. In works of Held and Hou [12] this cooling is provided by a radiation term $-d_\theta \theta$ which relaxes the temperature to a prescribed background temperature profile. In the asymptotic formulation of WTG this cooling term is $O(\epsilon^2)$ and hence doesn't affect the $O(1)$ temperature equation. If we assume the leading order temperature perturbation to be of the same order as the background temperature, the damping temperature becomes $O(1)$ and can participate in the $O(1)$ balance. With this we can modify the temperature equation to write

$$\Theta_t + w(1 + \Theta_z) = S_\theta - d_\theta \Theta$$

In the work of Biello and Majda, p_0 and θ_0 are the unknown terms and they had to go to $O(\epsilon)$ equations to get a closure for the system, but even that requires some assumption on the mean of the incoming and outgoing $O(\epsilon)$ meridional velocity from the system. Their choice of mean zero inflow or outflow, although closing the system, cannot be justified by physical or mathematical considerations of the full system.

The WTG or the MEWTG theories were obtained in a thin band of 500km meridional length scale around the equator and acts like a boundary layer theory. The theory also has the unknown temperature perturbation term which cannot be obtained as of yet but might be obtained from the solution of the equations valid outside the boundary layer. For this we need to determine the region of validity of the WTG solution and obtain a new set of asymptotically valid equations outside this region.

1.4.4. Region of validity of WTG. The main simplifying balance in the equatorial theory is the weak temperature gradient balance. This works so long as the perturbation temperature θ remains much weaker than the background temperature Θ. Due to the simplicity of the WTG equations, the scaling of the solutions with latitude is fairly simple to ascertain. Using this scaling we can find the latitude at which the WTG approximation breaks.

Since the solution depends on the heating prescribed, we will assume that it remains $O(1)$. The temperature equation, thus implies that $w = O(1)$ as well. The incompressibility equation gives us the following balance

$$v_y = O(-w_z)$$
$$v_y = O(1)$$

Hence, assuming a positive heating near the equator, the above equation implies that $v = O(y)$. The zonal wind, at least at the top of the troposphere is in balance with the coriolis force

$$vu_y = O(yv)$$
$$u_y = O(y)$$
$$u = O(y^2)$$

The order of terms in the incompressibility equation is

$$u_x = O(y^2)$$
$$v_y = O(1)$$
$$w_z = O(1)$$

This means that for $y > O(1)$, there is nothing to balance the zonal derivative of the zonal velocity and hence either $u_x = 0$ or we can increase the zonal length scales to get $u_x = O(1)$. The incompressibility equation tells us that at most, the order of each term is $O(1)$. With this we can write the order of the advective derivative as

$$\begin{aligned}\frac{D}{Dt} &= \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \\ &= O(1) + O(u_x) + O(v_y) + O(w_z) \\ &= O(1)\end{aligned}$$

The total derivative remains an $O(1)$ operator as we move out of the equatorial layer. Now let us look at scaling of the terms in the meridional momentum equation.

$$\frac{Dv}{Dt} + yu = -p_y$$

$\frac{Dv}{Dt} = O(v) = O(y)$ while the coriolis term $yu = O(y^3)$. This suggests that the only term that can balance the coriolis term is the pressure gradient terms. So

$$p_y = O(y^3)$$
$$p = O(y^4)$$

The total pressure perturbation p_{tot} is ϵp and hence scales as $p_{tot} = O(\epsilon y^4)$. Due to the hydrostatic balance temperature and pressure perturbations have the same scaling as a function of the latitude. Thus, $\theta = O(p) = O(y^4)$. Summarising the scaling of the variables, we have

$$u = O(y^2), \quad v = O(y), \quad w = O(1)$$
$$\theta = O(y^4), \quad p = O(y^4)$$

Using the above scales, we can write the equations at general latitude $y = \epsilon^{-\alpha} Y$ where $Y = O(1)$. To include x derivatives in play, we need $x = \epsilon^{-2\alpha} X$ where $X = O(1)$. The variables can be scaled as follows

$$u = \epsilon^{-2\alpha} U, \quad v = \epsilon^{-\alpha} V, w = W$$
$$\theta = \epsilon^{-4\alpha} \Theta, \quad p = \epsilon^{-4\alpha} P$$

With these transformations, the primitive equation eq.2.2 becomes

$$\frac{DU}{Dt} - YV = -P_X - dU$$
$$\epsilon^{2\alpha} \frac{DV}{Dt} - YU = -P_Y - \epsilon^{2\alpha} dV$$
$$\epsilon^{2-4\alpha} \frac{D\Theta}{Dt} + W = S_\theta - d\epsilon^{2-4\alpha} \Theta$$
$$\Theta = P_z$$
$$U_X + V_Y + W_Z = 0 \tag{1.27}$$

The above equation is valid for $\alpha < 1$ since we have used a Taylor's approximation for the coriolis term, $\sin(\epsilon y) = y$. For $\alpha > 1$, the dominant balance in the meridional momentum equation is the geostrophic balance. While this is different from the meridional equation we get in the WTG tropical theory, the terms in the balance are still present in the tropical theory. So the solutions of the system are present within the setting of the tropical theory. The dominant balance in the temperature equation is WTG, so long as $\alpha < 1/2$. At $\alpha = 1/2$, the full temperature equation needs to be used since the gradients of the perturbation are now of the same magnitude as the background temperature. Hence, the WTG approximation breaks down at $\alpha = 1/2$

The solutions coming from the tropics are no longer valid at $\alpha = 1/2$, which we will call the subtropics and a new set of equations needs to be derived at this latitude. Another interesting thing about this latitude is the magnitude of the temperature perturbation is now equal to the background temperature. Hence the solution of this new equation will help us to find the unknown temperature term used in the multiple scale tropical theory of eq.1.26.

1.5. Summary

The troposphere is modeled as an incompressible boussinesq fluid. The system was non-dimensionalised using scales valid for setting up a circulation in the tropics. All the non-dimensional parameters were represented in terms of a single small parameter, ϵ. An asymptotic expansion in multiple small parameters doesn't yield a unique solution, hence the need for a single small parameter.

Using the non-dimensionalised version of the equations, the asymptotic solution at the tropics was derived which is also called the weak temperature gradient approximation. Solutions for a prescribed heating profile were obtained which display the hallmarks of a Hadley circulation. Using scale analysis on the WTG solution, the poleward extent of the tropical layer was determined as $O(\epsilon^{-1/2})$ or $1500km$. A new set of asymptotic equations needs to be derived at this latitude which will describe the atmosphere's behavior in the subtropics. This has been done in Chapter 3.

CHAPTER 2

Shallow Water

2.1. Introduction

The wide variety of length scales in the atmosphere can be exploited to obtain various simplified models. These simplified models, though only applicable to the scales at which they were derived are helpful in filtering out the physics which is not contributing in a significant way to the phenomenon we are trying to study. Shallow water equations are an example of such a simplification. The vertical length scale of the troposphere is of the order of tens of kilometers while the horizontal length scales used in the study of large scale atmosphere are of the order of hundreds or thousands of kilometers. This clear scale separation in the horizontal and vertical length scales is used in the derivation of the shallow water equation. Many of the phenomenon like Rossby waves, Baroclinic instability, equatorial dynamics can be explained in the context of shallow water equations.

A global asymptotic theory of the atmosphere, should, at the leading order match the Hadley cell of the tropics and subtropics to a suitable mid latitude theory. Since the Hadley cell is an averaged circulation in the y-z plane and shallow water equations do not have the z direction, we no longer have a "circulation". Even though we no longer have a circulation, we should still be able to observe the angular momentum conserving zonal velocity observed in the equatorial part of the circulation. In this sense, this is the simplest model of the Hadley cell we could work with. This very simplistic model of the Hadley cell has been studied by Schneider [**34**], Held and Phillips [**13**] and Polvani and Sobel [**30**], the latter most of whom have studied the applicability and validity of the WTG approximation.

Polvani and Sobel found a good agreement of the WTG solution with the full solution near the equator but the solutions differed the farther we go from the equator. In this chapter we will derive the non dimensionalised version of the shallow water equations applicable for the atmosphere and use formal asymptotics and scaling arguments to determine the validity of the WTG model. In particular, we need to determine the latitude at which the WTG model ceases to be valid and derive a new set of balanced equations there.

2.2. Non dimensionalised equations

The non dimensionalisation proceeds in quite the same way as the non-dimensionalisation of the full 3D primitive equations we saw in the last chapter. Due to using shallow water equations, the total set of unknown variables reduces from five to three. The non dimensionalised shallow water equations for the earth are

$$\frac{Du}{Dt} - 2\Omega \sin\left(\frac{y}{R_E}\right) v = -g h_x - S_u$$
$$\frac{Dv}{Dt} + 2\Omega \sin\left(\frac{y}{R_E}\right) u = -g h_y - S_v$$
$$\frac{Dh}{Dt} + (H + h)\nabla . \vec{u} = S_h \tag{2.1}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

In the above set of equations, R_E is the radius of the earth, g is the acceleration due to gravity, $\vec{u} = (u, v)$ and $H + h$ is the total height of the troposphere. H can be thought of as the average height of the atmosphere and h, the deviations from this average. We non-dimensionalise according to the the scalings used in Biello and Majda [3] which leads to WTG scaling near the equator and QG scalings in the midlatitude. Time is non dimensionalised by $T = 1$ day, horizontal length scales by $L = 500$km and the vertical length by an as yet unknown H_0. This leads to a velocity on the scale of $v_0 = 5m/s$. The momentum equation becomes

$$\frac{Du}{Dt} - 2\Omega T \sin\left(y\frac{L}{R_E}\right) v = -\frac{gH_0}{v_0^2} h_x - S_u$$
$$\frac{Dv}{Dt} + 2\Omega T \sin\left(\frac{y}{R_E}\right) u = -\frac{gH_0}{v_0^2} h_y - S_v$$

The two non dimensional terms on the left side are

$$2\Omega T = \frac{2\Omega L}{v_0} \sim 1/0.1 = \epsilon^{-1}$$
$$\frac{L}{R_E} \sim 0.1 = \epsilon$$

The non dimensional term $v_0/(2\Omega L)$ is called the Rossby number(Ro). To get a balance between the coriolis term and the gradient term on the right side, we get the scaling of the height H_0

$$H_0 = \text{Ro H}\frac{L^2}{L_d^2}$$

where Ro is the Rossby number as defined above and $L_d = \sqrt{gH}/(2\Omega)$ is the Rossby deformation radius. With the non dimensionalisation of the height deviation h known, we can now write the non-dimensionalised set of shallow water equations

$$\frac{Du}{Dt} - \frac{\sin(\epsilon y)}{\epsilon} v = -\frac{h_x}{\epsilon} - S_u$$
$$\frac{Dv}{Dt} + \frac{\sin(\epsilon y)}{\epsilon} u = -\frac{h_y}{\epsilon} - S_v$$
$$\epsilon F\frac{Dh}{Dt} + (1 + \epsilon F h)\nabla\vec{u} = S_h \tag{2.2}$$

where $F = \frac{L^2}{L_d^2} = O(1)$ and the total height is given by $H_{total} = 1 + \epsilon F h$. In this chapter we will use $F = 1$. The total height is non-dimensionalised by H, the average height of the atmosphere.

2.3. Equatorial theory

There are two terms in equations that prevent us from writing a uniform asymptotic expansion. These are the coriolis term, $\sin(\epsilon y)$ and the heating term, S_h. The coriolis term is $O(1)$ when $y \sim O(\epsilon^{-1})$. This corresponds to the mid latitudes where theories like planetary geostrophy or quasi geostrophy are valid, depending, of course on the length scales chosen. In the tropics, where $y \sim O(1)$, the coriolis term becomes $O(\epsilon)$. The heating term S_h, on the other hand is stronger at the equator while growing weaker towards the poles. Due to these non uniform terms, a uniform asymptotic expansion of the solution is not feasible.

In this section we will look at the tropical theory which is valid for $y \sim O(1)$ latitudes. At this latitude, the coriolis parameter $\sin(\epsilon y)$ can be expanded in y using a taylor expansion. Besides this, the heating is also assumed to be $O(1)$. The term S_u and S_v are the zonal momentum and meridional momentum damping. S_h contains total heating and radiation term. The parameterisation used for the damping terms is as follows

$$S_u = -du$$

$$S_v = -dv$$

$$S_h = S - \epsilon d_h h \tag{2.3}$$

The above parameterisation is called Rayleigh damping for the momentum damping terms and Newtonian cooling for the damping term in the total heating term. The time scale of the radiation damping has been chosen to be of the same order as the advective time scales. These parameterisation have been used in various simplified models of the atmosphere as seen in Matsuno (1966) [25], Gill(1979) [10], Held and Philips [13]. With these assumptions, the equations for $y \sim O(1)$ are

$$\frac{Du}{Dt} - (y - \epsilon^2 y^3 + \ldots)v = -\frac{h_x}{\epsilon} - du$$
$$\frac{Dv}{Dt} + (y - \epsilon^2 y^3 + \ldots)u = -\frac{h_y}{\epsilon} - dv$$
$$\epsilon\frac{Dh}{Dt} + (1 + \epsilon h)(u_x + v_y) = S - \epsilon d_h h \tag{2.4}$$

As we can see in the momentum equations, the height gradient term is unbalanced. Hence the height perturbation in the tropical regime should be $O(\epsilon)$. This leads to the following asymptotic expansion for the solution $(u^\epsilon, v^\epsilon, h^\epsilon)$

$$u^\epsilon = u + \epsilon u_1 + \epsilon^2 u_2 + \ldots$$

$$v^\epsilon = v + \epsilon v_1 + \epsilon^2 v_2 + \ldots$$

$$h^\epsilon = \epsilon(h + \epsilon h_1 + \epsilon^2 h_2 + \ldots) \tag{2.5}$$

For notational simplicity, we have not used subscript 0 for the leading order terms. The total height is $H_{total} = 1 + \epsilon^2 h^\epsilon$

2.3.1. $O(1)$ solution. Plugging in the expansion at the end of the previous section into eq. 2.4 and taking the limit $\epsilon \to 0$ gives us the leading order equations

$$u_t + uu_x + vu_y - yv = -h_x - du$$
$$v_t + uv_x + vv_y + yu = -h_y - dv$$
$$u_x + v_y = S$$

(2.6)

a) Non axisymmetric case

This regime is the shallow water analogue of the WTG regime of the full 3D equations. Instead of the temperature gradients being neglected, here we are neglecting the height gradient terms in the continuity equation. The removal of the time derivative from the height equation transforms the equations from third order in time to second order.

These equations can be solved by using Helmholtz decomposition on the velocity. Decomposing the velocity into a stream function, ψ, and a potential, ϕ, we get

$$u = -\psi_y - \phi_x$$
$$v = \psi_x - \phi_y$$

Plugging this into the incompressibility equation gives us

$$\phi_{xx} + \phi_{yy} = -S \tag{2.7}$$

which is just a Poisson equation for the potential ϕ. To obtain an equation for the stream function ψ, we need to take the curl of the momentum equation to get the vorticity equation. The vorticity equation for our system is given by

$$\frac{D(\zeta + y)}{Dt} + S(\zeta + y) = -d\zeta$$
$$\zeta = v_x - u_y$$

(2.8)

where $\zeta + y$ is the absolute vorticity. Plugging in the Helmholtz decomposition into the vorticity gives us

$$\psi_{xx} + \psi_{yy} = \zeta \tag{2.9}$$

This is a Poisson equation for the stream function. The solution strategy is as follows. The vorticity equation is first time stepped to obtain the vorticity at the $(n+1)^{th}$ time step from the n^{th} time step vorticity. From the vorticity, the Poisson equation is inverted to obtain the stream function, ψ. In case the heating function is independent of time, the potential ϕ can be calculated at the initial time itself and will remain time independent throughout. If the heating is time dependent, then the Poisson equation for the potential needs to be solved at each time step. From the $(n+1)^{th}$ ψ and ϕ we obtain the time stepped velocity.

b) Axisymmetric case

The axis symmetric case provides an even more simplified version of the model. The axis symmetric case can also be thought to represent the top of the Hadley cell, Schneider [34], Held and Phillips [13]. The Rayleigh damping in this case is due to instabilities and turbulent processes. The equations in this case become

$$\begin{aligned} u_t + vu_y - yv &= -du \\ v_t + vv_y + yu &= -h_y - dv \\ v_y &= S \end{aligned} \tag{2.10}$$

The solution method of these equations is quite simple. The incompressibility equation gives us the meridional velocity. This is then used in the zonal momentum equation to compute the zonal velocity which is then used in the meridional momentum equation to compute the height perturbation, h.

If the heating is symmetric about the equator, the meridional velocity at the equator should be zero. To get a closed circulation the meridional velocity should be zero at some latitude, $y = y_H$.

This corresponds to the integral of the heating term from $y = 0$ to $y = y_H$ to be zero.

$$\int_0^{y_H} S dy = 0 \tag{2.11}$$

This constraint on the heating is ad hoc and doesn't feel like a natural constraint imposed through the equations themselves. A way out of this is to look at the asymptotic expansion of the height variable again. The expansion for h^ϵ was obtained by looking at the height gradient term in the momentum equations. Since it is unbalanced, we needed the perturbation to be $O(\epsilon)$. But, since these are gradient terms, we can still have a constant, $O(\epsilon^{-1})$ height perturbation term. With this the height variable h^ϵ can be written as

$$h^\epsilon = \epsilon^{-1} h_0 + \epsilon(h + \epsilon h_1 + \ldots) \tag{2.12}$$

We can also have an $O(1)$ constant height perturbation term, but that can always be absorbed into the $O(\epsilon^1)$ term. With this the $O(1)$ equations become

$$\begin{aligned} u_t + vu_y - yv &= -du \\ v_t + vv_y + yu &= -h_y - dv \\ (1 + h_0)v_y &= S - d_h h_0 \end{aligned} \tag{2.13}$$

and the constraint in eq. 2.11 becomes

$$\int_0^{y_H} (S - d_h h_0) dy = 0 \tag{2.14}$$

This constant height perturbation can be computed using continuity arguments as in Polvani and Sobel(2002) [30] but we have to assume the large latitude behaviour of h. The Hadley circulation model of Held and Hou(1980) [12] also uses a constraint like the above equation but since they are using the full model instead of the WTG model, they do not have to compute this constant. This approach of computing h_0 as given in Polvani and Sobel works as long as we assume that the WTG approximation works for the entirety of the Hadley cell. This assumption cannot be justified by scaling arguments as we will see later. Hence the computation of this constant needs to come from a sub tropical asymptotic theory applicable to the region where WTG theory breaks down.

2.3.2. Scale analysis. In this section we will look at the scaling of $O(1)$ solution (u, v, h) of eq.2.10. This will allow us to figure out the validity of WTG approximation as we move out of the $y \sim O(1)$ layer. For this let us look at a latitude $y = \epsilon^{-\alpha} Y$ where $\alpha \geq 1$ and $Y = O(1)$. We will also assume that the heating S remains $O(1)$ at least till $y \sim O(\epsilon^{-1})$. This assumption for the heating and the incompressibility equation give us a scaling for the meridional velocity

$$v_y = \epsilon^{\alpha} v_Y = S = O(1)$$

i.e the meridional velocity v scales as $v = \epsilon^{-\alpha} V$, where $V = O(1)$. The balance between the advection term vu_y and the coriolis term yv in the zonal momentum equation gives us a scaling of the zonal velocity

$$vu_y = Vu_Y$$

$$yv = \epsilon^{-2\alpha} YV$$

Hence the zonal velocity u scales as $u = \epsilon^{-2\alpha} U$ where $U = O(1)$. With these scalings, the highest term on the left side of the meridional momentum equation is the coriolis term

$$yu = \epsilon^{-3\alpha} YU$$

These can only be balanced by the pressure term h_y. This balance gives us

$$h_y = \epsilon^{\alpha} h_Y \sim \epsilon^{-3\alpha} YU$$

So the height perturbation scales as $h = \epsilon^{-4\alpha} H$ where $H = O(1)$. The scaling for the zonal variable x is also obtained from the incompressibility by assuming that the two terms in the velocity divergence have the same scale. This gives us

$$u_x = \epsilon^{2\alpha} U_x = O(v_y) = O(1)$$

i.e $x = \epsilon^{-2\alpha} X$ where $X = O(1)$. Plugging these scalings for (u, v, h) and x into eq.2.4 we get

$$U_t + UU_X + VU_Y - YV = -H_X - dU$$

$$\epsilon^{2\alpha}(V_t + UV_X + VV_Y) + YU = -H_Y - \epsilon^{2\alpha} dV$$

$$\epsilon^{2-4\alpha}\left(\frac{DH}{Dt} + H(U_X + V_Y)\right) + (U_X + V_Y) = S - \epsilon^{2-4\alpha} d_h H \tag{2.15}$$

where the total derivative is

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$
$$= \frac{\partial}{\partial t} + U\frac{\partial}{\partial X} + V\frac{\partial}{\partial Y}$$

The second equation above says that the total derivative is an $O(1)$ operator. The balance in the zonal momentum equation is the same as the WTG zonal momentum. For $\alpha \geq 1$, the balance in the meridional momentum equation is the geostrophic balance $YU = -H_Y$. These terms are already present in the WTG meridional momentum equation. Similarly, for $2 - 4\alpha > 0$, we get the same balance as in WTG height equation. So for $\alpha < 1/2$, the leading order equations are a subset of the WTG equations. Hence the solutions arising from the equatorial theory of this section remains valid. This is the same heuristic argument used in the matching theory of singular perturbations in Lagerstrom and Casten(1972) [20]. The equations 2.15, also suggest that when $y = O(\epsilon^{-1/2})$, WTG approximation breaks down due to the height perturbations becoming as strong as the assumed background height and we have to derive a new set of balanced equations. These equations and their properties are discussed in the next section.

2.4. Subtropical theory

It is clear from eq.2.15 that the WTG approximation can't be extended indefinitely. $y = O(\epsilon^{-1/2})$ also comes out as a latitude out of this equation, where the height advection terms in the height equation can no longer be neglected. From the scale analysis of the variables in the equatorial theory, at $\alpha = 1/2$, the scale of the variables is as follows

$$(u, v, h) = O((\epsilon^{-1}, \epsilon^{-1/2}, \epsilon^{-1}))$$
$$(x, y) = O((\epsilon^{-1}, \epsilon^{-1/2}))$$

We recast the variables with these scalings and put them back in the full set of equations to derive the equations valid for the subtropical latitude

$$(u, v, h) \to (\epsilon^{-1}u, \epsilon^{-1/2}v, \epsilon^{-1}h)$$
$$(x, y) \to (\epsilon^{-1}x, \epsilon^{-1/2}y) \tag{2.16}$$

$$u_t + uu_x + vu_y - (y - \epsilon y^3 + \ldots)v = -h_x - du$$
$$\epsilon(v_t + uv_x + vv_y) + (y - \epsilon y^3 + \ldots)u = -h_y - \epsilon dv$$
$$\frac{Dh}{Dt} + (1 + h)(u_x + v_y) = S - d_h h \tag{2.17}$$

The total height is given by

$$H_{total} = 1 + h$$

The dimensional scale of the variables is as follows. The zonal and meridional velocity are measured in $50m/s$ and $15m/s$ respectively. The zonal and meridional distances are measured in $5000km$ and $1500km$ respectively and the time is still measured on 1 day scale, the same as in the equatorial theory. For the solution given by $(u^\epsilon, v^\epsilon, h^\epsilon)$, we plug the following regular asymptotic expansion

$$u^\epsilon = u + \epsilon u^{(1)} + \epsilon^2 u^{(2)} \ldots$$
$$v^\epsilon = v + \epsilon v^{(1)} + \epsilon^2 v^{(2)} \ldots$$
$$h^\epsilon = h + \epsilon h^{(1)} + \epsilon^2 h^{(2)} \ldots$$

(2.18)

2.4.1. O(1) solution. Plugging the above ansatz into eq.2.17 and taking $\epsilon \to 0$ gives us the $O(1)$ equations. These are

$$u_t + uu_x + vu_y - yv = -h_x - du$$

$$yu = -h_y$$

$$(h_t + (hu)_x + (hv)_y) + u_x + v_y = S - d_h h \tag{2.19}$$

The zonal momentum and the height equations are the full equations. The only simplification is in the meridional momentum equation which reduces to geostrophic balance. The strength of the forcing, S is the same order as in the equatorial theory. Since we obtained this system as a poleward end of the Hadley cell solution from the equatorial theory, we can look at the zonally independent solution of eq. 2.19

$$u_t + vu_y - yv = -du$$

$$yu = -h_y$$

$$(h_t + (hv)_y) + v_y = S - d_h h \tag{2.20}$$

The above system is second order in time, but the two time dependent equations are not independent because the zonal velocity and height are related by the meridional geostrophy constraint. The geostrophic constraint can be used to combine the zonal momentum and the height equation to eliminate the time derivative. This gives us an equation for the meridional velocity, v, which is satisfied at all times. This is the shallow water analogue of the Sawyer-Eliassen equation. Taking the y derivative of the height equation, we get

$$(h_{yt} + (hv)_{yy}) + v_{yy} = S_y - d_h h_y$$

Multiplying the zonal velocity by y and adding it to the above equation we get

$$((yu + h_y)_t + (hv)_{yy} + yvu_y - y^2 v) + v_{yy} = S_y - ydu - d_h h_y$$

The time derivative above vanished because of meridional geostrophy. The remaining equation can be simplified to get an ODE for the meridional velocity.

$$(1+h)v_{yy} + 2v_y h_y - (u+y^2)v = S_y - (d-d_h)yu \tag{2.21}$$

We get a linear second order ODE for the meridional velocity which needs be satisfied at all times. Due to the absence of a vertical variable in the shallow water equation, the shallow water Sawyer-Eliassen equation is much simpler than its 3D counterpart. As we will see in the 3D subtropical theory chapter(DSD regime), the Sawyer-Eliassen equation for the the 3D model is a second order PDE in two dimensions. This brings up complications like having hyperbolic and elliptic regions, depending upon the zonal velocity and temperature(height analogue for the 3D system). When both hyperbolic and elliptic regions appear, the solution is not easy to obtain. But this problem doesn't occur here since the ODE is one dimensional.

Considering the heating to be symmetric about the equator, the meridional velocity at the equator should be zero. This gives us the equatorial($y = 0$) boundary condition for eq.2.21. Another assumption we will impose on the $O(1)$ heating is that it vanishes outside the $y = O(\epsilon^{-1/2})$ layer and is $O(\epsilon)$ as we move out of this layer. This is also supported by the quasi geostrophic and planetary geostrophy theories Pedlosky [27], Vaalis [43] used in mid latitudes which correspond to $y = O(\epsilon^{-1})$ latitude. Without a heating term to drive the system, the meridional velocity goes to zero for $y = O(\epsilon^{-\alpha})$ and $\alpha > 1/2$. This corresponds to $v \to 0$ as $y \to \infty$ boundary condition for eq.2.21. The x independent meridional velocity can be thought of as the leading order average meridional velocity which vanishes in the mid-latitudes due to zonal geostrophic balance.

As pointed out in the equatorial theory section, the asymptotic expansion of height can have an $O(\epsilon^{-1})$ constant term in it. But this term remained arbitrary in the WTG theory. Using the subtropical system of this section, we can now find the origin of that term and how to compute it. First, let us see why eq.2.20 necessitates the existence of that term. The height equation is

$$(h_t + h_y v + h v_y) + v_y = S - d_h h$$

Since the height perturbation here is $O(\epsilon^{-1})$, $h(t, y = 0)$ here corresponds to the $O(\epsilon^{-1})$ height term h_0 in WTG. So, an omission of the constant height term in WTG means $h(t, y = 0) = 0$. Also, due to the geostrophic constraint here, $h_y(t, y = 0) = 0$. Plugging these into the above equation gives us the WTG equation

$$v_y = S \quad \text{at } y = 0 \tag{2.22}$$

Since we already have $v = 0$ at $y = 0$ and $y = \infty$ as boundary conditions for the Sawyer Eliassen equation eq.2.21, the WTG condition at $y = 0$ makes the system over determined. Hence we cannot have $h(t, y = 0) = 0$ here or $h_0 = 0$ in WTG system. Now, let us look at the behaviour of the height equation as $y \to \infty$. As $y \to \infty$, $v = v_y \to 0$ and $S \to 0$. This leaves us with

$$h_t = -d_h h \tag{2.23}$$

Hence, in the steady state, $h \to 0$ as $y \to \infty$. Using this condition and the meridional geostrophy we can write the height as

$$h(y, t) = h_0 - \int_0^y su(s, t)ds$$

Using the $y \to \infty$ boundary condition for the height, we get

$$\begin{aligned} h_0 &= \int_0^\infty su(s, t)ds \\ h &= \int_y^\infty su(s, t)ds \end{aligned} \tag{2.24}$$

To solve the full $O(1)$ system, we need to solve the Sawyer-Eliassen equation at each time step. Using the meridional velocity obtained, we can time step the zonal momentum equation. Then using eq.2.24 we can compute the height. So, the system corresponds to performing only one time step which implies that the system is only first order in time.

2.4.1.1. *Results.* In this section we plot the velocity and height fields for a prescribed heating function S of the form

$$S = S_0 \exp(-(y/y_0)^2)$$

S_0 gives the strength of the heating function while y_0 gives the width of the heating function. The form of the function describes a heating concentrated near the equator and decaying rapidly as we move away from it. There are four parameters that can be varied in this system. These are the heating strength, the width of the heating and the two Rayleigh damping parameters, d and d_h. In this section we will look at the behaviour of the velocity and height while varying these parameters.

a): Changing the heating strength

The heating strength was varied while keeping the other parameters fixed. $y_0 = \sqrt{3}$ which corresponds to 2500km. The zonal velocity damping rate was 0.1 day^{-1} and the heating damping rate was 1 day^{-1}. The plots are given in Fig.2.1. An increase in the heating strength has the obvious effect of increasing the magnitude of the velocities and the height at the equator. On the other hand, if we consider the width of the Hadley cell to be from the equator to the latitude where the meridional and zonal velocity sharply decrease to zero, then there hasn't been an appreciable change in the width of the Hadley cell. The maxima of the zonal velocity and the meridional velocity also hasn't shifted in position.

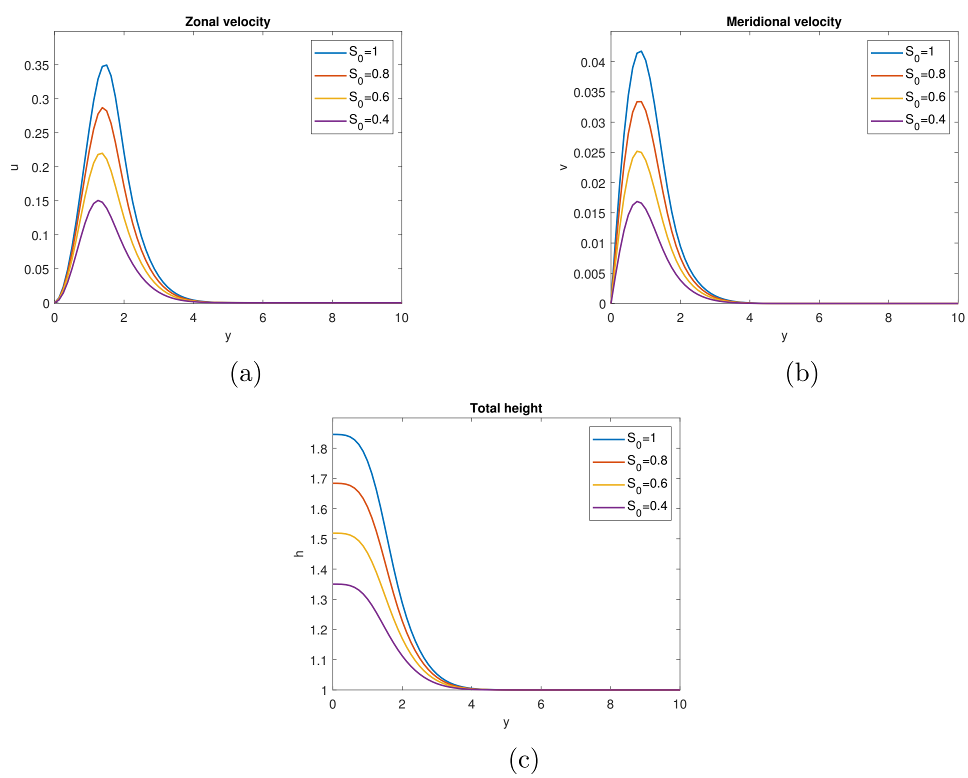
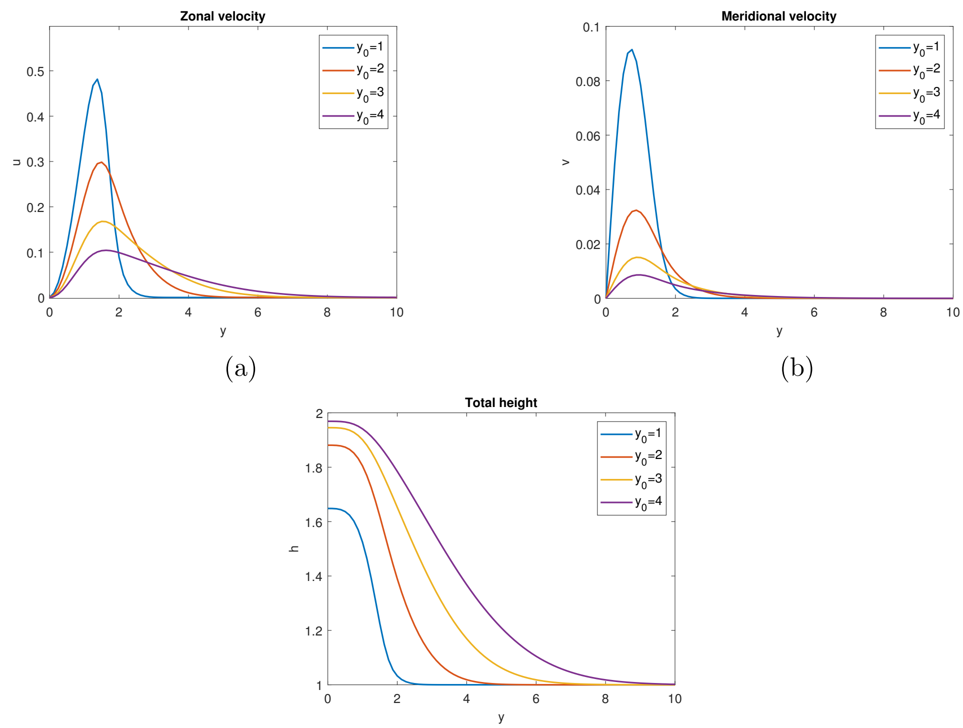


FIGURE 2.1. The heating amplitude S_0 was varied while keeping the other parameters fixed. $y_0 = \sqrt{3}$, $d = 0.1$, $d_h = 1$. a), b) and c) plot the zonal velocity, meridional velocity and the total height respectively. The meridional distance y is measured in units of 1500km.

b): Changing the heating width

The heating width was changed from $y_0 = 1$ to $y_0 = 4$ while keeping the other parameters fixed. This means that the heating is becoming less localised about the equator and is more spread out. These are plotted in Fig.2.2. The plots suggest that a more localised heating produces higher zonal and meridional velocity. The opposite happens with the total height. Even though a localised heating produces higher velocity, the velocity profiles also become localised and the width of the Hadley cell also shrinks.



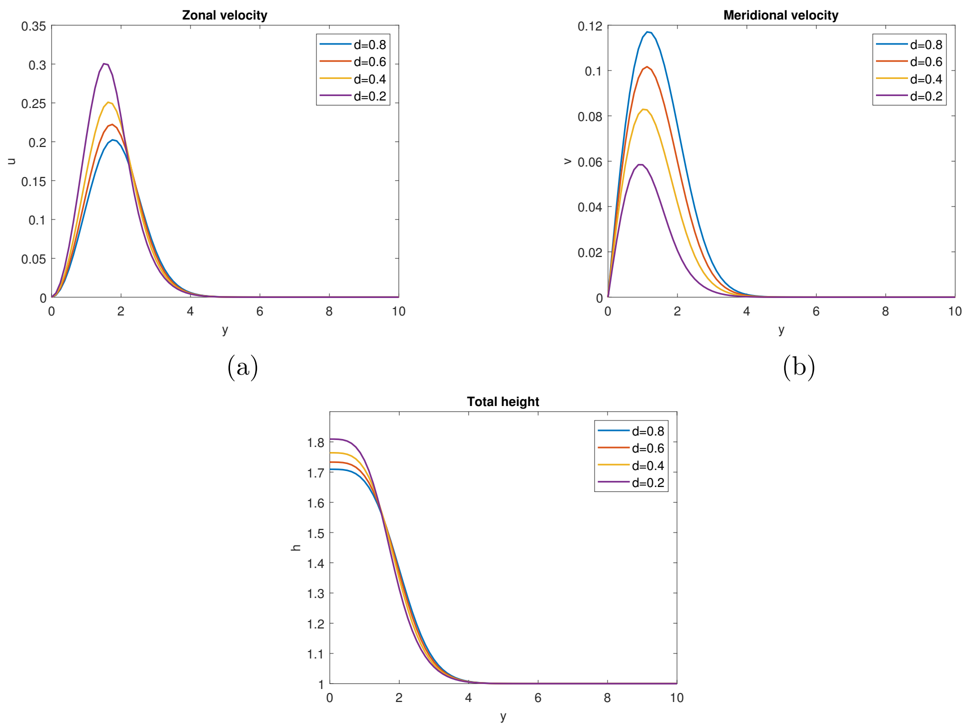
(a) (b) (c)

FIGURE 2.2. The heating width y_0 was varied while keeping the other parameters fixed. $S_0 = 1$, $d = 0.1$, $d_h = 1$. a), b) and c) plot the zonal velocity, meridional velocity and the total height respectively. The meridional distance y is measured in units of 1500km.

c): Changing the momentum damping

An increase in the zonal momentum damping has the obvious effect of decreasing the zonal momentum and hence increasing the height at the equator which follows from the geostrophic balance. The meridional velocity on the other hand has the opposite behaviour. It increases with an increase in the momentum damping. This can be explained by the Sawyer Eliassen equation where $-dyu$

plays the role of damping. A lower damping rate would imply lower damping, but since it also implies a higher zonal velocity u, there is a competition between the two and the zonal velocity wins out. So, a higher damping rate produces a lower zonal velocity and hence lower damping for the meridional velocity. Due to the same reason a lower damping shrinks the Hadley cell as well. These are plotted in Fig.2.3.



(a) (b) (c)

FIGURE 2.3. The momentum damping d was varied while keeping the other parameters fixed. $S_0 = 1$, $y_0 = \sqrt{3}$, $d_h = 1$. a), b) and c) plot the zonal velocity, meridional velocity and the total height respectively. The meridional distance y is measured in units of 1500km.

d): Changing the height damping

Decreasing the height damping results in an increase in all of the quantities. Both the strength and the width of the Hadley cell increase with a decrease in the height damping, d_h. The figures have been plotted in Fig.2.4

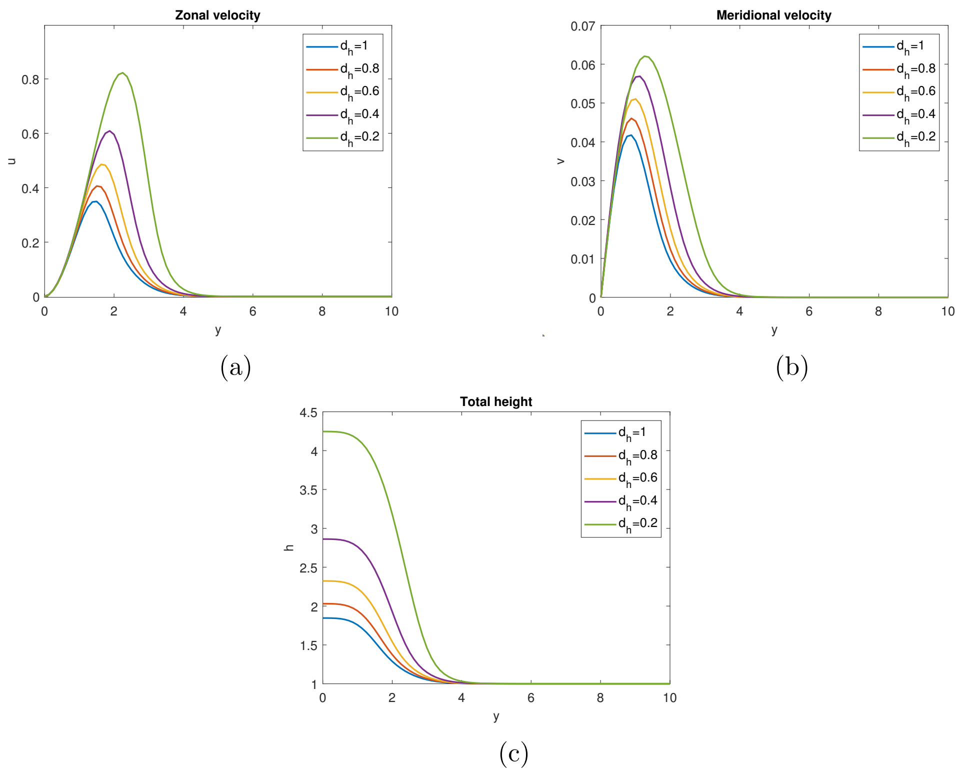


FIGURE 2.4. The height damping d_h was varied while keeping the other parameters fixed. $S_0 = 1$, $y_0 = \sqrt{3}$, $d = 0.1$. a), b) and c) plot the zonal velocity, meridional velocity and the total height respectively. The meridional distance y is measured in units of 1500km.

2.4.2. $O(\epsilon)$ solution. With zonally independent $O(1)$ solution and the ansatz in eq.2.18, the $O(\epsilon)$ system of equations in the subtropical theory is given by

$$u_t^{(1)} + uu_x^{(1)} + vu_y^{(1)} + v^{(1)}u_y - yv^{(1)} + y^3 v = -h_x^{(1)} - du^{(1)}$$

$$v_t + vv_y + yu^{(1)} - y^3 u = -h_y^{(1)} - dv$$

$$(h_t^{(1)} + uh_x^{(1)} + hu_x^{(1)} + (v^{(1)}h + h^{(1)}v)_y) + u_x^{(1)} + v_y^{(1)} = S^{(1)} - d_h h^{(1)} \tag{2.25}$$

$S^{(1)}$ is the second term in the asymptotic expansion of the heating in powers of ϵ, i.e $S_h = S + \epsilon S^{(1)} + \epsilon^2 S^{(2)}$. The solution method of the above linear system proceeds in the same way as the $O(1)$ system. An equation similar to the Sawyer-Eliassen equation can be derived so that the second equation in eq.4.21 is satisfied at all times.

For this section, we will use the $y \to \infty$ limits of the zeroth order system to get a simpler set of the above equations. Plugging in the behaviour of (u, v, h) at large y in the above equations we get

$$u_t^{(1)} - yv^{(1)} = -h_x^{(1)} - du^{(1)}$$
$$yu^{(1)} = -h_y^{(1)}$$
$$h_t^{(1)} + u_x^{(1)} + v_y^{(1)} = S^{(1)} - d_h h^{(1)} \tag{2.26}$$

The Sawyer-Eliassen equation for the above system is given by

$$v_{yy} - y^2 v + u_{xy} = S_y - yh_x - yu(d - d_h) \tag{2.27}$$

The solution of the homogeneous equation are given by parabolic cylinder functions. The large y system of $O(\epsilon)$ system, eq.2.26 is a linear system of PDEs. For zero forcing, it can be solved by plugging in wave solutions of the form

$$u^{(1)} = \tilde{u}(y) \exp(i(kx - \omega t))$$
$$v^{(1)} = \tilde{v}(y) \exp(i(kx - \omega t))$$
$$h^{(1)} = \tilde{h}(y) \exp(i(kx - \omega t))$$

Plugging this ansatz into eq.2.26 we get

$$-i\omega\tilde{u} - y\tilde{v} = -ik\tilde{h} - d\tilde{u}$$
$$y\tilde{u} = -\tilde{h}_y$$
$$-i\omega\tilde{h} + ik\tilde{u} + \tilde{v}_y = -d_h\tilde{h} \tag{2.28}$$

The zonal momentum and height equation are just algebraic equations. The y derivative only comes into play due to the meridional geostrophy. In any case, these equations should only be thought of as the limit of the $O(\epsilon)$ solution as we move out of the $y = O(\epsilon^{-1/2})$ layer.

2.4.2.1. *Scale analysis of the $O(\epsilon)$ solution.* To get the scaling of the $O(\epsilon)$ variables with latitude, we need to impose a few assumptions on the heating. We will assume that the leading order heating i.e S goes to zero as we move out of the $y = O(\epsilon^{-1/2})$ layer. In addition to this, we also assume that the $O(\epsilon)$ heating, i.e $\epsilon S^{(1)}$ remains $O(\epsilon)$ for all y. With this assumption and the height equation, we can get the scaling laws of the $O(\epsilon)$ velocities, $(u^{(1)}, v^{(1)})$. Let us write the latitude y as $y = \epsilon^{-\alpha} Y$ where $\alpha > -1/2$ and $Y = O(1)$. At this latitude, the $O(\epsilon)$ system simplifies to eq.2.26. The heating term $S^{(1)}$ needs to be balanced by the velocity divergence terms. This implies

$$
\begin{aligned}
v_y^{(1)} &= \epsilon^{\alpha} v_Y^{(1)} = O(1) \\
u_x^{(1)} &= O(1)
\end{aligned}
$$

The velocities, thus scale as $v^{(1)} = O(\epsilon^{-\alpha}) = O(y)$ and $u^{(1)} = O(1)$. The scaling for the height can be obtained from the geostrophic balance equation

$$
\begin{aligned}
h_y^{(1)} &= \epsilon^{\alpha} h_Y^{(1)} \\
y u^{(1)} &= \epsilon^{\alpha} h_Y^{(1)} \\
\epsilon^{-2\alpha} Y u^{(1)} &= h_Y^{(1)}
\end{aligned}
$$

Hence the $O(\epsilon)$ height scales as $h^{(1)} = O(\epsilon^{-2\alpha}) = O(y^2)$. While deriving the asymptotic equations for the subtropical theory, we had recast the variables with scaling appropriate for subtropical latitudes. Let us now put them back in the scales of the original shallow water equations of section 2.2

$$
\begin{aligned}
\mathcal{U} &= \epsilon^{-1} u^{\epsilon} = \epsilon^{-1}(u + \epsilon u^{(1)} + \epsilon^2 u^{(2)} + \ldots) \\
\mathcal{V} &= \epsilon^{-\frac{1}{2}} v^{\epsilon} = \epsilon^{-\frac{1}{2}}(v + \epsilon v^{(1)} + \epsilon^2 v^{(2)} + \ldots) \\
\mathcal{H} &= \epsilon^{-1} h^{\epsilon} = \epsilon^{-1}(h + \epsilon h^{(1)} + \epsilon^2 h^{(2)} + \ldots)
\end{aligned}
\tag{2.29}
$$

where $(\mathcal{U}, \mathcal{V}, \mathcal{H})$ are the variables in the scaling of eq.2.2. The meridional distances in this section correspond to $O(\epsilon^{-1/2})$ scale in eq.2.2. So the zonal distance, a general meridional latitude of

$O(\epsilon^{-\alpha})$,and the time variable in the scaling of eq.2.2 are

$$
\begin{aligned}
\mathbf{x} &= \epsilon^{-1}x \\
\mathbf{y} &= \epsilon^{-\frac{1}{2}-\alpha}y \\
\mathbf{t} &= t
\end{aligned}
\tag{2.30}
$$

The leading order quantities in eq.2.29 vanish if $\alpha > 0$, since the $O(1)$ heating term goes to zero. So, the scale of the variables is given by the first order terms. With this in mind and the scaling laws for the first order terms derived in this section, the scaling of the variables in the scale of eq.2.2 is

$$
\begin{aligned}
\mathcal{U} &= O(1) \\
\mathcal{V} &= O(\epsilon^{\frac{1}{2}-\alpha}) \\
\mathcal{H} &= O(\epsilon^{-2\alpha})
\end{aligned}
\tag{2.31}
$$

Writing the solution $(\mathcal{U}, \mathcal{V}, \mathcal{H})$ as $(u, \epsilon^{\frac{1}{2}-\alpha}v, \epsilon^{-2\alpha}h)$ and plugging this and the independent variable scaling of eq.2.30 into the primitive equation eq.2.2 we get the following

$$
\begin{aligned}
&\epsilon(u_t + \epsilon(u_x + vu_y)) - \epsilon^{-2\alpha}yv = \epsilon^{-2\alpha}h_x - du \\
&\epsilon(v_t + (uv_x + vv_y)) - yu = -h_y - \epsilon dv \\
&\epsilon^{1-2\alpha}(h_t + \epsilon(uh_x + vh_y)) + \epsilon(1 + \epsilon^{1-2\alpha}h)(u_x + v_y) = \epsilon S_h^{(1)}
\end{aligned}
\tag{2.32}
$$

The coriolis term $\sin(\epsilon y)$ has been approximated by ϵy. In case $\alpha = 1/2$, the y in the coriolis term needs to be replaced by $\sin(y)$. To get a balance with the time derivative term in the equation, we need to rescale time as $t \to \epsilon^{-2\alpha}t$. With this choice, for $0 \leq \alpha < 1/2$, we get the same balanced equations as we got from the extension of the subtropical theory in eq.2.26.

For $\alpha = 1/2$ i.e $\mathbf{y} = O(\epsilon^{-1})$, the balanced equations we get are different from the ones we obtain from the first order solution of the subtropical theory. This implies that the subtropical theory is only valid till $\mathbf{y} = O(\epsilon^{-1})$ and for this latitude we need to derive a new set of balanced equations.

2.5. Planetary geostrophy

As we saw in the previous section, the subtropical theory breaks down at the latitude of order ϵ^{-1}. This corresponds to meridional distances of order 5000km. So as to get a matching with the balanced equation valid for this latitude and the solution of the subtropical theory, we need to follow the scaling of the $O(\epsilon)$ solution of the subtropical theory. This scaling is given in eq.2.31 and for $\mathbf{y} = \epsilon^{-1}y$, this becomes

$$
\begin{aligned}
\mathcal{U} &= O(1) \\
\mathcal{V} &= O(1) \\
\mathcal{H} &= O(\epsilon^{-1})
\end{aligned}
\tag{2.33}
$$

So, at this latitude, we can write the variables $(\mathcal{U}, \mathcal{V}, \mathcal{H})$ as $(u, v, \epsilon^{-1}h)$ where (u, v, h) are $O(1)$ quantities. The scaling of the independent variables is $(\mathbf{x}, \mathbf{y}, \mathbf{t}) = \epsilon^{-1}(x, y, t)$. These correspond to 5000km in both the zonal and meridional directions and 10 days for the time scale. The heating is $O(\epsilon)$ and can be written as ϵS_h. With these scales, the primitive equation, eq.2.2 becomes

$$
\begin{aligned}
&\epsilon^2(u_t + uu_x + vu_y) - \sin(y)v = -h_x - \epsilon^2 du \\
&\epsilon^2(v_t + uv_x + vv_y) + \sin(y)u = -h_y - \epsilon^2 dv \\
&(h_t + (uh)_x + (vh)_y) + (u_x + v_y) = S_h - d_h h
\end{aligned}
\tag{2.34}
$$

The solution to the above equations can be written as an asymptotic series of the form

$$
\begin{aligned}
u^\epsilon &= u + \epsilon^2 u^{(1)} + \epsilon^4 u^{(2)} + \dots \\
v^\epsilon &= v + \epsilon^2 v^{(1)} + \epsilon^4 v^{(2)} + \dots \\
h^\epsilon &= h + \epsilon^2 h^{(1)} + \epsilon^4 h^{(2)} + \dots
\end{aligned}
\tag{2.35}
$$

2.5.1. O(1) solution. The leading order equations after plugging in the ansatz of eq.2.35 is

$$\sin(y)u = -h_y$$

$$\sin(y)v = h_x$$

$$(h_t + (uh)_x + (vh)_y) + (u_x + v_y) = S_h - d_h h \tag{2.36}$$

The above equations are called the planetary geostrophic equations and constitute a closed set of equations. The first two equations represent geostrophic balance and the third equation is a transport equation for the height. The scaling of the $O(\epsilon)$ solution of the subtropical layer shows that the time variable has to scale continuously from $O(1)$ at $y = O(\epsilon^{-1/2})$ to $O(\epsilon^{-1})$ at $y = O(\epsilon^{-1})$. This means that solutions from subtropical layer and PG in the intermediate region should match. However, as $\epsilon \to 0$, the solution from PG layer, being on a longer time scale would look stationary to the subtropical solution.

The time scale is set by the time scale of the advection terms $u\partial_x$ and $v\partial_y$. With the scales chosen at the beginning of this section, this comes out to be $O(\epsilon)$. This is why we have to chose a longer time scale. If we want an $O(1)$ time scale, we need to increase the magnitude of the velocity by an order in ϵ, i.e $(u, v) = O(\epsilon^{-1})$.

2.5.2. Shock formation in the height equation. The PG, height equation is a non linear advection equation and, using the geostrophic balance, can be written entirely in the height variable. The horizontal velocities, using the geostrophic balance can be written as

$$u = -\frac{h_y}{f}$$

$$v = \frac{h_x}{f} \tag{2.37}$$

where $f = \sin(y)$. Plugging this in the height equation we get

$$h_t - \left(\frac{h_y}{f}(1+h)\right)_x + \left(\frac{h_x}{f}(1+h)\right)_y = S_h - d_h h \tag{2.38}$$

Simplifying the above equation gives us

$$h_t - \frac{f_y}{f^2}(1+h)h_x = S_h - d_h h \tag{2.39}$$

The above equation implies that the height advection is non-linear and only travels in the zonal direction. The wave speed is given by $-\frac{f_y}{f^2}(1+h)$ and is always negative, implying a westward moving wave. The above equation can be solved implicitly for $S_h = 0$ and bears resemblance to the damped Burger's equation. We will use the method of characteristics to find the solution. The characteristic parameterized by the time coordinate t, is given by

$$\frac{dx}{dt} = -\frac{f_y}{f^2}(1+h)$$

while the corresponding equation along the characteristic transforms into

$$\frac{dh}{dt} = -d_h h$$

For the initial condition $h(t=0,x,y) = \tilde{h}(x,y)$, the solution is given by

$$h = \tilde{h}\left[x + \frac{f_y}{f^2}\left(t + \frac{h}{d_h}(e^{d_h t} - 1)\right), y\right]e^{-d_h t} \tag{2.40}$$

Taking an x derivative of the above equation gives us

$$h_x\left[1 - \frac{f_y}{d_h f^2}\tilde{h}_x(1 - e^{-d_h t})\right] = \tilde{h}_x e^{-d_h t} \tag{2.41}$$

Hence, the x derivative becomes infinite when

$$t^* = -\frac{1}{d_h}\ln\left(1 - \frac{f^2 d_h}{f_y \tilde{h}_x}\right) \tag{2.42}$$

For shocks to exist, the argument inside the logarithm should be less than 1. This is true if the derivative of the initial condition is positive somewhere in the domain. Given a periodic domain, this will always happen assuming the initial state $h \in \mathscr{C}^1$. For a real solution to exist, the argument inside the logarithm also needs to be bigger than zero. This gives us

$$\tilde{h}_x > \frac{f^2}{d_h}f_y \tag{2.43}$$

As $y \to 0$ or in the subtropical limit of the solution, the above condition for shock becomes $h' > 0$ and the time to develop shock $t^* \to 0$. This implies that shocks form instantly. Although the scaling, and hence the PG theory is not valid the scaling of the shock time is valid in some intermediate region between the subtropics and the midlatitudes. Taking the limit of eq.2.42 as $y \to 0$ we obtain the following for the scaling of the time taken to form shocks

$$t^* \sim \frac{f^2}{f_y \tilde{h}_x}$$
$$= \frac{\sin^2(y)}{\cos(y)\tilde{h}_x} \tag{2.44}$$

Using the geostrophic balance and the solution of the height equation, we can write the solution of the leading order zonal and meridional velocity. These come out to be

$$u = -\frac{(\tilde{h}_x[t + h(e^{d_h t} - 1)/d]F + \tilde{h}_x)e^{-d_h t}}{1 - \frac{\tilde{h}_y f_y}{f^2 d_h}(1 - e^{-d_h t})}$$
$$v = \frac{\tilde{h}_x f e^{-d_h t}}{f^2 - f_y \tilde{h}_x (1 - e^{-d_h t})/d_h} \tag{2.45}$$

In the zonal velocity expression, $F = \frac{d}{dy}\left(\frac{f_y}{f^2}\right)$. To match this with subtropical theory, we transform the coordinates from PG to the subtropical coordinates and look at the behaviour of the velocities. The transformation is as follows

$$y \to \epsilon^{1/2} y$$
$$t \to \epsilon t$$

This reverts the meridional and time scaling back to 1500km and 1 day. With this, the zonal and meridional velocity from the PG solution scale as

$$u \sim -\frac{\tilde{h}_x(1 + \tilde{h})t}{\epsilon y^2 (y^2 - \tilde{h}_x t)}$$
$$v \sim \frac{\tilde{h}_x y}{\sqrt{\epsilon}(y^2 - \tilde{h}_x t)} \tag{2.46}$$

As we can see from these expressions, the scale of the velocities match exactly with the scale of the velocities derived in the sub tropical section. When a heating term is added to the height equation, the scaling of the solution as $y \to 0$ is not as evident since an analytic solution is no longer available.

2.5.3. Transformed time variable. The previous section matches the leading order solution of PG with the leading order terms in the subtropical theory. In this section we will look at the case when the leading order subtropical solutions are considered to be zonally symmetric which

results in them tending to zero as we move out of the subtropics. In this case we need to match the first order subtropical solution with the leading order PG solution.
The scaling coming from the $O(\epsilon)$ subtropical theory matches with the $O(1)$ PG scales. The only difference is in the scaling of the time variable. This is due to the fact that the leading order velocities are $O(1)$ while the distances are $O(\epsilon^{-1})$. The leading order solution of the subtropical theory as we move out of the $y = O(\epsilon^{-1/2})$ are zero. Using this the $O(\epsilon)$ equations in the subtropical theory are

$$u_t^{(1)} - yv^{(1)} = -h_x^{(1)} - du^{(1)}$$

$$yu^{(1)} = -h_y^{(1)}$$

$$h_t^{(1)} + u_x^{(1)} + v_y^{(1)} = S^{(1)} - d_h h^{(1)}$$

The height equation suggests that the zonal velocity remains $O(1)$ while $v^{(1)} = O(y)$. $O(1)$ zonal velocity and the geostrophic wind equation suggests that $h^{(1)} = O(y^2)$. The scale of meridional distance is $y = O(\epsilon^{-1/2})$. So, corresponding to the scales of the non dimensionalised primitive equations, the scales of the variables are

$$\mathcal{U} \sim 1$$

$$\mathcal{V} \sim \epsilon y$$

$$\mathcal{H} \sim \epsilon y^2$$

The above scaling is only valid for $y >> \epsilon^{-1/2}$. To get the above scaling baked into the variables, we can transform them as

$$\mathcal{U} = u$$

$$\mathcal{V} = \sin(\epsilon y) v$$

$$\mathcal{H} = \sin^2(\epsilon y) \frac{h}{\epsilon} \tag{2.47}$$

The above transformations give us the scaling valid for both the subtropical and PG latitudes. Plugging this into the primitive equations(for $y >> \epsilon^{-1/2}$), eq.2.2, we get the following

$$u_t + \epsilon u u_x + \sin(\epsilon y) v u_y - \frac{\sin^2(\epsilon y)}{\epsilon} v = -\sin^2(\epsilon y)\frac{h_x}{\epsilon} - du$$
$$\sin(\epsilon y)(v_t + u v_x + v(\sin(\epsilon y) v)_y) + \frac{\sin(\epsilon y)}{\epsilon} u = -\frac{(\sin^2(\epsilon y) h)_y}{\epsilon^2} - d\sin(\epsilon y) v$$

(2.48)

$$\sin^2(\epsilon y)[h_t + \epsilon(uh)_x + \sin(\epsilon y)(vh)_y + 3\epsilon\cos(\epsilon y) vh] + (1+\sin(\epsilon y)h)(\epsilon u_x + (\sin(\epsilon y)v)_y) = \epsilon S_h$$

For a general latitude $y = \epsilon^{-\alpha} Y$ with $1/2 < \alpha \leq 1$ the operator

$$\sin(\epsilon y)\partial_y \sim \epsilon^{1-\alpha} Y \frac{\partial}{\epsilon^{-\alpha}\partial Y}$$
$$= O(\epsilon)$$

So, for $\alpha < 1$, the advective operator is always $O(\epsilon)$. The derivative of the height with respect to time is $O(\epsilon^{2-2\alpha})$, which will dominate all the other terms in the height equation for $\alpha > 1/2$ and we will not be able to get a balanced equation. To get a balance in this equation we need to transform the time variable so that the total time derivative becomes $O(\epsilon)$. The time derivative term in the height equation can be written, for $\alpha < 1$, as

$$\sin^2(\epsilon y) h_t \approx \epsilon^2 y^2 h_t$$

To make the above term $O(\epsilon)$, $\forall y$, we transform the time variable into

$$t = \frac{\sin^2(\epsilon y)}{\epsilon}\tau \tag{2.49}$$

With the above transformation, the derivatives become

$$\frac{\partial}{\partial t} = \frac{\epsilon}{\sin^2(\epsilon y)}\frac{\partial}{\partial \tau}$$
$$\frac{\partial}{\partial y} = \frac{\partial}{\partial y} - 2\epsilon\frac{\tau\cos(\epsilon y)}{\sin(\epsilon y)}\frac{\partial}{\partial \tau}$$

For $\tau = O(1)$, the τ derivative formally scales as $O(1/(\epsilon y^{-2}))$. This matches with $O(1)$ time scale of the subtropical theory when $y = O(\epsilon^{-1/2})$ and $O(\epsilon^{-1})$ time scale of PG when $y = O(\epsilon^{-1})$. The y derivative on the other hand scales as $O(y^{-1})$. With the transformations in this section, the leading order equations for $\epsilon^{-1/2} << y << \epsilon^{-1}$ are

$$v = h_x$$
$$u = -(2h + yh_y - 2\tau h_\tau)$$
$$h_\tau + u_x + v_y = S_h$$

These are the same equations one would get from the subtropical equation set. These stop becoming valid when $y = O(\epsilon^{-1})$. For $y = \epsilon^{-1}Y$ we get the following, at the leading order

$$v = h_x$$
$$u = -\big(2\cos(Y)h + \sin(Y)h_Y - 2\tau\cos(Y)h_\tau\big)$$
$$\begin{aligned} h_\tau + &\sin^2(Y)(uh)_x + \sin^3(Y)(vh)_Y - 2\tau\sin^2(Y)\cos(Y)(vh)_\tau \\ &+ 3\cos(Y)vh + (1+\sin(Y)h)\big(u_x + (\sin(Y)v)_y - 2\tau\cos(Y)v\big) = S_h \end{aligned} \tag{2.50}$$

2.6. Summary

To simplify our full 3D system, we used the shallow water approximation to reduce the dimensionality of the system by one. Instead of 5 variables, now we only had to contend with 3 variables. Using the non-dimensionalisation used for the full system, we derived an analogous version of the shallow water equations. We used WTG approximation near the equator to derive the system valid for tropical dynamics. Using the scaling behaviour of the leading order solution, we found that the system could only be extended till $O(\epsilon^{-1/2})$ or around 1500 km from the tropics. Hence a new regime was derived at this latitude and the tropopause height at the equator was determined as a matching condition between the tropical and the sub tropical regimes. To obtain the polar limit of the sub tropical regime, we had to go up the asymptotic expansion and look at the $O(\epsilon)$ solution. This gives us $O(\epsilon^{-1})$ as the poleward limit of the subtropical regime. The scale of the variables leads to planetary geostrophy as the midlatitude theory. This results in the scales of the subtropical and midlatitude regimes to match but the time scales in the two theories are off by an order of ϵ. A variable time scale version of the planetary geostrophy has been discussed as a possible remedy for this discrepancy in the time scales but the resulting equations get highly non linear. The analysis of these equations has been left as a possible avenue for future studies.

CHAPTER 3

Subtropical Theory

The weak temperature gradient(WTG) approximation can be used to model a thermally driven Hadley cell. Polvani and Sobel [30] used the WTG approximation in a shallow water model to show the agreement between the WTG model and the full set of equations. The WTG solution agrees very well with the full model near the equator but starts to diverge from the actual solution the farther away we move. From the solutions of the WTG approximation, the flow was also observed to become gravitationally unstable near the poleward end of the Hadley cell. Hence WTG approximation works well for the ascending branch of the Hadley cell but fails to provide accurate results for the subtropical descending branch. In the following discussion we will briefly restate the derivation given at the end of the introductory chapter and determine the latitude at which the WTG approximation breaks down.

We start with our system of equations in the MEWTG scaling.

$$\begin{aligned}
\frac{Du}{Dt} - \frac{\sin(\epsilon y)}{\epsilon} v + \frac{p_x}{\epsilon} &= S_u \\
\frac{Dv}{Dt} + \frac{\sin(\epsilon y)}{\epsilon} u + \frac{p_y}{\epsilon} &= S_v \\
\epsilon \frac{D\theta}{Dt} + w &= S_\theta - \epsilon d_\theta \theta \\
p_z &= \theta \\
u_x + v_y + w_z &= 0 \qquad (3.1)
\end{aligned}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \qquad (3.2)$$

The horizontal velocities have been non-dimensionalised by $5m/s$ and the vertical velocity by $5cm/s$. The (x, y) distances are measured in 500km units and the vertical distance in units of 5km. Time is measured in units of 1 day.

Starting from the equator, there is a balance among the advection terms and the coriolis force in the zonal momentum equation i.e. $vu_y \sim yv$. This gives us the $u \sim y^2$ scaling of the Hadley cell. The balance between the coriolis term and the pressure term in the meridional momentum equation gives us a scaling for the pressure and in turn a scaling for the temperature.

$$p_y \sim \epsilon y u$$

$$p \sim \theta \sim \epsilon y^4 \tag{3.3}$$

In a zonally symmetric, thermally driven Hadley cell, we have the following balance in the incompressibility equation

$$v_y \sim w_z$$

Using the above balance and the temperature scaling we can find the latitude at which the weak temperature gradient approximation breaks down. The scale of the perturbation temperature advection terms in the temperature equation is

$$\epsilon v\theta_y \sim \epsilon w\theta_z \sim O(\epsilon^2 y^4 w)$$

The scale of the background temperature advection is given by w. The two will be of equal magnitude when $\epsilon y^4 = O(1)$ or $y = O(\epsilon^{-1/2})$.

3.1. Derivation of the subtropical theory

The latitude $y = O(\epsilon^{-1/2})$ sets the scale of both u and θ at $O(\epsilon^{-1})$. Let us assume the following scaling for the meridional velocity and the meridional distance

$$v \to \epsilon^{\alpha} v$$

$$y \to \epsilon^{-1/2} Y + \epsilon^{\beta} y$$

Since we want the latitude to be $O(\epsilon^{-1/2})$, the meridional length scale should be less than the latitude. This sets a constraint on β, i.e $\beta \geq -1/2$. To get the time and advective derivative in the total derivative to be of the same order we need

$$\partial_t \sim u\partial_x \sim v\partial_y \sim w\partial_z \tag{3.4}$$

This gives us the following scales for the time and length scales

$$\begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \epsilon^{\beta-\alpha} \\ \epsilon^{\beta-\alpha-1} \\ \epsilon^{\beta} \\ \epsilon^{0} \end{bmatrix}$$

For the dependent variables we have the following scaling

$$\begin{bmatrix} u \\ v \\ w \\ p \\ \theta \end{bmatrix} = \begin{bmatrix} \epsilon^{-1} \\ \epsilon^{\alpha} \\ \epsilon^{\alpha-\beta} \\ \epsilon^{-1} \\ \epsilon^{-1} \end{bmatrix}$$

A further balance between the advective terms and the coriolis force gives us $\beta = -1/2$. So we can combine the meridional distance into a single variable $y \to \epsilon^{-1/2}y$. The above scaling gives us the following equations

$$\begin{aligned} \frac{Du}{Dt} - yv + p_x &= S_u \\ \epsilon^{2(\alpha+1)}\frac{Dv}{Dt} + yu + p_y &= 0 \\ \frac{D\theta}{Dt} + w &= S_\theta \\ \theta &= p_z \\ u_x + v_y + w_z &= 0 \end{aligned}$$

The leading order will satisfy meridional geostrophy provided $\alpha > -1$. If this constraint is satisfied, we get the following equations at the leading order

$$\begin{aligned}
&\frac{Du}{Dt} - yv + p_x = S_u \\
&yu + p_y = 0 \\
&\frac{D\theta}{Dt} + w = S_\theta \\
&\theta = p_z \\
&u_x + v_y + w_z = 0
\end{aligned} \tag{3.5}$$

The above equations have the same structure as the leading order equation set of IMMD by Biello and Majda [2] and the semi-geostrophic equations used in frontogenesis by Eliassen [9] and Hoskins and Bretherton [14]. Zonally symmetric and linear versions of these equations have been widely used(Leovy [21], Dunkerton [6]) to model circulations. When $\alpha = 0$ we also get the same scaling of the variables as in IMMD. When $\alpha = -1/2$ we get the same 1 day time scale and 5000km x scale as in the long wave MEWTG theory.

3.1.1. Two interesting scalings. The previous section shows that there are infinite scalings that basically give us the same leading order equations as IMMD. The only difference between these infinite set of perturbation equations is in the choice of α or the choice of meridional velocity scale.

Taking inspiration from the scaling in QG and MEWTG, we can choose $\alpha = 0$, keeping the order of meridional velocity the same in all these regimes i.e 5m/s. This gives us the following scales for the variables

$$\begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} = O \begin{bmatrix} \epsilon^{-1/2} \\ \epsilon^{-1/2} \\ \epsilon^{-1/2} \\ \epsilon^{0} \end{bmatrix}, \qquad \begin{bmatrix} u \\ v \\ w \\ p \\ \theta \end{bmatrix} = O \begin{bmatrix} \epsilon^{-1} \\ \epsilon^{0} \\ \epsilon^{1/2} \\ \epsilon^{-1} \\ \epsilon^{-1} \end{bmatrix}$$

which leads to the following set of equations

$$\frac{Du}{Dt} - yv + p_x = S_u$$
$$\epsilon^2 \frac{Dv}{Dt} + yu + p_y = 0$$
$$\frac{D\theta}{Dt} + w = S_\theta$$
$$\theta = p_z$$
$$u_x + v_y + w_z = 0 \tag{3.6}$$

This is the long time and long wave theory from Biello and Majda [2]. The heating is lower than the heating in MEWTG by a factor of $\epsilon^{-1/2}$. If we want the subtropical theory and the equatorial theory to have the same order of magnitude of heating i.e $O(1)$ heating, we can no longer have a $O(1)$ meridional velocity. To see this we can make use of the weak temperature gradient approximation. At the equator and in the intermediate region between the equator and the subtropics i.e for $y = o(\epsilon^{-1/2})$, the vertical velocity and heating should be of the same order of magnitude. By using the incompressibility equation we have

$$v = -\int_0^y w_z dy$$

If w_z remains $O(1)$, the meridional velocity scales as y. This implies that in the subtropics, where $y = O(\epsilon^{-1/2})$, v is $O(\epsilon^{-1/2})$. This gives us $\alpha = 1/2$ which in turn gives us the following scaling of the variables

$$\begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} = O\begin{bmatrix} \epsilon^0 \\ \epsilon^{-1} \\ \epsilon^{-1/2} \\ \epsilon^0 \end{bmatrix}, \qquad \begin{bmatrix} u \\ v \\ w \\ p \\ \theta \end{bmatrix} = O\begin{bmatrix} \epsilon^{-1} \\ \epsilon^{-1/2} \\ \epsilon^0 \\ \epsilon^{-1} \\ \epsilon^{-1} \end{bmatrix}$$

The x and t scaling above is the same as the planetary length scale(5000 km) and 1 day time scaling from MEWTG. We will call this scaling the diurnal subtropical dynamics(DSD). The equations corresponding to these scalings are

$$\begin{aligned}
&\frac{Du}{Dt} - yv + p_x = S_u \\
&\epsilon\frac{Dv}{Dt} + yu + p_y = 0 \\
&\frac{D\theta}{Dt} + w = S_\theta \\
&\theta = p_z \\
&u_x + v_y + w_z = 0 \qquad (3.7)
\end{aligned}$$

3.1.2. Energy equation. Let us work with the leading order equations 3.5 in DSD. Multiplying the zonal velocity equation by u, the temperature equation by θ and the meridional geostrophy by v we get

$$\begin{aligned}
&\frac{D(u^2/2)}{Dt} - yuv + up_x = uS_u \\
&yuv + vp_y = 0 \\
&\frac{D(\theta^2/2)}{Dt} + w\theta = \theta S_\theta
\end{aligned}$$

Adding the three and using the hydrostatic balance to replace $w\theta$ by wp_z we get

$$\frac{D}{Dt}(u^2/2 + \theta^2/2) + \nabla.(\vec{u}p) = uS_u + \theta S_\theta \qquad (3.8)$$

where $\vec{u} = (u, v, w)$ is the three dimensional velocity vector. Denoting $u^2/2 + \theta^2/2$ by the energy E, we can write the above equation in conservative form as

$$\begin{aligned}
&\frac{\partial E}{\partial t} + \nabla.(\vec{u}(E + p)) = uS_u + \theta S_\theta \\
&E = \frac{u^2 + \theta^2}{2} \qquad (3.9)
\end{aligned}$$

In a bounded volume V, without any outflow, we can integrate the energy equation and use the divergence theorem to get rid of the advection term. This gives us

$$\frac{d}{dt}\int_V E dV = \int_V uS_u + \theta S_\theta \qquad (3.10)$$

Without any forcing, the energy will remain conserved within a bounded region with no outflow.

3.1.3. PV equation. The potential vorticity equation of the full system is given by

$$\frac{D(\vec{\omega}.\nabla\Theta)}{Dt} = (\vec{\omega}.\nabla S_\theta) + \nabla\Theta.(\nabla \times \vec{F}) \tag{3.11}$$

where

$$\vec{\omega} = -v_z\hat{i} + u_z\hat{j} + (v_x - u_y + \frac{\sin(\epsilon y)}{\epsilon})\hat{k}$$

$$\vec{F} = (S_u, S_v, \frac{\theta}{\epsilon})$$

$$\Theta = \epsilon\theta + z \tag{3.12}$$

With the scaling used in DSD , ($\alpha = 1/2$), we get the following expression for the leading order PV

$$Q = u_z\theta_y - (\theta_z + 1)(u_y - y) \tag{3.13}$$

The leading order terms in the forcing equation on the right are

$$\vec{\omega}.\nabla S_\theta = \big(U_z S_{\theta,y} - (U_y - y)S_{\theta,z}\big)$$

$$\nabla\Theta.(\nabla \times \vec{F}) = \theta_y S_{u,z} - (\theta_z + 1)S_{u,y} \tag{3.14}$$

Using the above relations, the PV equation at the leading order is

$$\frac{DQ}{Dt} = \big(U_z S_{\theta,y} - (U_y - y)S_{\theta,z}\big) + \theta_y S_{u,z} - (\theta_z + 1)S_{u,y} \tag{3.15}$$

W can use a potential formulation of (u, θ) using the thermal wind balance.

$$y(u - y^2/2) = yM = \phi_y$$

$$\theta + z = \Theta = -\phi_z$$

Using this, the potential vorticity becomes

$$Q = M_z\Theta_y - M_y\Theta_z$$

$$= \phi_{zz}\left(\frac{\phi_y}{y}\right)_y - \frac{\phi_{yz}^2}{y}$$

The PV in DSD is non linear in the potential function ϕ, which makes inverting the potential vorticity equation to obtain u and θ a difficult task. This can be contrasted with the PV in QG, which is linear in the stream function Ψ, and can be easily inverted.

3.2. Solution of zonally independent DSD

If we want to model the zonally averaged Hadley cell, we need to look at the x independent version of the DSD equations.

$$\begin{aligned} & u_t + vu_y + wu_z - yv = S_u \\ & yu = -p_y \\ & \theta_t + v\theta_y + w\theta_z + w = S_\theta \\ & \theta = p_z \\ & v_y + w_z = 0 \end{aligned} \tag{3.16}$$

The geostrophic balance and the hydrostatic balance can be used to write the thermal wind relation

$$yu_z + \theta_y = 0$$

The geostrophic wind condition can be leveraged to obtain a relation between the three velocities and the potential temperature which has no time derivative in it. The derivation of this equation is outlined here.

Taking the z derivative of the momentum equation and multiplying it by y we get

$$(yu_z)_t + yv_zu_y + yvu_{yz} + yw_zu_z + ywu_{zz} - y^2v_z = yS_{u,z} \tag{3.17}$$

Taking the y derivative of the temperature equation gives us

$$\theta_{yt} + v_y\theta_y + v\theta_{yy} + w_y\theta_z + w\theta_{yz} + w_y = S_{\theta,y} \tag{3.18}$$

Adding the above two equations and using the geostrophic wind constrain we can get rid of the time derivative and get the following

$$yv_z(u_y - y) - 2yv_yu_z - vu_z + w_y(\theta_z + 1) = yS_{u,z} + S_{\theta,y}$$

The incompressibility allows us to use a stream function ψ such that $(v, w) = (\psi_z, -\psi_y)$. Plugging this in the above equation gives us a second order PDE in ψ

$$y\psi_{zz}(u_y - y) - 2y\psi_{yz}u_z - \psi_{yy}(\theta_z + 1) - \psi_z u_z = yS_{u,z} + S_{\theta,y} \tag{3.19}$$

This is a version of the Sawyer Eliassen equation(Sawyer [32]) with the beta effect. The above PDE can be elliptic, parabolic or hyperbolic depending upon the sign of the discriminant, which in this case is given by

$$\begin{aligned} \Delta &= -y(-yu_z^2 - (\theta_z + 1)(u_y - y)) \\ &= -yQ \end{aligned} \tag{3.20}$$

where Q is the potential vorticity in DSD as derived in the previous section. In the northern hemisphere, the regions with $Q > 0$, $Q < 0$ and $Q = 0$ are elliptic, hyperbolic and parabolic respectively.

3.2.1. Boundary conditions. Let us assume that the domain is a rectangular box $[0, L] \times [0, H]$ in the $y - z$ plane. $y = 0$ and $y = L$ denote the equator and the poleward boundary respectively. $z = 0$ and $z = H$ denote the ground and the top of the troposphere respectively. The top and bottom boundary conditions are the easiest to specify. The no penetration boundary condition implies that the vertical velocity should be zero. This means that $\psi_y = 0$ at $z = 0$ and $z = H$ i.e. ψ is a constant at the top and bottom. These two constants can be different.

The boundary condition at the meridional boundaries is a bit less obvious. For the poleward boundary condition we can look towards the geostrophic theory which says $v = p_x/f$. Hence the zonally averaged v should be zero at the poleward boundary. This means that $\psi_z = 0$ and hence ψ should be a constant at $y = L$. Since ψ is a constant at the top, right and the bottom boundary, these three constants must be the same. If we assume the meridional velocity to be zero at the equator, this would make ψ a constant at the equator. So $v = 0$ boundary condition at the equator implies that $\psi = c$ everywhere along the boundary. We can also look at the WTG equatorial theory near y=0 to come up with an alternate boundary condition. The Sawyer-Eliassen equation as $y \to 0$ is simply

$$\psi_{yy} = -S_{\theta,y} \tag{3.21}$$

Integrating this once we get

$$\begin{aligned} \psi_y &= -S_\theta(y,z) + f(z) \\ \implies w &= S_\theta(y,z) - f(z) \end{aligned} \tag{3.22}$$

The first equation above provides the Neumann boundary condition at y=0. The function f(z) is still unknown. Applying WTG approximation near the equator we have

$$w = S_\theta(y,z) \tag{3.23}$$

Using the above two relations we have

$$f(z) = 0 \tag{3.24}$$

Hence the boundary condition at $y = 0$ should be

$$\psi_y(0,z) = -S_\theta(0,z) \tag{3.25}$$

We haven't yet discussed the boundary conditions for the zonal velocity. Assuming that we have $v = 0$ at the equator, the steady state zonal velocity equation at $y = 0$ is

$$wu_z = -du \tag{3.26}$$

Since $w = 0$ at the top and bottom, for a non zero damping d, we will have $u = 0$ at the top and bottom. Since the equation is an ODE, $u = 0$ is the only possible solution. The temperature is obtained by integrating the geostrophic wind balance equation.

3.2.2. Solution of the Sawyer-Eliassen equation. The Sawyer Eliassen equation is easy to solve when the potential vorticity is positive, i.e the system is elliptic. A Gaussian heating profile near the origin has been chosen

$$S_\theta = S\sin(Z) * \exp(-Y/3)^2; \tag{3.27}$$

For the momentum damping, a Rayleigh surface drag is used

$$S_u = -d\exp(-z/h)u \tag{3.28}$$

Two types of model zonal velocities have been used to compute the $v - z$ stream function.

Constant Shear:

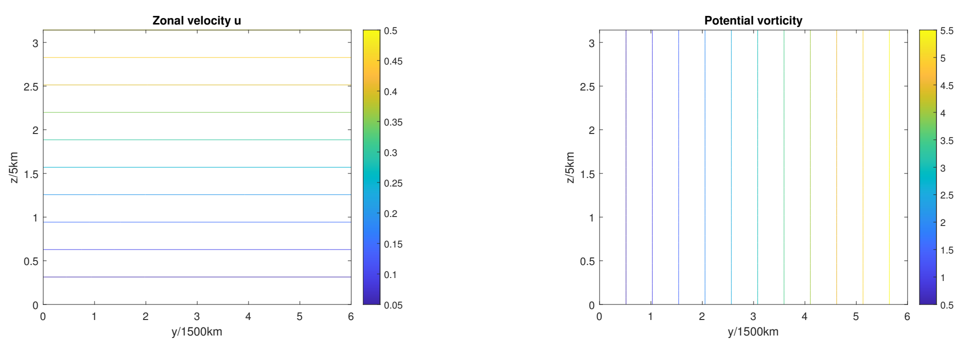
$$u = \lambda z$$

$$\theta = -\lambda\frac{y^2}{2}$$

The potential vorticity is

$$Q = y(1-\lambda^2) \tag{3.29}$$

We get an elliptic equation when $\lambda < 1$. Fig 3.1 shows the zonal velocity profile and the corresponding potential vorticity for $U_{max} = 25\ m/s$. The solution of the meridional and vertical velocity and the corresponding stream function for two different U_{max} have been plotted in Fig 3.2. These correspond to a maximum zonal velocity of 25m/s and 50m/s respectively.



(a) (b)

FIGURE 3.1. a)Zonal velocity with a constant linear shear(λ) for a maximum zonal velocity of 25 m/s and b) the corresponding potential vorticity.

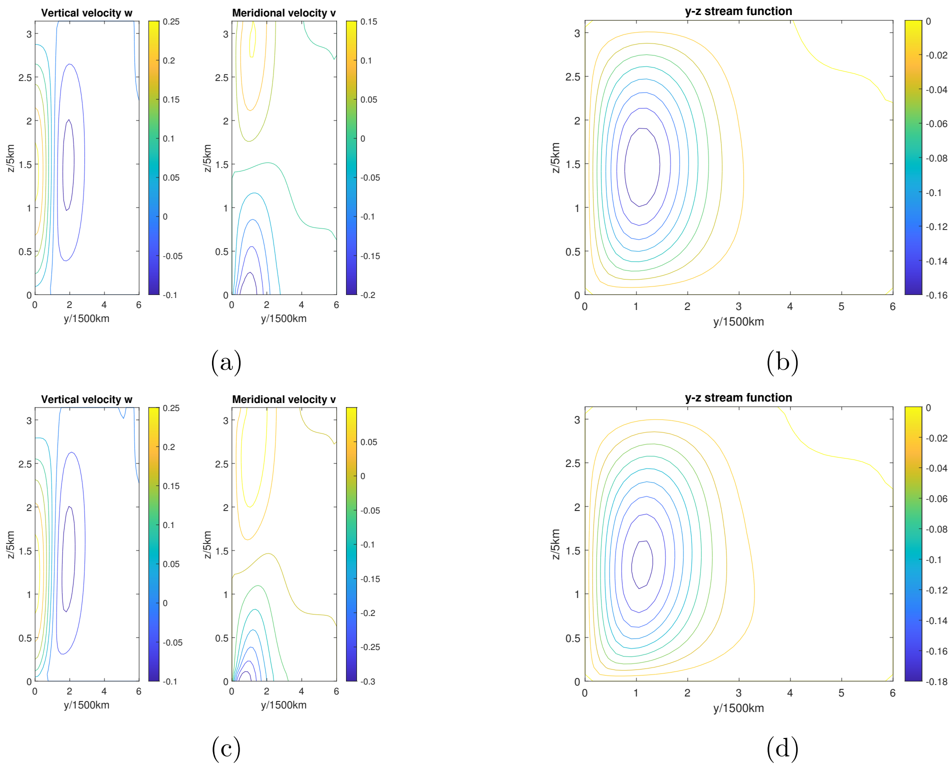


FIGURE 3.2. Zonal velocity with a constant linear shear(λ) has been used. a), b) has $U_{max} = 25m/s$. c), d) has $U_{max} = 50m/s$. Dirichlet boundary condition has been used at all boundaries.

Meridionally varying shear:

The following zonal velocity profile was chosen

$$u = \lambda z \frac{y^2}{2} \exp(-y^2/3)$$

This is an example of a realistic jet profile with a jet maximum around 3000km. The velocity profile and potential vorticity have been plotted for two different values of λ in Fig 3.3. The values of λ were chosen such that the maximum zonal velocity is 25 m/s and 50 m/s in the two cases. The corresponding solution for the stream function, meridional and vertical velocity have been plotted in Fig 3.4. Numerical error can be seen in subfigures c) and d) of Fig 3.4 near the equator. These errors correspond to the part of the domain where the flow has negative potential vorticity. Since negative potential vorticity means that the equations are hyperbolic, an elliptic solver is unable to deal with these areas.

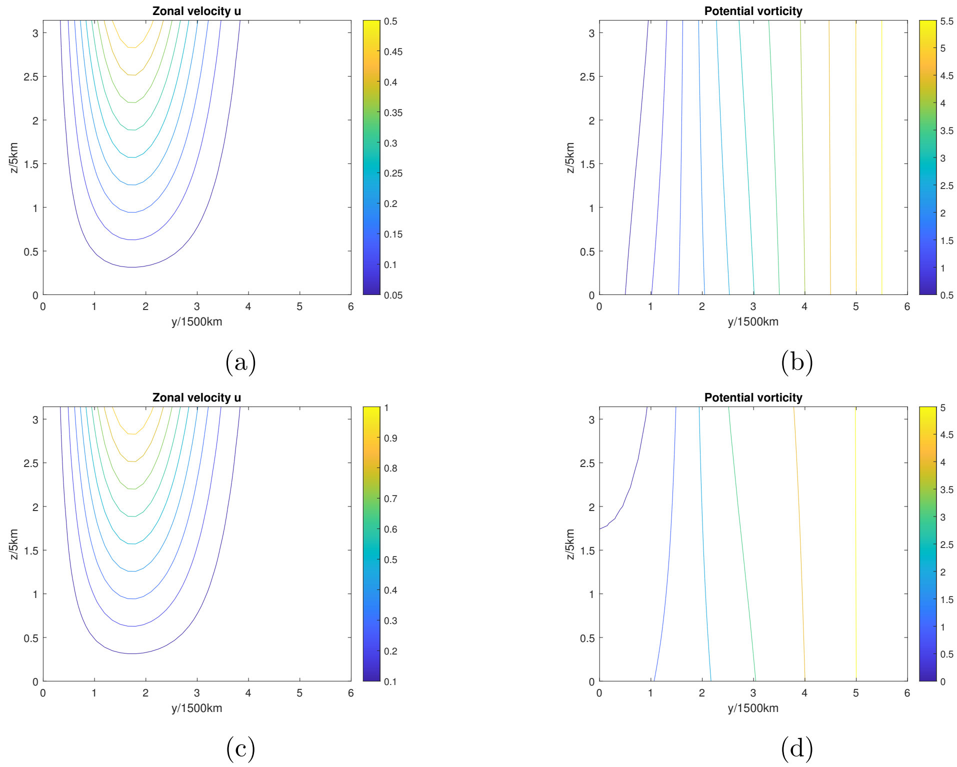


FIGURE 3.3. a), c) Zonal velocity and b), d) the corresponding potential vorticity in the meridionally varying case. λ is set so as to give a maximum velocity of 25m/s in a) and 50m/s in c)

3.2.3. Effect of heating. The following Gaussian heating field has been used for all the runs

$$S_\theta = S\sin(z)\exp(-(y/d)^2)$$

Increasing the strength of heating increases the strength of the circulation, but is not required to obtain a circulation at all. Even with zero heating, we still get a circulation although, it is very weak. The vertical velocity, meridional velocity and the stream function have been plotted in Fig. 3.5.

3.2.3.1. *Solutions with changed boundary condition at the equator.* For this section, instead of a zero v boundary condition at the equator, we have used the WTG boundary condition discussed in section 3.2.1. The solutions have been plotted below in Fig 3.6.

With the WTG boundary condition at $y = 0$, the meridional flow direction is the reverse of what is expected. Instead of a positive meridional velocity at the top and negative at the bottom of the troposphere we are getting the exact opposite.

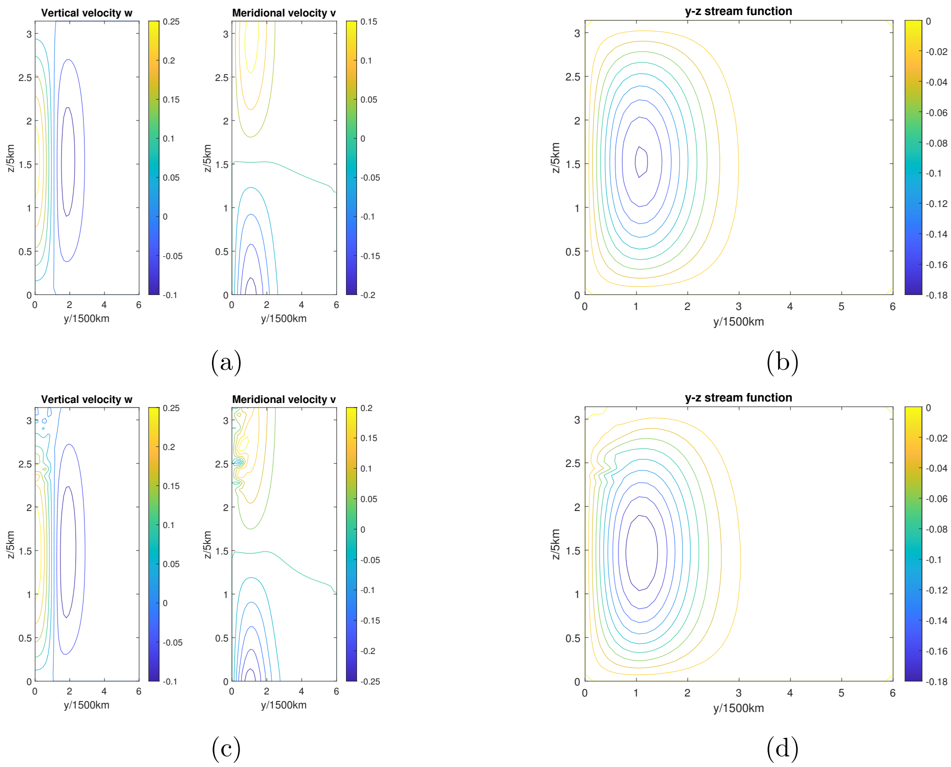
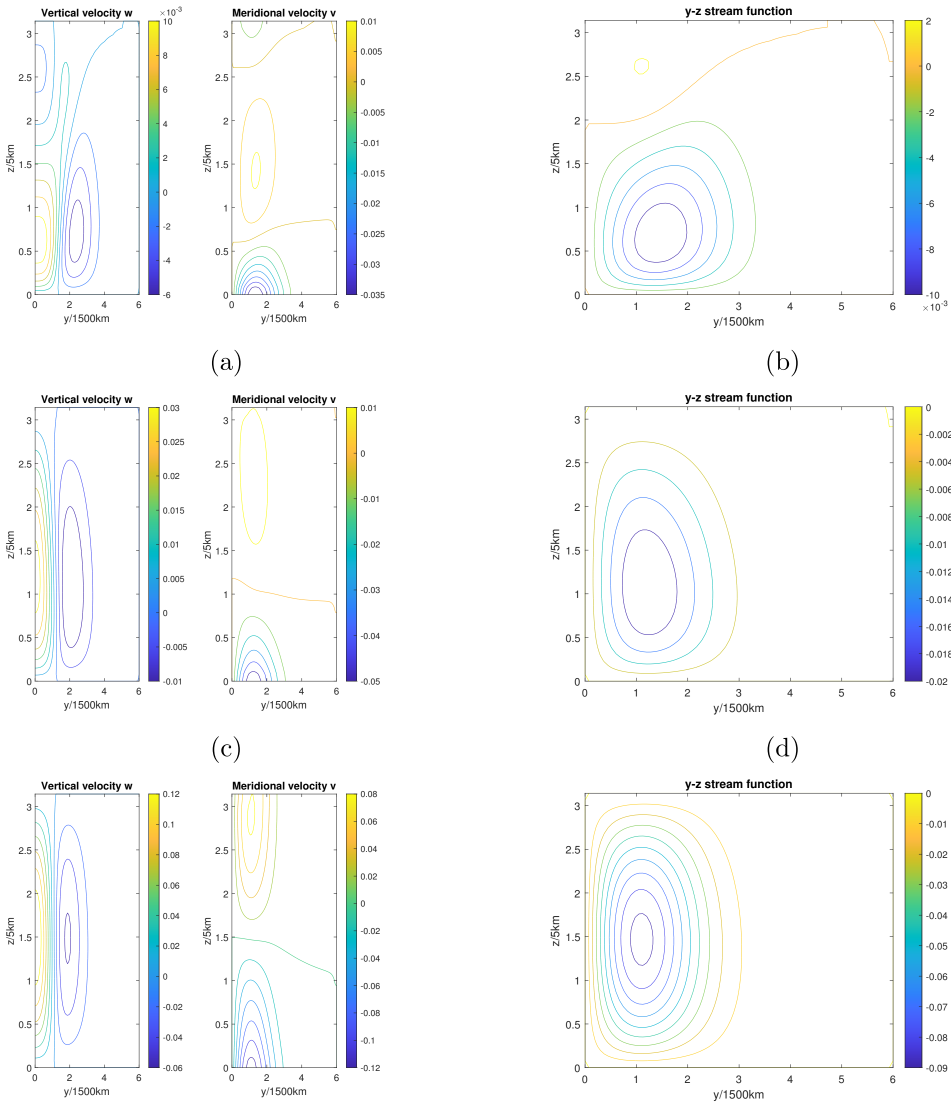


FIGURE 3.4. Zonal velocity with a meridionally varying shear has been used. a), b) has $U_{max} = 25m/s$. c), d) has $U_{max} = 50m/s$. Dirichlet boundary condition has been used at all boundaries.

Using WTG as a boundary condition for DSD seems to be problematic. In the symmetric heating case at least, we have an overdetermined set of boundary conditions coming from the equatorial theory. The meridional velocity at the equator is zero while the vertical velocity is given by the WTG approximation. So, the equatorial boundary layer theory gives us both the vertical velocity and the meridional velocity via the incompressibility relation. The elliptic operator we get out of the Sawyer Eliassen equation only requires one boundary condition at the equator side. If we give it the $v = 0$ boundary condition, WTG relation will not be satisfied and when the $\psi_y = -S_\theta$ condition is given, the $v = 0$(Dirichlet BC) condition won't be satisfied.

The vertical velocity at the equator with Dirichlet boundary condition has been plotted in Fig. 3.7 for different values of the zonal velocity shear, λ. The vertical velocity at the equator is always smaller than that predicted by WTG. In the $S_0 = 1$ case, the forcing is dominated by the heating term and there is not much variation with increasing U_{max}. The variation is appreciable when the heating is reduced. This discrepancy between the WTG equatorial theory and the subtropical



(a) (b)

(c) (d)

(e) (f)

FIGURE 3.5. The maximum heating at the equator S has been varied. The meridionally varying zonal velocity profile with $U_{max} = 25m/s$ has been used in all of them.

theory arises because we used $\theta = 0$ at the $y = 0$ boundary for both models. As we will see in the matching section later, the way out of this is to think of $\theta(t, y = 0)$ coming out of the subtropical solution as a boundary condition which then feeds into the equatorial WTG theory. For this we need to look at time dependent solutions of the subtropical DSD model.

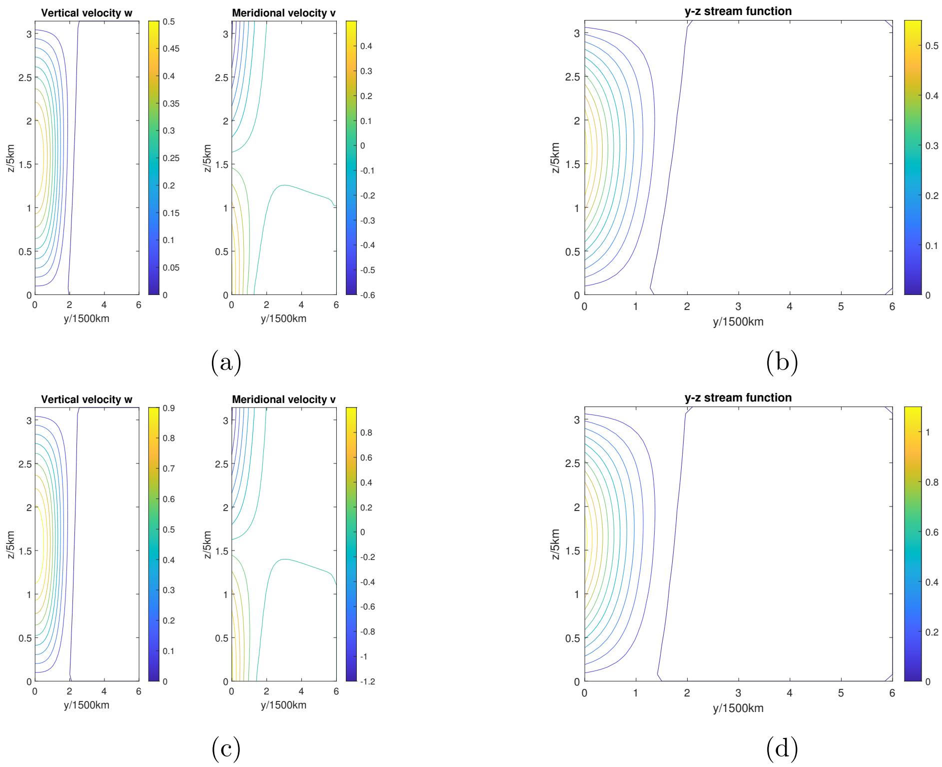


FIGURE 3.6. The meridionally varying zonal velocity profile with $U_{max} = 25m/s$ has been used. The heating parameter S has been kept at 0.5 in a) and b) and 1 in c) and d).

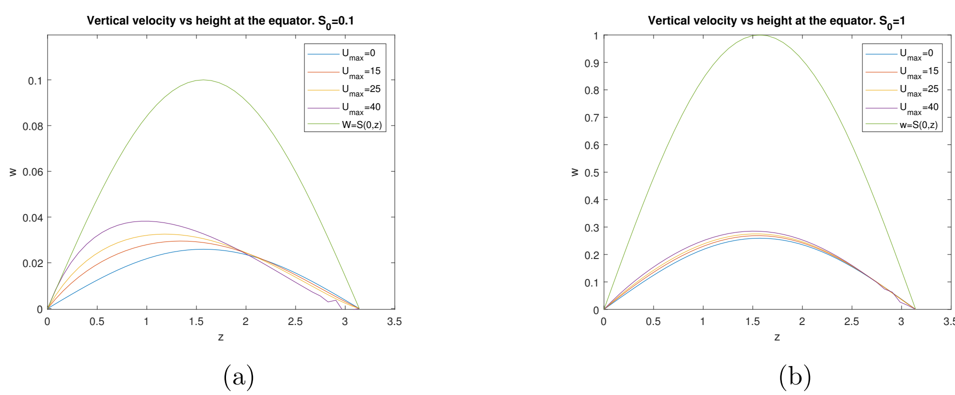


FIGURE 3.7. The vertical velocity at the equator for different values of the maximum zonal wind. The meridionally varying zonal wind profile has been chosen. The WTG vertical velocity has been plotted in green. S_0 denotes the amplitude in the heating profile chosen

3.3. Time dependent solution

3.3.1. Explicit scheme. The solution strategy is to solve the sawyer Eliassen equation at each time step to obtain the stream function and then time step the zonal velocity equation.

$$y\psi^n_{zz}(u^n_y - y) - 2y\psi^n_{yz}u^n_z - \psi^n_{yy}(\theta^n_z + 1) - \psi^n_z u^n_z = yS_{u^n,z} + S_{\theta^n,y} \tag{3.30}$$

Using the stream function we can calculate the zonal velocity at the next time step.

$$u^{n+1} = u^n - dt(v^n u^n_y + w^n u^n_z - yv^n + du^n) \tag{3.31}$$

Once we obtain the zonal velocity at the n+1 step, we can use the geostrophic wind equation to get the temperature.

$$\theta^{n+1} = \int_0^y y u^{n+1}_z dy + \theta^{n+1}_0(z,t)$$

θ_0 can be thought of as the correction to the background heat profile at the equator. To obtain this function we turn to the WTG approximation at the equator

$$w(\theta_{0,z} + 1) = S_\theta$$

$$w = \frac{S_\theta}{\theta_{0,z} + 1} \quad \text{at y=0} \tag{3.32}$$

The vertical velocity we obtain from the sawyer Eliassen equation won't satisfy the above equation for any general θ_0. Hence we have to select a θ_0 that gives us a w that satisfies the sawyer Eliassen equation and the above relation at the same time. To obtain this function we use an iterative process. We initialise the θ_0 function to be the zero function and solve the sawyer Eliassen equation. Then, using the vertical velocity obtained, we solve eq.3.32 at the equator. Using this new sounding we solve the sawyer eliassen equation to get the vertical velocity again. This procedure will be continued till we get the desired amount of error in θ_0

3.3.2. Semi implicit scheme. The previous section looked at a completely explicit time scheme to obtain the time varying solution. Explicit schemes require small time steps and are unstable for large time steps. Implicit solutions on the other hand are always stable. Since the equation set is non-linear, an implicit scheme would require us to solve a non linear equations at each time step. To remedy this, a mixed implicit explicit scheme has been used. The coriolis forcing in the zonal momentum equation and the background temperature advection terms are implicit and all the others are explicit. The scheme can be written as follows

$$u^{n+1} = u^n - dt(v^n u_y^n + w^n u_z^n - yv^{n+1} + d(z)u^n)$$

$$\theta^{n+1} = \theta^n - dt(v^n \theta_y^n + w^n \theta_z^n + w^{n+1} + d_\theta \theta^n)$$

(3.33)

The thermal wind and incompressibility constraints are

$$yu_z^{n+1} + \theta_y^{n+1} = 0$$

$$v_y^{n+1} + w_z^{n+1} = 0$$

It is difficult to solve for the time stepped quantities here so we use a predictor corrector method to solve this set of equations. For the predictor quantities, we times step the explicit version of the equations and denote the predictor quantities by a * superscript.

$$u^* = u^n - dt(v^n u_y^n + w^n u_z^n - yv^n + d(z)u^n)$$

$$\theta^* = \theta^n - dt(v^n \theta_y^n + w^n \theta_z^n + w^n + d_\theta \theta^n)$$

(3.34)

The above predictor quantities won't satisfy the two constraints and so we need correction terms to remedy this

$$u^{n+1} = u^* + \delta u$$

$$\theta^{n+1} = \theta^* + \delta\theta$$

Subtracting eq.3.34 from first two equations of eq.3.33 we get the correction quantities.

$$y\delta u_z + \delta\theta_y = dt(y^2\Delta v_z - \Delta w_y)$$

$$\Delta v = v^{n+1} - v^n$$

$$\Delta w = w^{n+1} - w^n$$

Δv and Δw also satisfy the incompressibility condition. Chosing a stream function such that $(\Delta v, \Delta w) = (-\phi_z, \phi_y)$ we get

$$y\delta u_z + \delta\theta_y = -dt(y^2\phi_{zz} + \phi_{yy})$$

To make (u^{n+1}, θ^{n+1}) satisfy the thermal wind relation

$$y^2\phi_{zz} + \phi_{yy} = \frac{1}{dt}(yu_z^* + \theta_z^*) \tag{3.35}$$

The above equation is an elliptic equation and can be solved given boundary conditions. We have used $w = 0$ at the top and bottom and $v = 0$ at $y = 0$ and $y = \infty$ respectively. Since we are dealing with mean quantities, $v \to 0$ is supported by geostrophic balance. Once ϕ is obtained we can find the $(.)^{n+1}$ quantities from that.

3.3.3. Steady state solutions. In this section we consider two forms of heating profiles and the steady state solutions obtained. One of the profiles is the Gaussian profile centered at the equator and another with heating at the equator and cooling towards the poleward side.

$$S^{(1)} = S_0 \sin(z) e^{-(y/3)^2}$$

$$S^{(2)} = \frac{S_0}{2}\sin(z)[\cos(\pi y/L) + 2\cos(2\pi y/L)]$$

where S_0 is the amplitude of the heating and L is the length of the domain in the meridional direction. Both the profiles are symmetric about the equator and we need only solve the elliptic equation 3.35 in one of the hemispheres. In both the heating cases considered, a steady Hadley cell develops given a high enough temperature and momentum damping. A circulation is present no matter how small the heating amplitude is unlike the case studied by Plumb and Hou [29] who use an off equator thermal forcing. When the damping terms are too small, the Sawyer Eliassen equation becomes hyperbolic and the solution scheme described in the previous section doesn't work. Dunkerton [7] has solved the system by assuming the terms that make the discriminant of the Sawyer-Eliassen equation negative to be zero. The hyperbolic region gives rise to symmetric instability(Dunkerton [5], Stevens [41]) if the meridional time derivatives are retained. We will take a look at the hyperbolic regions in a later section.

The system was solved in a rectangular box of height 15 km and width 9000 km in the northern hemisphere. $y = 0$ and $y = 9000$ boundary correspond to the equator and the $y \to \infty$ boundary. The boundary conditions used are the same as those used in section 2.1. 40 and 60 points were used in the vertical and horizontal direction respectively. The momentum damping profile from eq. 3.28 with a time scale of 3 days was used. A constant temperature damping of 3 days timescale was used as well. Semi-Explicit scheme was used for time stepping instead of the explicit scheme as it provided smoother solutions. The time stepping was run until a steady state was achieved.

The zonal velocity, the stream function, potential temperature and the potential vorticity have been plotted in Fig 3.8 and Fig 3.9 for different values of the heating amplitude S_0 and each of the two heating profiles. As expected, in both the heating profiles the strength of the circulation increases as we increase the heating amplitude. The trade wind magnitude also increases with heating. The main difference between the heating profile 1 and 2 is the presence of a cooling region on the poleward side in profile 2. This cooling region gives rise to a weaker counter-clockwise circulation on the poleward side in addition to the clockwise Hadley circulation on the equator-ward side. This counter-clockwise circulation results in the easterly zonal velocity at the tropopause for y greater than ~ 5000km.

The zonal velocity and the meridional velocity variation with changing heating amplitude has been plotted in Fig 3.10. Increasing the heating amplitude pushes the maxima position of the zonal velocity slightly towards the pole while it has the opposite effect on the meridional velocity. The maxima of the meridional velocity is also the position where the wind starts going downwards. So the downward moving part of the circulation moves equatorwards with increased heating. This change is of the order of hundreds of kilometers.

The above effects are true for both the heating profiles. The area where they differ is the dynamics caused by the presence of cooling region in profile 2. In profile 1, the tropopause meridional velocity is not zero inside the domain and only goes to zero as we approach the poleward boundary while in profile 2 it actually goes to zero inside the domain.

The plot of the total temperature profile at the equator with varying heating amplitude has been plotted in Fig 3.11. It is apparent from the plots that the total temperature at the equator is not simply $\Theta = z$ but there is a correction to this. Let us call this correction $\tilde{\theta}(z,t)$. Then by the geostrophic wind balance, the total temperature should be

$$\Theta = z + \tilde{\theta}(z,t) + \int_0^y s u_z(s,z,t)\,ds \tag{3.36}$$

We will see the importance of this term in the section discussing matching with the equatorial weak temperature gradient. For now, we can see that the temperature at the equator has to be computed from the subtropical theory and thus the solution of WTG theory can only be obtained by solving the DSD region first. On the other hand, the total temperature at the poleward end comes out to be very close to the background temperature profile $\Theta = z$.

3.3.4. The hyperbolic region. The cases plotted in the previous section have all been where the Sawyer-Eliassen equation remains elliptic. The cases where the equation becomes hyperbolic, the solution scheme blows up. This happens if the damping is reduced or the heating amplitude is increased. This development of hyperbolic region can be seen at the top of the equator from the potential vorticity plots in Fig 3.8. The sign of the potential vorticity flips from positive for small time to negative if we keep on time stepping the solution. The negative potential vorticity represents the hyperbolic region. These hyperbolic regions develop first at the equator, near the tropopause. A Taylor expansion of the potential vorticity near the equator sheds light on this behaviour. We know that at the equator, the zonal velocity has a $y^2/2$ profile at the tropopause. So, the zonal velocity can be written as

$$u = \frac{y^2}{2} f(z) + \dots$$

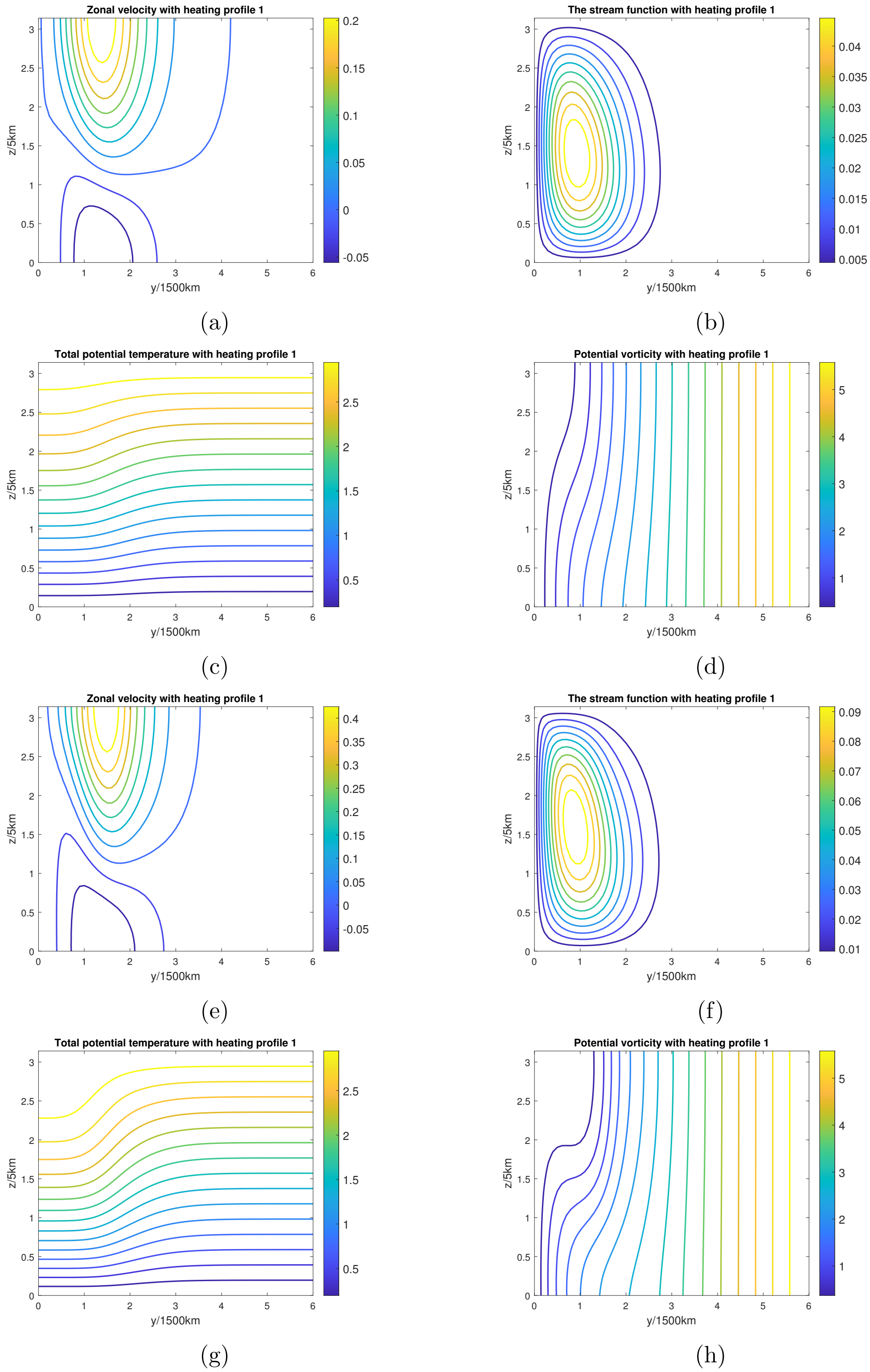


FIGURE 3.8. The zonal velocity, stream function, total potential temperature and potential vorticity for heating profile 1 with varying heating amplitude. $S_0 = 0.5$ for a), b), c), d) and $S_0 = 1$ for e), f), g), h).

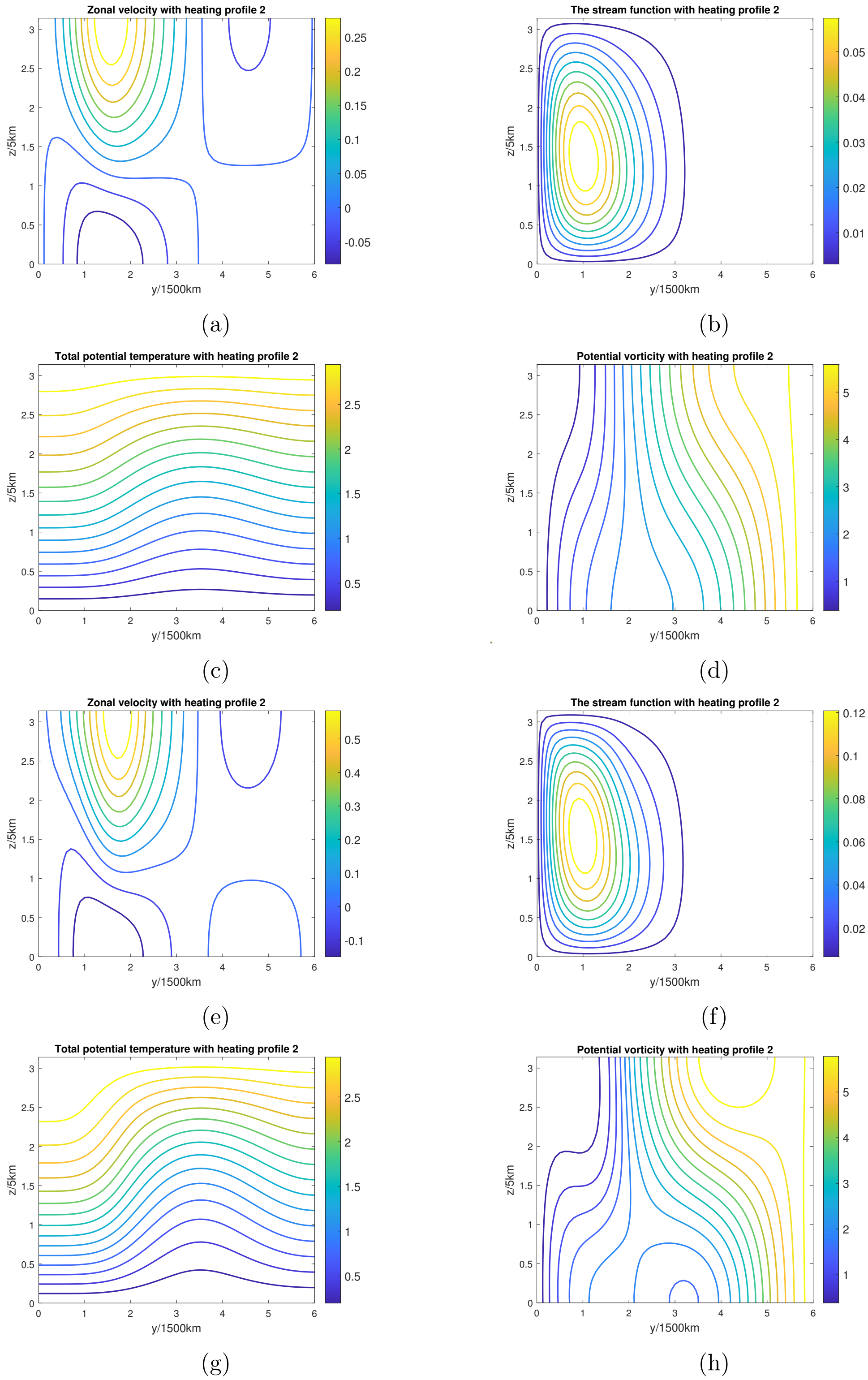
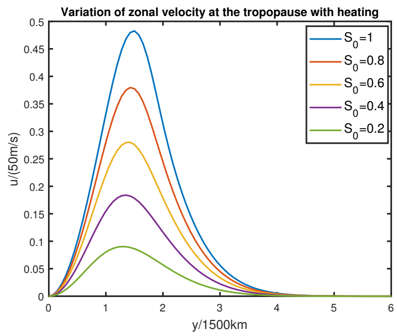
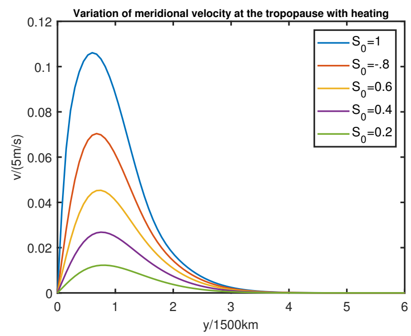


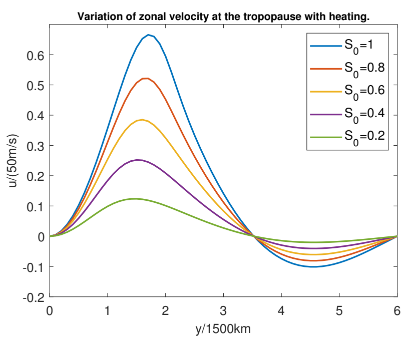
FIGURE 3.9. The zonal velocity, stream function, total potential temperature and potential vorticity for heating profile 2 with varying heating amplitude. $S_0 = 0.5$ for a), b), c), d) and $S_0 = 1$ for e), f), g), h).



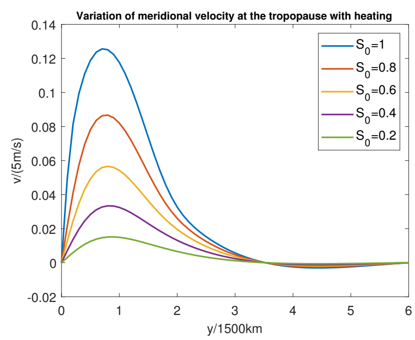
(a)



(b)

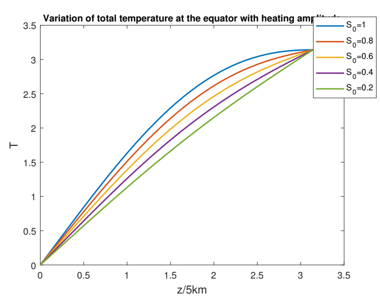


(c)

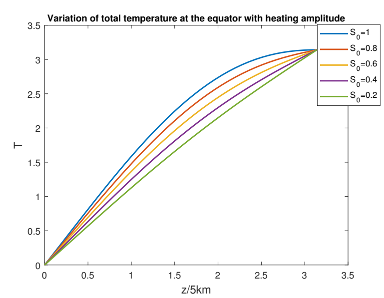


(d)

FIGURE 3.10. Effect of variation in the heating amplitude on the zonal velocity and the meridional velocity at the tropopause. a), b) have been computed for profile 1 and c), d) have been computed with profile 2



(a)



(b)

FIGURE 3.11. Effect of variation in the heating amplitude on the total temperature at the equator. a) has been compute for profile 1 and b) has been computed for profile 2

where $f(z) = 1$ at the top of the troposphere in the steady, no damping case. The potential vorticity then becomes

$$Q = -y(u_z)^2 + \Theta_z(y - u_y)$$

$$= -\frac{y^5}{4}(f'(z))^2 + y\Theta_z(1 - f(z)) + \dots \tag{3.37}$$

Θ_z is the total temperature gradient, which can be seen to be positive from Fig3.11. This is also required for the atmosphere to be statically stable. Near the tropopause, $f(z)$ is very close to 1 and the potential vorticity is dominated by the first term which is always negative. This shows that the Sawyer-Eliassen equation will always be hyperbolic at the equator. Now, if we add damping to the mix, $f(z)$ no longer equals 1 at the tropopause and the second term dominates the first term in the potential vorticity expansion making the system elliptic.

This y^2 behaviour of the zonal velocity is only valid near the equator. Away from the equator, $u << y^2$ and the potential vorticity is dominated by $y\Theta_z$ terms which is positive, provided the atmosphere remains statically stable.

3.4. Matching with WTG

The DSD system of equations fail as we move towards the equator. In particular, the meridional geostrophic balance no longer holds. The perturbation temperature at the equator is also much weaker and the advection terms in the temperature equation are much smaller than the other terms. This gives us the weak temperature gradient approximation.The temperature equation in eq. 3.1 is

$$\epsilon^2\left(\theta_t + v\theta_y\right) + w(z + \epsilon^2\theta)_z = S_\theta - \epsilon^2 d\theta \tag{3.38}$$

Hence the leading order equation obtained from this should be

$$w\frac{d\overline{\Theta}}{dz} = S_\theta$$

This is the weak temperature gradient approximation that has been widely used. The $\overline{\Theta}$ term is the background potential temperature which is assumed to be linear here i.e $\overline{\Theta} = z$. This corresponds to choosing a constant Brunt-Vaisala frequency at the equator. This has been done to compute the vertical velocity at low latitudes in the Hadley cell model of Held and Hou [**12**] and to study tropical boundary layers by Schneider and Lindzen [**35**]. But as we have seen from our exploration of DSD, this background stratification at the equator cannot be defined a priori. The background temperature at the equator needs to be obtained from the solution of leading order DSD. This relates to the correction potential temperature that we have used in eq.3.36. So the total temperature in the WTG approximation should be

$$\Theta = z + \tilde{\theta}(z,t) + \epsilon^2\theta + \dots \tag{3.39}$$

$$\overline{\Theta} = z + \tilde{\theta}(z,t) \tag{3.40}$$

With this in mind the complete set of zonally independent equations in WTG approximation are

$$u_t + vu_y + wu_z - yv = -du$$

$$v_t + vv_y + wv_z + yu = -p_y$$

$$\theta = p_z$$

$$\tilde{\theta}_t + \epsilon^2(\theta_t + v\theta_y + w\theta_z) + w\frac{d\overline{\Theta}}{dz} = S_\theta - d\tilde{\theta} - \epsilon^2 d\theta$$

$$v_y + w_z = 0$$

Solution method: The temperature equation becomes an equation for the vertical velocity giving us

$$w\frac{d\overline{\Theta}}{dz} = S_\theta - d\tilde{\theta} - \tilde{\theta}_t \tag{3.41}$$

where $\overline{\Theta}$ and $\tilde{\theta}$ are known quantities obtained through solving the subtropical DSD equations. Once we have the vertical velocity, the incompressibility equation is used to obtain the meridional velocity v. Then the zonal momentum equation is used to time-step u. Once all three components of velocity have been solved for, the meridional velocity equation gives us the pressure and hence the temperature using the hydrostatic balance.

3.5. Summary

We studied the subtropical extension(DSD) of the WTG theory described for the tropics in the introductory chapter. This subtropical theory is the 3D analogue of the subtropical theory derived in the shallow water system as the scaling of both the zonal and the meridional velocity are the same. Like the shallow water case, the matching between the subtropical DSD and the tropical WTG provides us with the background stratification at the equator. Without going through the

DSD solution, the closure of the WTG theory requires us to assume a temperature stratification at the equator.

The zonally invariant version of DSD has been used to model the Hadley cell. Unlike the WTG model, this doesn't require us to put special constraints on the background heating other than it vanishing at infinity. A closed cell is obtained whenever the Sawyer-Eliassen equation remains elliptic.

A semi implicit finite difference method was used for the solution of the DSD system, which allows us to use bigger time steps than could be used in the explicit scheme. The explicit scheme requires us to solve the Sawyer Eliassen operator which can become hyperbolic , parabolic or elliptic depending on the temperature and zonal velocities. The operator we need to solve in the semi implicit case is always elliptic. Despite this we run into cases when the system becomes unstable. This might be due to the system of equations as a whole becoming ill posed. The treatment of the mixed hyperbolic-elliptic system has been left as a future endeavour.

This mixed hyperbolic-elliptic behaviour of the 3D subtropical theory can be contrasted with the shallow water case where the problem is always well posed since the Sawyer Eliassen equation is 1D and is easily solvable. The hyperbolic region appears whenever the heating amplitude is strong or the damping is very weak. So, the 3D system prevents us from using very low values of the damping rates while the shallow water model poses no such limitations.

CHAPTER 4

Baroclinic Instability Study

The models of the Hadley cell considered in the previous chapters, both WTG and DSD, begin with assumptions that warrant a closer inspection. The WTG model requires explicit forcing where the integrated heating balances the integrated cooling. Additionally, a WTG model requires zonal momentum damping even in the upper troposphere, without which the zonal momentum would show an angular momentum conserving (βy^2) behaviour at latitudes up to the poleward end of the Hadley cell. This would make the velocity in the jet substantially larger that what is observed in the atmosphere. Although, the damping term can be justifiably large near the surface due to surface friction, it should be negligible near the tropopause.

The DSD model relaxes the assumption of balanced latent heating and radiative cooling but retains the damping sources to get a closed Hadley cell. In both models, we consider axisymmetric circulation. This is justifiable as long as the zonally varying fluctuations remain small compared to the mean flow. In other words, the mean flow should be stable. The problem of instability in jets has been much studied and the instabilities fall under the categories of barotropic, or baroclinic instability. Baroclinic instability has been proposed as the main phenomenon which limits the angular momentum conserving increase of the jet velocity with latitude.

Charney [4], Eady [8] and Philip [28] studied baroclinic instabilities arising due to the vertical shear in jets. It is the fluxes generated due to the growth and breaking of the unstable modes which provide the damping necessary to limit the increase of the strength of the jet. The damping used in our models is an attempt to parameterize these fluxes. The two layer model can be used to give an estimate of the poleward estimate of the Hadley cell, Held [31]. Stone [42] has used the theory of baroclinic adjustment to attempt to predict the decaying behaviour of the Hadley cell at its poleward terminus. Baroclinic adjustment is the phenomenon of convective adjustment where the mean flow of the fluid adjusts to the state at which it is marginally stable to fluctuations. Unlike the two layer model, a critical shear for instability is not present in the continuously stratified model.

Works by Schneider and Walker([38], [44]), Schneider([37] have related the poleward extent of the Hadley cell to a supercriticality parameter which is closed related to baroclinic instability.
Eady studied the instability in a continuously stratified atmosphere which is always unstable under a linearly sheared zonal velocity profile. It is the two layer baroclinic model of Philips, which provides a cut off on the shear, above which the flow becomes unstable.
We will start our study with a simple meridional velocity profile and use a damping term which relaxes the zonal velocity to the marginal stability state given by the two layer model.
The asymptotics of the Hadley cell, studied in the previous chapters require that the physics of the descending branch of the Hadley Circulation should be modeled using the DSD theory rather than quasi-geostrophy. Instead of using the QG model to study jet instabilities, we will use the DSD framework discussed in the previous chapter to study these instabilities. Modified asymptotic formulations of the subtropical theory will be used to model the interaction of the momentum and temperature fluxes with the mean fields.

4.1. Baroclinic instability models

Held and Hou derived an estimate for the latitudinal extent of the Hadley cell by assuming the cell to be energetically closed and incorporating a radiative cooling which balances the heating at the equator. The parameter dependence obtained by them does not correspond to those observed in macroturbulent simulations(Walker and Schneider [44]). The poleward terminus of the Hadley cell is thought to be the result of the jet becoming unstable due to baroclinic instability. When the vertical shear in the jet becomes large enough, it becomes unstable to fluctuations and the fluxes due to these fluctuations provide a mechanism for a decay in the shear. In this section we will use pre-existing models of baroclinic instability to parameterize a damping force which relaxes the jet to a marginally stable state. We will look at two models of baroclinic instability, the Eady and Philips model, and their implication about this damping force.

4.1.1. Flux in the Eady Problem. In the Eady problem, a vertically sheared zonal wind profile $u = \Lambda z$ is assumed and its linear instability is studied on an f plane. The width and height of the channel are L and H respectively. The details of the solution and the growth rates can be found in Eady [8], Vallis [43]. Since the β effect is absent, this is the simplest model of baroclinic instability in a continuously stratified atmosphere. The horizontal velocities can be represented by a stream function such that $(u, v) = (\psi_y, -\psi_x)$. The stream function for the horizontal flow is given by

$$\psi = \mathrm{Re}\big(\Phi(z)\sin ly e^{ik(x-ct)}\big)$$
$$\Phi(z) = A\cosh\mu\hat{z} + B\sinh\mu\hat{z} \tag{4.1}$$

where $l = n\pi/L$, $\mu^2 = L_d^2(k^2 + l^2)$ and $\hat{z} = z/H$. With this stream function we can calculate the velocity, temperature and the consequent fluxes. These are as follows

$$u' = \mathrm{Re}\big(\Phi(z)l\cos(ly)e^{ik(x-ct)}\big)$$
$$v' = -\mathrm{Re}\big(i\Phi(z)k\sin(ly)e^{ik(x-ct)}\big)$$
$$\theta' = \mathrm{Re}\big(\Phi'(z)\sin(ly)e^{ik(x-ct)}\big) \tag{4.2}$$

We can write c as $c_r + i\sigma/k$. Since the constants in $\Phi(z)$ depend on the initial condition, we can write it as $A(z)e^{i\phi(z)}$. This will give us the following $u'v'$

$$u' = A(z)l\cos ly\cos(k(x-c_rt)+\phi(z))e^{\sigma t}$$
$$v' = A(z)k\sin ly\sin(k(x-c_rt)+\phi(z)) + e^{\sigma t}$$
$$u'v' = \frac{1}{4}A^2(z)kl\sin(2ly)\sin(2k(x-c_rt)+\phi(z))e^{2\sigma t}$$
$$\overline{u'v'} = 0 \tag{4.3}$$

The zonally averaged momentum flux in the Eady problem is zero. The temperature flux, $\theta'v'$, on the other hand can be non zero

$$\theta' = \sin ly e^{\sigma t}\big(A'(z)\cos(k(x-c_rt)+\phi(z)) - A(z)\phi'(z)\sin(k(x-c_rt)+\phi(z))\big)$$
$$\overline{\theta'v'} = \frac{1}{2}A^2(z)\phi'(z)\sin^2(ly)e^{2\sigma t} \tag{4.4}$$

The undamped Eady problem is always unstable no matter how small the shear λ is, although the corresponding growth rate will be smaller as well. This doesn't help us in using the principle of baroclinic adjustment to define a marginally stable background profile for which the flow is stable. This problem is rectified by using a two layer geostrophic model which does have a limit above which the flow becomes unstable. We will look at this model in the next section.

4.1.1.1. *Philip's two layer model.* Philip's two layer model of baroclinic instability considers two layers of rotating fluid on the β plane with velocity U_1 and U_2 respectively. The total depth of the system is H. In this case, unlike the Eady problem, instabilities only occur if

$$\frac{U_1 - U_2}{H} > \frac{\beta L_d^2}{4H}$$
$$L_d = \frac{NH}{f} \tag{4.5}$$

The term $\frac{U_1 - U_2}{H}$ can be viewed as the shear $\Lambda = u_z$. Hence the critical shear(u_z^c), can be written as

$$u_z^c = \frac{\beta L_d^2}{2H}$$

Expanding the terms in the above expression for the coriolis factor $f = 2\Omega \sin(y/R)$ as in the previous sections, we get

$$u_z = \frac{2\Omega \cos{(y/R)}(NH)^2}{4HR(2\Omega \sin{(y/R)})^2} \tag{4.6}$$

We non-dimensionalise this by the MEWTG parameters. $NH = 50m/s$ can be written as $\epsilon^{-1} u_0$ where $u_0 = 5m/s$ is used to non-dimensionalise the horizontal velocity. This gives us the following expression for the non-dimensionalised critical shear

$$u_z = \frac{\epsilon^{-2}}{4y^2} \tag{4.7}$$

From, the equatorial section on the Hadley cell, we know that the zonal velocity increases as $u \sim y^2$. Since we have an $O(1)$ height, u_z also scales as y^2. Plugging this in the above expression gives us the order of the latitude when the Hadley cell shear equals the critical shear of baroclinic instability.

$$u_z \sim y^2 \sim \frac{\epsilon^{-2}}{y^2}$$
$$y^4 \sim \epsilon^{-2}$$
$$y \sim \epsilon^{-1/2}$$

Coincidentally, this also matches with the latitude where the WTG approximation breaks down and we have to use the DSD theory. Hence if we want to approximate the Hadley cell by the much simpler WTG theory, we have to account for this instability. For this we will use a damping term in the zonal velocity equation to relax the zonal velocity to the marginally stable state. This term is given by

$$F_b = -\alpha(u - u_{top})^+ u \tag{4.8}$$

This term only turns on, when the velocity exceeds the critical velocity. Since the condition for the criticality deals with the shear u_z, we have to approximate the velocity at the top by somehow. This can be done by approximating the shear as

$$u_z \sim \frac{u^{top} - u^{bottom}}{H}$$

Since u^{bottom} is much smaller than the velocity at the top of the troposphere and $H = 1$ at the top of the troposphere, u_z can be further approximated by

$$u^{top} \approx u_z^c$$

The damping force in eq.4.8 becomes

$$F_b = -\alpha\left(u - \frac{\epsilon^{-2}}{4y^2}\right)^+ u \tag{4.9}$$

α has the dimension of inverse length. Increasing α increases the effectiveness of the damping force to reduce the zonal velocity to the marginal stability value.

4.2. Baroclinic instability damping model

To use the WTG model to model the entirety of the Hadley cell, we have to somehow incorporate the effects of baroclinic instability. We do this by incorporating a damping force using the principle of baroclinic adjustment which relaxes the zonal velocity to a state of marginal stability. In this section we will look at the effects of the damping force in eq.4.9 on the zonal velocity at the tropopause.

Since the vertical velocity is zero at the tropopause with a fixed height, the steady zonal momentum equation is given by

$$vu_y - yv = -du - F_b \tag{4.10}$$

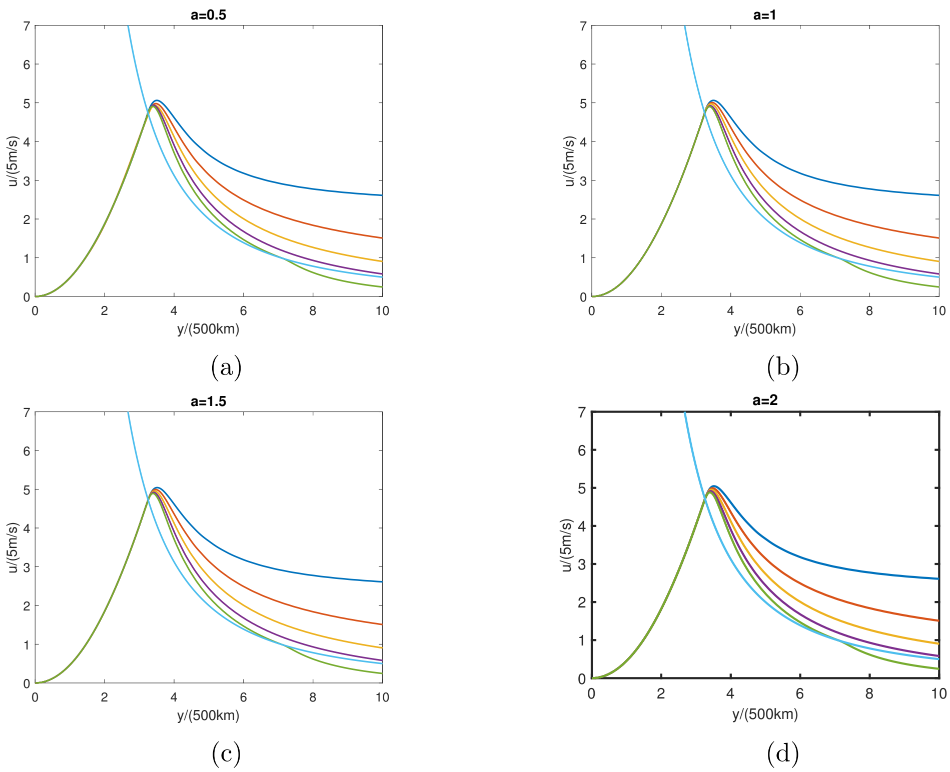


FIGURE 4.1. The values of a, the region of linear growth is 0.5, 1, 1.5 and 2 in a), b), c) and d) respectively. $V_0 = 1$,$b = 3$, $d = 0.1$ and $\alpha = 0.5$

where F_b is the baroclinic damping force. Since we are using the WTG model, the meridional velocity, v is computed using the incompressibility equation and the vertical velocity which is in turn computed using the heating. In this section we use the following function to model the meridional velocity at the tropopause

$$v = \begin{cases} V_0 \frac{y}{a} & x \leq a \\ V_0 & a < x \leq b \\ V_0 \left(\frac{y}{b}\right)^{-s} & x > b \end{cases}$$

We have used the non-dimensionalisation of WTG which was used in the introductory chapter. Horizontal velocity is measured in units of 5 m/s and distances are measured in units of 500km. The meridional velocity chosen has a linear increase till $y = a$ followed by a constant velocity till $y = b$. $s > 0$ so that the meridional velocity decays to zero as $y \to \infty$. V_0 is the maximum meridional velocity in the region.

The zonal velocity profile depends on the length of the linear growth of the meridional velocity, the decay length scale b, the maximum meridional velocity and the damping parameters α and d. The effects of changing these parameters on the zonal velocity are summarized below

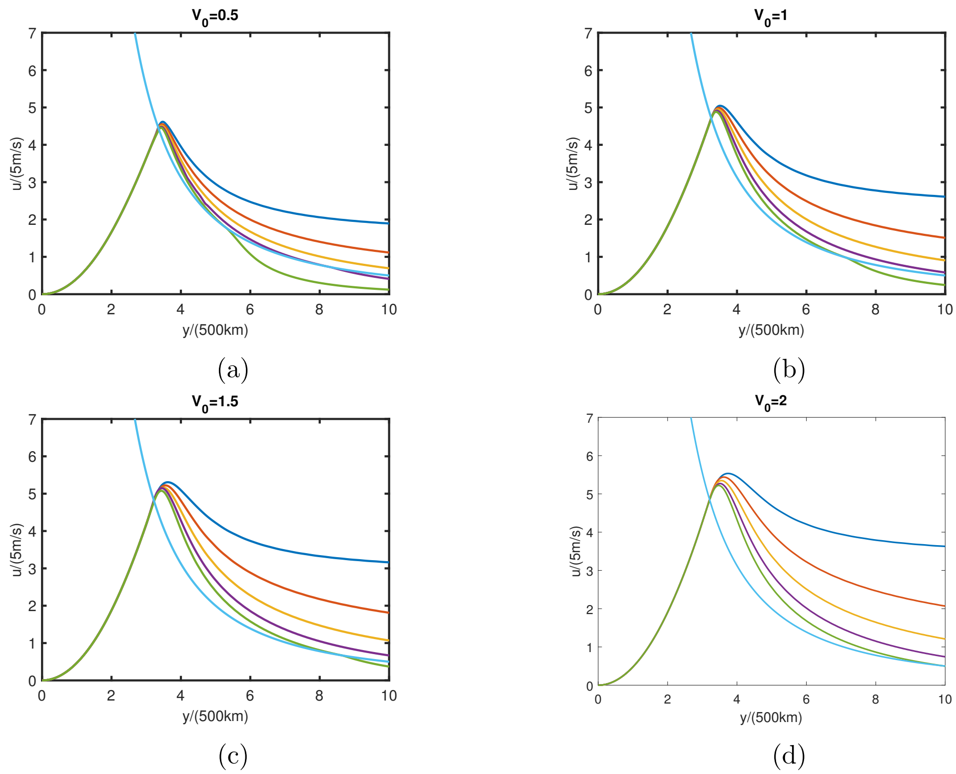


FIGURE 4.2. The values of maximum meridional velocity, V_0 is 0.5,1,1.5 and 2 in a), b),c) and d) respectively. $a = 1$,$b = 3$, $d = 0.1$ and $\alpha = 0.5$

- **Effect of** a: Changing the length of the linear region doesn't effect the behaviour of u that much. The plots , Fig.4.1, look identical.
- **Effect of** V_0: Increasing V_0, Fig.4.2, increases the zonal velocity as a whole. This is because the meridional advection term dominates the Rayleigh damping and the baroclinic damping term.
- **Effect of** b: b controls the length of the region of non decreasing v. Increasing b, Fig. 4.3, results in changing the maximum value that the zonal velocity can reach. But this maximum is not unbounded. For $b = 4$ and $b = 5$, the maximum is the same after which the velocity starts to decrease. The decay rate for the profile chosen will be higher the

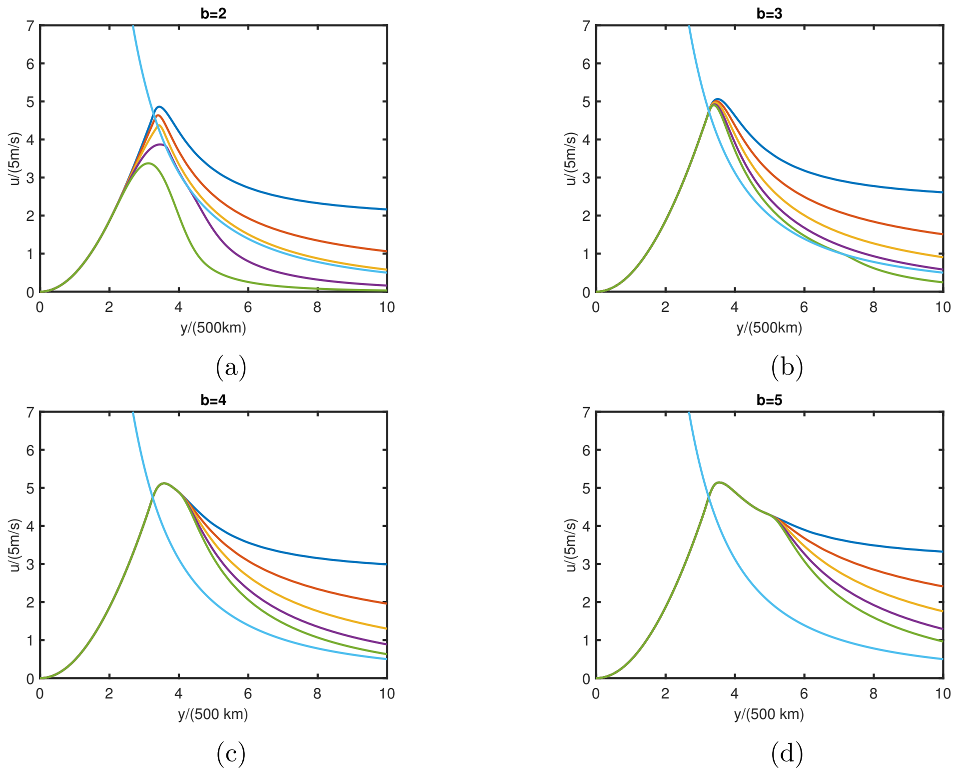


FIGURE 4.3. The region of constant meridional velocity(b) is till 2, 3, 4 and 5 in a), b),c) and d) respectively. $a = 1$, $V_0 = 1$, $d = 0.1$ and $\alpha = 0.5$

lower the value of b is. This is why the decay in u for $b = 4$ is more pronounced than $b = 5$.

- **Effect of** α: Increasing α, Fig.4.4, increases the decay of the zonal velocity to make it closer to the baroclinically stable limit.

4.2.1. Asymptotic behaviour of the zonal velocity. We want to look at the $y \to \infty$ behaviour of the solution to

$$vu_y - yv = -du - \frac{(u - u^*)^+ u}{L} \tag{4.11}$$

where L is the decay length scale due to the dissipative action of baroclinic instability. As $y \to \infty$, u_y goes to zero and we are left with the following equation

$$u^2 - u(u^* - Ld) - Lyv = 0 \tag{4.12}$$

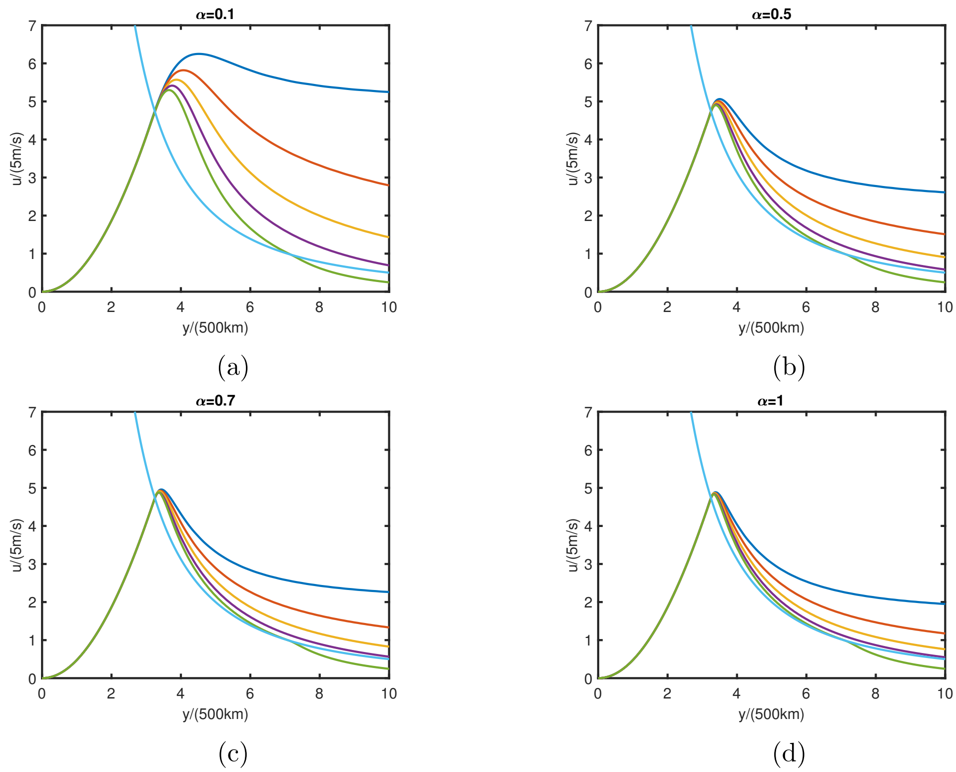


FIGURE 4.4. The values of the baroclinic damping parameter, α is 0.1, 0.5, 0.7, and 1 in a), b),c) and d) respectively. $a = 1$, $b = 3$, $V_0 = 1$ and $d = 0.1$

Assuming that the meridional velocity, v decays as $V_0 y^{-s}$ we get the following

$$u^2 - u(u^* - Ld) - LV_0 y^{1-s} = 0 \tag{4.13}$$

Since the y derivative of u drops off as $y \to \infty$, the equation is no longer a differential equation but transforms into a quadratic algebraic equation. The two solutions of the equation are given by

$$\begin{aligned} u &= \frac{1}{2}\left(u^* - Ld + \sqrt{(u^* - Ld)^2 + 4LV_0 y^{1-s}}\right) \\ &= \frac{1}{2}\left(\frac{\epsilon^{-2}}{2y^2} - Ld + \sqrt{\left(\frac{\epsilon^{-2}}{2y^2} - Ld\right)^2 + 4LV_0 y^{1-s}}\right) \end{aligned}$$

(4.14)

We only consider the positive root, since the other solution will be negative and will not match with the positive zonal velocity as $y \to 0$. Near the QG latitude, $y \sim O(\epsilon^{-1})$, so that $u^* = O(1)$. For $s = 1$, all the terms in the expression are $O(1)$. As s increases, the second term in the square root goes to zero much faster. So, for the values of s greater than 1, the asymptotic value of u in becomes

$$u = u^* - Ld \tag{4.15}$$

Since the above value is less than the baroclinic limit, the baroclinic damping term turns off as soon as the zonal velocity touches the baroclinic limit curve. After this happens, the only surviving terms in the zonal momentum equation are the coriolis term and the Rayleigh damping term. A balance between these two gives us the following

$$u = \frac{V_0 y^{1-s}}{d} \qquad \text{for } s > 1 \tag{4.16}$$

So, for a decaying meridional velocity, the asymptotic behaviour of the zonal velocity doesn't depend on the baroclinic decay rate of our model but only on the Rayleigh damping rate.

4.3. Instability in DSD

The Hadley cell model in the previous section attempts to incorporate the effects of instability into the mean flow dynamics by parameterizing those effects as a damping force. There are two drawbacks with the approach being used. The first is that we have used the Weak temperature Gradient model for the entire Hadley cell. As we have seen in the DSD section, WTG model cannot be used at higher latitudes, especially when the jet reaches its maximum velocity. Another problem is the model used to study the baroclinic instabilities in the jet itself. The baroclinic instability models described in the beginning of the chapter, like the Eady and Phillips model both work under the framework of the Quasi-Geostrophic (QG) theory. This poses the same problem that using the WTG model does, in that the QG model is not applicable at the jet maxima. While meridional geostrophy is satisfied by the jet, zonal geostrophy is not. To remedy both these issues, we will study the instabilities using the DSD system and look at the effect the instabilities have on the mean flow.

In the following sections we will go over the instabilities in the leading order mean flow under the presence of a zonal flow with vertical shear. These types of flows typically produce baroclinic instabilities in fluids. In the mean DSD theory, these types of flows have an unbounded growth rate as the wave number increases. This seems to be a problematic behaviour for the theory, but we need to remember that the mean DSD theory is a long wave perturbation theory and hence is not valid when the wave number becomes large or the length scales become too small. To overcome this issue, we will move onto a spatio-temporal multiscale version of DSD to further analyses the instabilities.

4.3.1. Baroclinic instability in mean DSD. We begin by using the non-dimensional model described in the chapter on the DSD theory. Instead of using the meridional variation of the coriolis parameter, we will be using the f-plane version of the theory to simplify the calculations. The vertical extent of the model is from $z = 0$ to $z = \pi$. We will assume that the horizontal domain is doubly periodic in both the zonal and meridional direction. Due to the inherent asymmetry in the length scales, these periods correspond to 15000km and 1500km in the zonal and meridional length scales respectively.

Taking inspiration from the Eady problem, we will study the instabilities in the zonal flow with a constant vertical shear, $U = \Lambda z$. The thermal wind balance gives us, $\Theta_y = -f\Lambda$. $(v, w) = (0, 0)$ is the steady state solution of the unforced and unheated system, just as in the Eady instability case. With this, the linearised DSD system of equation is given by

$$u_t + \Lambda z u_x + w\Lambda - fv = -p_x$$

$$fu = -p_y$$

$$\theta_t + \Lambda z \theta_x - f\Lambda v + w = 0$$

$$\theta = p_z$$

$$u_x + v_y + w_z = 0 \tag{4.17}$$

To make sure that the meridional geostrophy constraint is satisfied at all times, we proceed in a way similar to the derivation of the Sawyer-Eliassen equation. Taking the y derivative of the first equation and x derivative of the second and subtracting the two, we get

$$u_{yt} + \Lambda z u_{yx} + \Lambda w_y + f w_z = 0 \tag{4.18}$$

Using the geostrophic wind condition, $\theta_y = -fu_z$ in the temperature equation we get

$$fu_{zt} + \Lambda f z u_{zx} + f\Lambda v_y - w_y = 0 \tag{4.19}$$

Taking a y derivative and using incompressibility to substitute for v_y

$$f u_{zyt} + \Lambda f z u_{zxy} - f\Lambda(u_{yx} + w_{yz}) - w_y = 0 \tag{4.20}$$

Taking a z derivative of eq.4.18 and subtracting from eq.4.20 we get

$$2f\Lambda u_{xy} + 2f\Lambda w_{yz} + f^2 w_{zz} + w_{yy} = 0 \tag{4.21}$$

This is the linearised form of the Sawyer-Eliassen equation. The above equation can be written in the operator form as follows

$$\mathcal{O}(w) = -2f\Lambda u_{xy}$$

where the operator $\mathcal{O}$ is a 2D second order differential operator given by

$$\mathcal{O} = f^2 \frac{\partial^2}{\partial z^2} + 2f\Lambda \frac{\partial^2}{\partial y \partial z} + \frac{\partial^2}{\partial y^2} \tag{4.22}$$

Depending upon the value of the coriolis parameter, f and the shear Λ, the above operator can be elliptic, hyperbolic or parabolic. This behaviour is determined by the discriminant of the operator, which is given by

$$\Delta = f^2(1 - \Lambda^2) \tag{4.23}$$

For $\Delta > 0$, $\Delta = 0$ and $\Delta < 0$, the operator is elliptic, parabolic and hyperbolic respectively. Since eq.4.21 is a linear equation, it can admit wave like solutions of the form $u = u(z)\exp(i(kx + ly - \omega t)$. We haven't resolved the vertical coordinate into Fourier terms since the full set of linearised equations depend on z and a Fourier transform along the vertical will involve convolutions and only complicate the solution. Plugging the wave solution into eq.4.21 we get

$$-2f\Lambda klu + 2if\Lambda lw' + f^2 w'' - l^2 w = 0 \tag{4.24}$$

This gives us an expression for $u(z)$ that we can plug in eq.4.18. With this we get the following equation

$$\left(-\frac{\omega}{k} + \Lambda z\right)(f^2 w_{zz} + 2if\Lambda l w_z - l^2 w) = 2f\Lambda(i\Lambda l w + f w_z) \tag{4.25}$$

Let us define the following operators

$$A(w) = f^2 w_{zz} + 2if\Lambda l w_z - l^2 w$$

$$B(w) = 2f\Lambda(i\Lambda l w + f w_z) \tag{4.26}$$

With this the eigenvalue equation can be written as

$$-\frac{\omega}{k}A(w) = (B - \Lambda z A)w \tag{4.27}$$

The operators do not depend on k. Hence, for a fixed l, ω is just a linear function of k. This linear dependence implies that for unstable flows, we can keep increasing the value of the zonal wavenumber to get an ever increasing growth rate. These catastrophically unstable modes only exist when we increase the zonal wavenumber which means we are looking at small scale fluctuations. Since the DSD theory is a long wave approximation at the subtropics, these solutions are not valid asymptotic approximation of the real flow.

4.4. Multiple scale theory

The eigenvalue equation obtained for the mean field instability in DSD lead us to the fact that the growth rate can be unbounded if the flow is unstable. Since this happens at small length scales and the mean theory is only valid for large length scales, the method of multiple scales is the perfect remedy for this. We will use the single scaled version of IMMD described in chapter 2 instead of the DSD theory. Both the theories are analogous but only differ in the scaling of the variables. The main advantage of using IMMD instead of DSD is that the multi scale version of DSD leads to fractional powers of ϵ while the multi scale IMMD uses whole powers. Apart from this, the multi scale version of DSD will follow the same procedure.

The single scale IMMD equations are as follows

$$\frac{Du}{DT} - yv + p_x = S_u$$
$$\epsilon^2 \frac{Dv}{DT} + yu + p_y = 0$$
$$\frac{D\theta}{DT} + w = S_\theta$$
$$\theta = p_z$$
$$u_X + v_y + w_z = 0 \tag{4.28}$$

where the time scale, T and the zonal length scale, X correspond to 3 days and 15000km respectively. We use multiple scales an order of ϵ lesser than the longer time and length scale. This corresponds to shorter time scale, $t = 8$ hr and shorter length scale , $x = 1500$ km. Introducing new variables

$$x = \epsilon^{-1} X$$
$$t = \epsilon^{-1} T$$

The derivatives transform as

$$\frac{\partial}{\partial T} = \epsilon^{-1} \frac{\partial}{\partial t} + \frac{\partial}{\partial T}$$
$$\frac{\partial}{\partial X} = \epsilon^{-1} \frac{\partial}{\partial x} + \frac{\partial}{\partial X}$$

With this, we get the multi-scale IMMD equations as described in Biello & Majda([3]). The asymptotic expansion used by the authors was as follows

$$u^\epsilon = U + \epsilon(u_1' + \overline{u_1}) + \epsilon^2(u_2' + \overline{u_2}) \ldots$$
$$v^\epsilon = \epsilon(v_1' + \overline{v_1}) + \epsilon^2(v_2' + \overline{v_2}) + \ldots$$
$$w^\epsilon = \epsilon(w_1' + \overline{w_1}) + \epsilon^2(w_2' + \overline{w_2}) + \ldots$$
$$p^\epsilon = P + \epsilon(p_1' + \overline{p_1}) + \epsilon^2(p_2' + \overline{p_2}) + \ldots$$
$$\theta^\epsilon = \Theta + \epsilon(\theta_1' + \overline{\theta_1}) + \epsilon^2(\theta_2' + \overline{\theta_2}) + \ldots$$

(4.29)

The leading order zonal velocity, pressure and temperature do not depend on the small scale variables. The barred quantities represent the zonal mean while the dashed one represent the fluctuations

$$\overline{f}(t,T,X,y,z) = \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} f(t,T,x,X,y,z)dx$$
$$f = \overline{f} + f'$$

Let us look at the $O(1)$ zonal momentum equation after we plug in the above expansion

$$(U_T + u'_{1,t} + \overline{u}_{1,t}) + U(U_X + u'_{1,x}) + (v'_1 + \overline{v_1})U_y + (w'_1 + \overline{w_1})U_z - y(v'_1 + \overline{v_1}) = -P_X - p'_{1,x} \tag{4.30}$$

Taking the zonal and time average of the above equation, we get the long wave Hadley cell regime equation given by

$$U_T + UU_X + \overline{v_1}U_y + \overline{w_1}U_z - y\overline{v_1} = -P_X \tag{4.31}$$

Subtracting the zonal mean of eq.4.30 from the full equation we get an equation for the fluctuation

$$u'_{1t} + Uu'_{1x} + v'_1 U_y + w'_1 U_z - yv'_1 = -p'_{1x} \tag{4.32}$$

The fluctuation equation is a short length and time scale linear equation and will allow us to study the small scale instabilities which were not allowed in the mean instability theory.

Even though this system allows us to study the effect of the mean flow on the instability, there is no way for the flux generated due to the instabilities to affect the mean flow since the order of the flux terms like $(v'_1 u'_{1,y})$ is higher than the mean by an order of ϵ. To increase the magnitude of the flux terms, we can either promote both the fluctuation terms by $O(\epsilon^{1/2})$ or use a shorter y length scale which will promote the magnitude of the derivative in the flux term. We will now look at these two cases.

4.4.1. Increased fluctuation magnitude. To increase the magnitude of the fluctuation flux terms we will increase the magnitude of the fluctuation terms by $O(\epsilon^{1/2})$. This amounts to using the following asymptotic expansion

$$
\begin{aligned}
u^\epsilon &= U + \sqrt{\epsilon}(u_1' + \overline{u_1}) + \epsilon(u_2' + \overline{u_2}) + \epsilon^{3/2}(u_3' + \overline{u_3}) \\
v^\epsilon &= \sqrt{\epsilon}v_1' + \epsilon(v_2' + \overline{v_2}) + \epsilon^{3/2}(v_3' + \overline{v_3}) \\
w^\epsilon &= \sqrt{\epsilon}w_1' + \epsilon(w_2' + \overline{w_2}) + \epsilon^{3/2}(w_3' + \overline{w_3}) \\
p^\epsilon &= P + \sqrt{\epsilon}(p_1' + \overline{p_1}) + \epsilon(p_2' + \overline{p_2}) + \epsilon^{3/2}(p_3' + \overline{p_3}) \\
\theta^\epsilon &= \Theta + \sqrt{\epsilon}(\theta_1' + \overline{\theta_1}) + \epsilon(\theta_2' + \overline{\theta_2}) + \epsilon^{3/2}(\theta_3' + \overline{\theta_3})
\end{aligned}
$$

(4.33)

The above ansatz is plugged in eq.4.28. The derivation of the equations obtained till order $\epsilon^{3/2}$ is given in the appendix. Here, we take a look at the terms in the zonal momentum equation.

$$
\begin{aligned}
u_t^\epsilon &= \sqrt{\epsilon}u_{1t}' + \epsilon(U_T + u_{2t}') + \epsilon^{3/2}(u_{1,T}' + u_{3t}' + \overline{u}_{1T}) + O(\epsilon^2) \\
u^\epsilon u_x^\epsilon &= \sqrt{\epsilon}Uu_{1x}' + \epsilon(UU_X + (u_1' + \overline{u_1})u_{1x}' + Uu_{2x}') + \epsilon^{3/2}(U\overline{u}_{1X} + (u_1' + \overline{u_1})(U_X + u_{2x}') + (u_2' + \overline{u_2})u_{1x}') \\
v^\epsilon u_y^\epsilon &= \sqrt{\epsilon}v_1'U_y + \epsilon((\overline{v_2} + v_2')U_y + v_1'(u_1' + \overline{u}_{1,y})) + \epsilon^{3/2}(v_1'(u_{2y}' + \overline{u}_{2y}) + (\overline{v_2} + v_2')(u_{1y}' + \overline{u_{1y}}) + (\overline{v_3} + v_3')U_y) \\
yv^\epsilon &= \sqrt{\epsilon}yv_1' + \epsilon(yv_2' + y\overline{v}_2) + \epsilon^{3/2}(v_3' + \overline{v}_3)
\end{aligned}
$$

(4.34)

$$
p_x^\epsilon = \sqrt{\epsilon}p_{1x}' + \epsilon(P_X + p_{2x}') + \epsilon^{3/2}(p_{1X}' + \overline{p}_{1X} + p_{3x}')
$$

The term $w^\epsilon u_z^\epsilon$ follows the same pattern as $v^\epsilon u_y^\epsilon$. Collecting $O(\sqrt{\epsilon})$ terms gives the equation for the fluctuation

$$
u_{1t}' + Uu_{1x}' + v_1'U_y + w_1'U_z - yv_1' = -p_{1x}' \tag{4.35}
$$

Similarly, collecting the $O(\epsilon)$ terms and taking their mean and subtracting from the full equation to get the fluctuations, we get the following two equations

$$U_T + UU_X + \overline{v_2}U_y + \overline{w_2}U_z + \overline{v_1' u_{1y}'} + \overline{w_1' u_{1z}'} - y\overline{v_2} = -P_X$$

(4.36)

$$u_{2t}' + ((u_1' + \overline{u_1})u_{1x}' + Uu_{2x}') + (v_2' U_y + (v_1' u_{1y}')' + v_1' \overline{u}_{1y}) + (w_2' U_z + (w_1' u_{1z}')' + w_1' \overline{u}_{1z}) - yv_2' = -p_{2x}'$$

Doing the same procedure for all the five variables, we get the following five equations for the leading order fluctuations

$$\begin{aligned}
&u_{1t}' + Uu_{1x}' + v_1' U_y + w_1' U_z - yv_1' = -p_{1x}' \\
&v_{1t}' + Uv_{1x}' + yu_1' = -p_{1y}' \\
&\theta_{1t}' + U\theta_{1x}' + v_1' \Theta_y + w_1' \Theta_z + w_1' = 0 \\
&\theta_1' = p_{1z}' \\
&u_{1x}' + v_{1y}' + w_{1z}' = 0
\end{aligned} \tag{4.37}$$

The above equations are the same equations as obtained in Biello & Majda([3]) although the magnitude of the fluctuations are not the same. These equations are linear and describe the dynamics of the leading order fluctuations at the synoptic scale under the presence of a strong planetary scale background jet, $U(T, X, Y, z)$. The equations for the planetary scale mean background flow is given by

$$\begin{aligned}
\frac{DU}{DT} - y\overline{v_2} &= -P_X - \overline{v_1' u_{1y}'} - \overline{w_1' \theta_{1z}'} \\
yU &= -P_Y \\
\frac{D\Theta}{DT} + \overline{w_2} &= S - \overline{v_1' \theta_{1y}'} - \overline{w_1' \theta_{1z}'} \\
\Theta &= P_z \\
U_X + \overline{v}_{2y} + \overline{w}_{2z} &= 0
\end{aligned} \tag{4.38}$$

where

$$\frac{D}{DT} = \partial_T + U\partial_X + \overline{v_2}\partial_y + \overline{w_2}\partial_z$$

The flow is advected purely by the mean flow at planetary zonal scales. These equations are the same as the DSD equations obtained for the subtropical Hadley cell but differ in the fact that the fluctuations are also allowed to interact with the mean flow in the form of momentum and temperature flux terms. Since the evolution of the mean quantities is on the large time scale, while the leading order fluctuation equation evolves on the faster time scale, we need to go to higher order equations to get the large time evolution of the fluctuations. This happens at $O(\epsilon^{3/2})$ and the corresponding equations are derived in the appendix.

Since these higher order equations get too complicated, we will not be analysing them here. In the coming sections we will pose the fluctuation system of eq.4.37 as an eigenvalue problem to study the instability of subtropical jet like profiles. We will use the eigenfunctions of the unstable modes to study the behaviour of the flux terms in eq.4.38

4.4.1.1. *Wave solutions of IMMD fluctuations.* Let the background zonal velocity and temperature be given by $U(y,z)$ and $\Theta(y,z)$ respectively. The geostrophic wind condition connecting them is

$$yU_z = -\Theta_y$$

Plugging in wave solutions $\sim e^{ik(x-\sigma t)}$ in the fluctuation equation eq.4.37, we get

$$\begin{aligned} -ik\sigma u + ikUu + U_y v + U_z w - yv &= -ikp \\ -ik\sigma v + ikUv + yu &= -p_y \\ -ik\sigma\theta + ikU\theta + \Theta_y v + \Theta_z w + w &= 0 \\ iku + v_y + w_z &= 0 \end{aligned} \tag{4.39}$$

Transforming the meridional and vertical velocity as

$$\begin{aligned} v &= ikv' \\ w &= ikw' \end{aligned} \tag{4.40}$$

Using the above transformation we get

$$
\begin{aligned}
(U-\sigma)u + U_y v' + U_z w' - y v' &= -p \\
-k^2 (U-\sigma) v' + y u' &= -p_y \\
(U-\sigma)\theta + \Theta_y v' + \Theta_z w' + w' &= 0 \\
\theta &= p_z \\
u + v'_y + w'_z &= 0 \qquad (4.41)
\end{aligned}
$$

In case of stable solutions, i.e real σ we can assume the eigenfunctions are real. Due to the transformations in eq.4.40, the meridional and vertical velocity lead the zonal velocity and temperature in phase by $\pi/2$. The eigenfunctions can be written as

$$
\begin{aligned}
u &= \tilde{u}(y,z) e^{ik(x-\sigma t)} \\
\theta &= \tilde{\theta}(y,z) e^{ik(x-\sigma t)} \\
v &= i\tilde{v}(y,z) e^{ik(x-\sigma t)} \\
w &= i\tilde{w}(y,z) e^{ik(x-\sigma t)}
\end{aligned}
$$

4.4.1.2. *Barotropic and Baroclinic splitting of fluctuations.* Let the mean zonal velocity be given by

$$U = U_0(x,y) + U_1(x,y)\cos z$$

This gives us the following background temperature profile

$$
\begin{aligned}
\overline{\theta} &= -\Theta(x,y)\sin z \\
\Theta(x,y) &= \int_0^y sU(x,s)ds \qquad (4.42)
\end{aligned}
$$

Splitting the fluctuations into barotropic and first baroclinic modes we get the following

$$u' = u_0 + u_1 \cos z$$

$$v' = v_0 + v_1 \cos z$$

$$w' = w_1 \sin z$$

$$p' = p_0 + p_1 \cos z$$

$$\theta' = -p_1 \sin z$$

Plugging this in the IMMD fluctuation equations and only looking at the barotropic and first baroclinic component we get,

$$u_{0,t} + U_0 u_{0,x} - U_1 u_{1,x}/2 + U_{0,y} v_0 - U_{1,y} v_1/2 + U_1 w_1/2 - y v_0 = -p_{0,x}$$

$$v_{0,t} + U_0 v_{0,x} - U_1 v_{1,x}/2 + y u_0 = -p_{0,y}$$

$$u_{0,x} + v_{0,y} = 0$$

$$u_{1,t} + U_0 u_{1,x} - U_1 u_{0,x} + U_{0,y} v_1 - U_{1,y} v_0 - y v_1 = -p_{1,x}$$

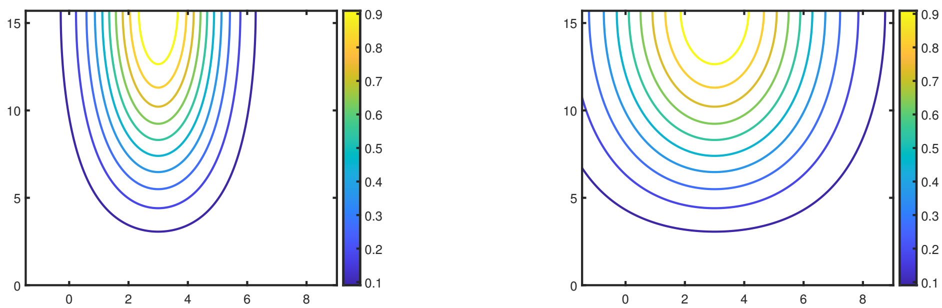
$$v_{1,t} + U_0 v_{1,x} - U_1 v_{0,x} + y u_1 = -p_{1,y}$$

$$p_{1,t} + U_0 p_{1,x} + v_1 y U_1 - w_1 = 0$$

$$u_{1,x} + v_{1,y} + w_1 = 0 \tag{4.43}$$

The first three equations are for the barotropic modes and the latter four are for the first baroclinic components. The baroclinic instability equations can be used to substitute for the vertical velocity w_1 in the baroclinic pressure equation. The three barotropic equations can be condensed into a single vorticity equation using a stream function for the horizontal velocities. This reduces the set of equations to four.

The equations were solved using a Fourier transform in the x and y direction and truncating the y Fourier transform to get the eigensolutions for a range of values of the zonal wavenumber k. The solutions have been discussed in the next section.



(a) (b)

FIGURE 4.5. U0=U1=0.5 for both the profiles. The jet maxima is fixed at $y_0 = 2$ or $y = 3000$km in dimensional terms. The length scale is chosen as L=1 for a) and L=1.75 for b) .

4.4.1.3. *Results.* The truncated instability equation set of eq.4.43 have been solved with the following background zonal jet

$$\overline{U} = (U_0 - U_1 \cos z) \exp(-(y - y_0)^2 / 2L^2) \quad \text{:Gaussian profile}$$

The jet is centered at $y = y_0$ and decaying away from it to simulate the behaviour of the subtropical jet obtained in the subtropical theory section. When the parameters U_0 and U_1 are the same, it results in a vanishing zonal velocity at the ground. The corresponding temperature has been computed using the geostrophic wind constraint. The solutions plotted in Fig.4.6 have been plotted for $U_0 = U_1 = 1$ and $y_0 = 2$. This corresponds to a vertical velocity of 50m/s at the tropopause and the maximum jet maxima at 3000km. A contour plot of the background zonal velocity has been plotted in Fig.4.5. The eigensolution with the highest growth rate have been plotted for $L = 1.75$ or approximately 2600km in dimensional terms. The wave momentum and temperature fluxes for the fastest growing modes have been plotted in Fig.4.7. Since the jet is exponentially decaying away from its maximal region, the wave activity is confined to that region as well. This is evident from both the eigensolution plots and the flux contour plots. The flux terms -$(vu)_y$ and -$(v\theta)_y$ act as forcing and heating term in the zonally averaged momentum and temperature equation, eq4.38. Let us call these fluxes F_u and F_θ. Both the fluxes are negligible on the equator-ward side of the jet. Both the temperature and momentum terms F_θ and F_u are structurally made up of two parts with opposite signs. Near the jet maxima, on the poleward side, F_u and F_θ act as positive forcing

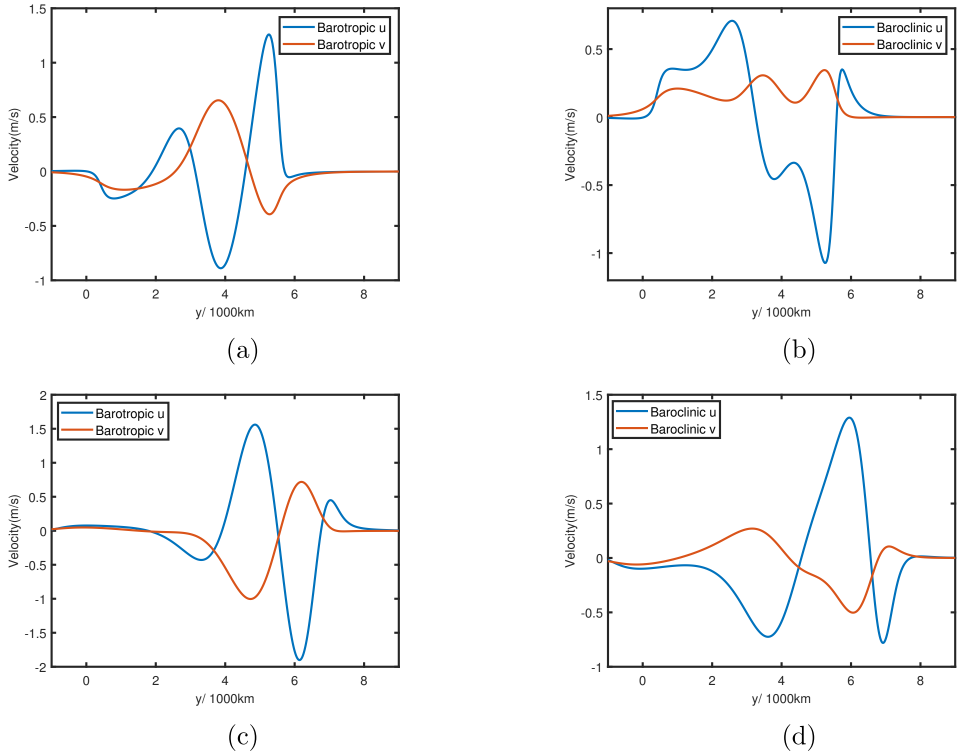


FIGURE 4.6. U0=U1=0.5. k=6 mode has been used in all of the cases. a), b) L=1. c), d) L=1.75 .

and cooling respectively. This region is followed by negative momentum forcing and heating region. In the DSD model of the Hadley cell, S_θ is very low on the poleward side of the jet. Including the effect of the eddy terms implies that all the heating is being provided by the F_θ term. The heating on the poleward side and cooling on the equatorward side will generate a counter-clockwise circulation which should impede the growth of the Hadley cell jet and thus close the cell closer than without the inclusion of these flux terms. Since the flux terms are only affecting the poleward side of the jet, this would result in a sharpening of the jet, a result comparable with the QG models of baroclinic instability.

The above picture of the Hadley cell is a qualitative result obtained by looking at the eddy flux terms. The time scale of the eddy scale phenomenon is faster than the zonally averaged Hadley cell by an order of ϵ. This combined with the fact that the eddy equations are linear means that there is no saturation of the exponentially growing modes. Due to this they can't be inserted into the Hadley cell equations to simulate their effect on the zonally averaged flow. Nonetheless,

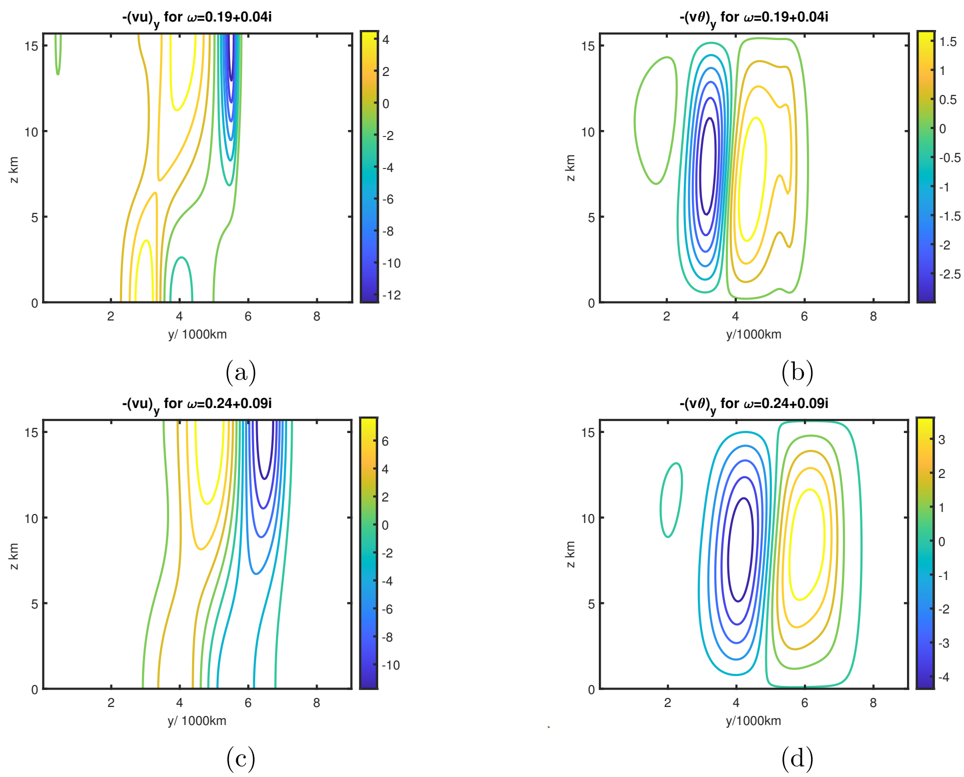


FIGURE 4.7. U0=U1=0.5. k=6 mode has been used in all of the cases. a), b) L=1. c), d) L=1.75 .

this time scale separation of the eddy modes and the zonally averaged flow means that in the real atmosphere, the zonally averaged flow must be near the marginal stability threshold.

4.4.2. Fully multiple scale theory. The aim of this study was to provide an asymptotic framework to get the fluctuation flux terms to interact with the mean flow. In a regular asymptotic expansion, the fluxes are smaller than the mean flow by an order of ϵ and hence, formally appear at equations of different order. In the previous section, the order of the fluxes was increased by increasing the magnitude of the fluctuation terms by an order of $\epsilon^{-1/2}$. This works in increasing the magnitude but the fluctuation equations thus obtained are linear and modulate over a faster time scale than the mean. Any unstable modes generated will quickly become unbounded and won't be able to interact with the slow time scale mean flow. If the fluctuation flow saturated, the fluxes could interact with the mean flow but this is prevented by the fluctuation system being linear.
In this section we will look at the second way in which we can increase the magnitude of the fluctuation fluxes. This is done by using an asymptotic ansatz modulating at a smaller length

scales. This will increase the magnitude of the derivative of the fluxes, making it formally the same order as the mean flow. Since the fast time modulations of the fluctuation terms occurs at the same order as the slow time scale modulations of the mean flow, these fluxes will also interact with the fluctuation flow, making their evolution non-linear. So, a multiple scale approach seems to simultaneously resolve the issue of increasing the magnitude of the fluxes as well as making the fluctuation equations non-linear.

4.4.2.1. *Multi-scale IMMD.* In the subtropical Hadley cell chapter, depending on the heating magnitude, we derived two scalings valid for the subtropical region. These were the IMMD and DSD scalings. Multiple scale asymptotic techniques have been applied on these two scalings to derive models describing the interaction of the flux with the mean fields. We first apply the technique to IMMD equations.

The IMMD model equations are as follows

$$
\begin{aligned}
&\frac{Du}{DT} - Yv + p_X = S_u \\
\epsilon^2 &\frac{Dv}{DT} + Yu + p_Y = 0 \\
&\frac{D\theta}{DT} + w = S_\theta \\
&\theta = p_Z \\
&u_X + v_Y + w_Z = 0
\end{aligned}
\tag{4.44}
$$

We denote the large scale time and space variables by T and $\vec{X} = (X, Y, Z)$ and the small scale variables as t and $\vec{x} = (x, y, z)$. The scales of the small scale variables is given by

$$t = \epsilon^{-1} T, \quad \vec{x} = \epsilon^{-1} \vec{X}$$

The long and short time scale correspond to 3 days and 8hrs respectively. The large scale zonal, meridional and vertical distances are scaled by 15000km,1500km and 5 km respectively. The small scale zonal, meridional and vertical distances are scaled by 1500km, 150km and 500m respectively.

The corresponding derivatives transform as

$$\frac{\partial}{\partial T} = \epsilon^{-1}\frac{\partial}{\partial t} + \frac{\partial}{\partial T}$$
$$\frac{\partial}{\partial X} = \epsilon^{-1}\frac{\partial}{\partial x} + \frac{\partial}{\partial X}$$

The (Y, Z) derivatives transform in the same way. We define the spatial and temporal averages of a function $f(t, T, \vec{x}, \vec{X})$ as follows

$$\langle f \rangle = \lim_{t^* \to \infty} \frac{1}{2t^*} \int_{-t^*}^{t^*} f(t, T, \vec{x}, \vec{X}) dt$$
$$\overline{f} = \lim_{L \to \infty} \frac{1}{(2L)^3} \int_{-L}^{L} \int_{-L}^{L} \int_{-L}^{L} f(t, T, \vec{x}, \vec{X}) dxdydz$$

Using the above definitions, we can define the spatial and temporal fluctuations as

$$\tilde{f} = f - \langle f \rangle$$
$$f' = f - \overline{f}$$

where, by definition $\tilde{f}$ and f' have vanishing temporal and spatial averages respectively. For a multiscale solution, we look at expansions of the form

$$f^{\epsilon} = f_0(\epsilon^{-1}T, T\epsilon^{-1}\vec{X}, \vec{X}) + \epsilon f_1(\epsilon^{-1}T, T\epsilon^{-1}\vec{X}, \vec{X}) + \epsilon^2 f_2(\epsilon^{-1}T, T\epsilon^{-1}\vec{X}, \vec{X}) + \dots$$

When $T = O(1)$, the fast time scale $t = O(\epsilon^{-1})$. For the consecutive terms in the expansion to remain ordered in magnitude, we need ϵf_i to be $o(1)$. In other words the functions f_i can only grow sublinearly(Kevorkian and Cole [15], Majda [22], Majda and Klein [24])with the fast time variable t. Similar argument holds for each of the small spatial scale as well. This sublinear growth implies the following for the averages of the fast time and small scale derivatives of the functions

$$\langle f_t \rangle = \overline{\nabla_x f} = 0$$

We want to model the interaction of the fluxes generated due to the smaller magnitude flow with the large scale, large magnitude zonal flow. For this, we make the leading order zonal velocity, pressure and temperature to be independent of the small scale variables and use the following ansatz in eq.4.51 to get a multi scale model

$$
\begin{aligned}
u^\epsilon &= U(T,\vec{X}) + \epsilon(u_1'(t,T,\vec{x},\vec{X}) + \overline{u}(t,T,\vec{X})) + O(\epsilon^2) \\
v^\epsilon &= (v'(t,T,\vec{x},\vec{X}) + \overline{v}(t,T,\vec{X})) + O(\epsilon) \\
w^\epsilon &= (w'(t,T,\vec{x},\vec{X}) + \overline{w}(t,T,\vec{X})) + O(\epsilon) \\
p^\epsilon &= P(T,\vec{X}) + \epsilon(p'(t,T,\vec{x},\vec{X}) + \overline{p}(t,T,\vec{X})) + \epsilon^2(p_2'(t,T,\vec{x},\vec{X}) + \overline{p_2}(t,T,\vec{X})) + O(\epsilon^3) \\
\theta^\epsilon &= \Theta(T,\vec{X}) + \epsilon(\theta'(t,T,\vec{x},\vec{X})) + \overline{\theta}(t,T,\vec{X}) + O(\epsilon^2) \qquad (4.45)
\end{aligned}
$$

To get the background temperature to be independent of any fluctuation terms, p' has to be independent of z. After plugging in the above expansion and taking the spatial and time averages we get the equations governing the large time evolution of the mean fields. Subtracting the spatial means gives us the fast time evolution of the fluctuations. Subtracting the spatio-temporal average from the spatial average gives us the fast time evolution of the leading order planetary scale anomalies. These three sets of equations have been listed below and given the appropriate names from Biello and Majda [3]

a): **Hadley cell-fluctuation interaction theory**

$$
\begin{aligned}
&\frac{DU}{DT} - YV = -P_X - \langle\overline{F_u}\rangle \\
&YU = -P_Y \\
&\frac{D\Theta}{DT} + W = \langle\overline{S_\theta}\rangle - \langle\overline{F_\theta}\rangle \\
&P_Z = \Theta \\
&U_X + V_Y + W_Z = 0 \qquad (4.46)
\end{aligned}
$$

where $V = \langle\overline{v}\rangle$ and $W = \langle\overline{w}\rangle$ and the total derivative represents the advection by the leading order mean flow over the large scale spatial and temporal variables

$$
\frac{D}{DT} = \frac{\partial}{\partial T} + U\frac{\partial}{\partial X} + V\frac{\partial}{\partial Y} + W\frac{\partial}{\partial Z}
$$

The flux terms F_u and F_θ are given by

$$
\begin{aligned}
F_u &= v'u'_y + w'u'_z \\
F_\theta &= v'\theta'_y + w\theta'_z
\end{aligned}
\tag{4.47}
$$

b): **Fast fluctuation dynamics**

$$
\begin{aligned}
&\frac{\overline{D}u'}{\overline{D}t} + v'U_Y + w'U_Z - Yv' = -p'_x - (F_u)' \\
&p'_y = 0 \\
&\frac{\overline{D}v'}{\overline{D}t} + Yu' = -(p'_Y - p'_{2,y}) - (F_v)' \\
&\frac{\overline{D}\theta'}{\overline{D}t} + v'\Theta_Y + w'\Theta_Z + w' = S'_\theta - (F_\theta)' \\
&\theta' = p'_Z + p'_{2,z} \\
&v'_y + w'_z = 0
\end{aligned}
\tag{4.48}
$$

F_v is the meridional velocity flux and is given by

$$
F_v = v'v'_y + w'v'_z \tag{4.49}
$$

where the total derivative is given by

$$
\frac{\overline{D}}{\overline{D}t} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + \overline{v}\frac{\partial}{\partial y} + \overline{w}\frac{\partial}{\partial z}
$$

c): **Fast planetary scale anomalies**

$$
\begin{aligned}
&\tilde{\bar{u}}_t + \tilde{\bar{v}}U_Y + \tilde{\bar{w}}U_Z = -\tilde{\overline{F_u}} \\
&\tilde{\bar{v}}_t + Y\tilde{\bar{u}} = -\tilde{\bar{p}}_Y - \tilde{\overline{F_v}} \\
&\tilde{\bar{\theta}}_t + \tilde{\bar{v}}\Theta_Y + \tilde{\bar{w}}\Theta_Z + \tilde{\bar{w}} = \tilde{\overline{S_\theta}} - \tilde{\overline{F_\theta}} \\
&\tilde{\bar{\theta}} = \tilde{\bar{p}}_Z \\
&\tilde{\bar{v}}_Y + \tilde{\bar{w}}_Z = 0
\end{aligned}
\tag{4.50}
$$

The model described by the three sets of equations in eq.4.46,4.48 and 4.50 describes the dynamics of the Hadley cell and the small scale fluctuations. The set of equations in 4.46 describe the slow evolution of the leading order mean fields in the Hadley cell. Other than the inclusion of the fluxes, the equations obtained are identical to those discussed in the subtropical theory section. These flux terms are generally parameterized as a damping force, like the Rayleigh damping or a viscous damping. It should be noted that meridional geostrophy is still maintained and there are no flux terms interacting with the mean meridional flow.

The interaction between the Hadley cell and the flux terms was also present in the theory derived in the previous section, but the interaction between the fluctuation and fluxes was absent. As a result the fluctuation equations were linear and no saturation could be attained for growing modes. In the fully multiscale theory, the fluctuation equations, like the mean flow are also non-linear and will have a limiting solution for $T \sim O(1)$ or as the fast time scale grows to infinity.

Finally, eq.4.50 describes the fast time evolution of the planetary scale anomalies. All the planetary anomaly fields have a vanishing time average and are only advected by the zonal field in the meridional and vertical directions. Again, the equations obtained, unlike those obtained in [3] are non-linear.

4.4.2.2. *Multi scale DSD theory.* The derivations in this section follow in quite the same way as in the previous section. We use the same asymptotic ansatz as eq.4.45. The DSD equations are as follows

$$\begin{aligned}
&\frac{Du}{DT} - Yv + p_X = S_u \\
\epsilon &\frac{Dv}{DT} + Yu + p_Y = 0 \\
&\frac{D\theta}{DT} + w = S_\theta \\
&\theta = p_Z \\
&u_X + v_Y + w_Z = 0
\end{aligned} \tag{4.51}$$

Instead of an ϵ^2 in the meridional equation, DSD has a factor of ϵ. The zonal velocity is scaled by 50m/s, the meridional velocity by 15m/s and the vertical velocity by 5m/s. The heating here is higher by a factor of $\epsilon^{1/2}$ compared to IMMD. The slow time scale, T, here is 1 day which corresponds to a 2.4 hr fast time scale. The large scale zonal distance is scaled by 5000km, while the meridional and vertical distance scalings are the same as in IMMD at 1500km and 5km respectively. The small scale zonal, meridional and vertical distance is then 500km,150km and 500m respectively. After plugging in the multi scale expansion and performing the averaging procedures, we obtain the following three sets of equations

a): **Hadley cell-fluctuation interaction theory**

$$
\begin{aligned}
&\frac{DU}{DT} - YV = -P_X - \langle \overline{F_u} \rangle \\
&YU = -P_Y - \langle \overline{F_v} \rangle \\
&\frac{D\Theta}{DT} + W = \langle \overline{S_\theta} \rangle - \langle \overline{F_\theta} \rangle \\
&P_Z = \Theta \\
&U_X + V_Y + W_Z = 0
\end{aligned}
\tag{4.52}
$$

where $V = \langle \overline{v} \rangle$ and $W = \langle \overline{w} \rangle$ and the total derivative represents the advection by the leading order mean flow over the large scale spatial and temporal variables

$$
\frac{D}{DT} = \frac{\partial}{\partial T} + U\frac{\partial}{\partial X} + V\frac{\partial}{\partial Y} + W\frac{\partial}{\partial Z}
$$

The flux terms F_u, F_v, and F_θ are given by

$$
\begin{aligned}
&F_u = v'u'_y + w'u'_z \\
&F_v = v'v'_y + w'v'_z \\
&F_\theta = v'\theta'_y + w\theta'_z
\end{aligned}
\tag{4.53}
$$

b): **Fast fluctuation dynamics**

$$
\begin{aligned}
&\frac{\overline{D}u'}{\overline{D}t} + v'U_Y + w'U_Z - Yv' = -p'_x - (F_u)' \\
&\frac{\overline{D}v'}{\overline{D}t} + Yu' = -p'_y - (F_v)' \\
&\frac{\overline{D}\theta'}{\overline{D}t} + v'\Theta_Y + w'\Theta_Z + w' = S'_\theta - (F_\theta)' \\
&\theta' = p'_Z + p'_{2,z} \\
&v'_y + w'_z = 0
\end{aligned}
\tag{4.54}
$$

where the total derivative is given by

$$\frac{\overline{D}}{\overline{D}t} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + \overline{v}\frac{\partial}{\partial y} + \overline{w}\frac{\partial}{\partial z}$$

c): **Fast planetary scale anomalies**

$$\tilde{\tilde{u}}_t + \tilde{\tilde{v}}U_Y + \tilde{\tilde{w}}U_Z = -\overline{\tilde{\tilde{F}}_u}$$

$$\tilde{\tilde{v}}_t + Y\tilde{\tilde{u}} = -\overline{\tilde{\tilde{F}}_v}$$

$$\tilde{\tilde{\theta}}_t + \tilde{\tilde{v}}\Theta_Y + \tilde{\tilde{w}}\Theta_Z + \tilde{\tilde{w}} = \overline{\tilde{\tilde{S}}_\theta} - \overline{\tilde{\tilde{F}}_\theta}$$

$$\tilde{\tilde{\theta}} = \tilde{\tilde{p}}_Z$$

$$\tilde{\tilde{v}}_Y + \tilde{\tilde{w}}_Z = 0 \tag{4.55}$$

The slow time evolution of the leading order mean flow is given by eq.4.52. The only difference compared to the IMMD case is the meridional momentum equation. Meridional geostrophy is no longer satisfied as the meridional momentum flux terms are bigger in the DSD scaling. For the fast fluctuation dynamics, eq.4.54, the order ϵ^2 pressure doesn't participate in the meridional momentum equation. It is calculated from the hydrostatic balance and only participates in the higher order momentum balances. The fast evolution of planetary scale anomalies, eq.4.55 sees no effect of pressure on the zonal or meridional momentum. The only difference between the multi scale IMMD and multi scale DSD arises in the meridional momentum equation, as might be expected considering the single scaled equations only differ in the meridional momentum equation.

4.5. Summary

Using the principle of baroclinic adjustment and the Philip's two layer model, the effects of instability were modeled as a damping force which relaxes the flow back to the marginal stable flow. This damping was then plugged into the WTG model to obtain the solution for the zonal velocity at the tropopause.

In the second part of the chapter, instabilities formed due to a zonal jet were studied in the DSD system of chapter 3. Thinking of the momentum fluxes as forcing/damping and the temperature fluxes as heating/cooling, these fluxes should generate a circulation counter to the Hadley cell at the poleward terminus of the Hadley ell. Using a multi-scale approach, an attempt was made to incorporate the fluxes into the mean flow system. Since the instability system is linear, the flux terms, in case of instabilities do not saturate. Due to the fluxes modulating at a faster time scale than the mean flow, instabilities will cause singularities in the mean flow. This implies that the mean flow must self organize into a state of marginal stability. A fully 3D multi-scale system has also been derived with the fluxes interacting with both the mean flow and the instabilities. The instability system is no longer linear in this case and the solution can reach saturation, making an interaction with the mean flow possible. Unfortunately, the full system is a collection of 15 equations and numerical solution becomes difficult to obtain. nevertheless, the qualitative properties of the system have been discussed.

APPENDIX A

Faster Geostrophy

The scaling of the zonal velocity in the subtropical theory leads us to seek solution with faster zonal velocity in the mid-latitudes. QG theory has $O(1)$ zonal velocity, which is an order of ϵ slower than the zonal velocity in IMMD or DSD theory. To this end, we use an asymptotic ansatz of the following form

$$\begin{aligned}
u^\epsilon &= \frac{U(y,z)}{\epsilon} + u + \epsilon u_1 + \dots \\
v^\epsilon &= v + \epsilon v_1 + \dots \\
w^\epsilon &= w + \epsilon w_1 + \dots \\
p^\epsilon &= \frac{\Pi(y,z)}{\epsilon} + p + \epsilon p_1 + \dots \\
\theta^\epsilon &= \frac{\Theta(y,z)}{\epsilon} + \theta + \epsilon \theta_1 + \dots
\end{aligned} \tag{A.1}$$

Plugging this in the non dimensionalised primitive equations, at leading order, we get meridional geostrophy and the hydrostatic balance

$$\begin{aligned}
\Pi_y &= -fU \\
\Theta &= \Pi_z
\end{aligned} \tag{A.2}$$

At the next order in ϵ, we get the equations $O(1)$ variables

$$\begin{aligned}
Uu_x + U_y v + U_z w - fv + p_x &= -dU \\
Uv_x + fu + p_y &= 0 \\
U\theta_x + v\Theta_y + w(1+\Theta_z) &= 0 \\
\theta &= p_z \\
u_x + v_y + w_z &= 0
\end{aligned} \tag{A.3}$$

We can also include the time derivatives if we rescale the time as $t \to \epsilon\tau$ where τ corresponds to time scale of the order of 2.4 hours.

A.1. Zonally invariant solution

A zonally invariant version of the equations gives us the following two equations

$$\begin{aligned} v(U_y - f) + wU_z &= -dU \\ v\Theta_y + w(1 + \Theta_z) &= 0 \end{aligned} \tag{A.4}$$

Alongside this, we have the incompressibility and the geostrophic wind constraints

$$\begin{aligned} fU_z + \Theta_y &= 0 \\ v_y + w_z &= 0 \end{aligned}$$

We use the following transformations

$$U = M + fy \tag{A.5}$$

$$\tilde{\Theta} = \Theta + z \tag{A.6}$$

The geostrophic wind constraint becomes

$$fM_z + \tilde{\Theta}_y = 0$$

M and $\tilde{\Theta}$ represent the total angular momentum and the total background temperature. Defining a potential function Φ such that $\Phi_y = fM$ and $\Phi_z = -\tilde{\Theta}_y$. The transformations in eq.A.6 along with the potential function can be plugged in eq.A.4 to obtain the following system

$$\begin{aligned} v\Phi_{yy} + w\Phi_{yz} &= -d(\Phi_y + f^2 y) \\ v\Phi_{yz} + w\Phi_{zz} &= 0 \end{aligned} \tag{A.7}$$

We will get a similar system of equations for the steady state, zonally invariant IMMD or DSD system. The only difference will be the inclusion of a source term in the temperature equation, since the heating is considered to be larger in the subtropics. The eq.A.7 can be solved for v and w to give us the following

$$v = -fd\frac{\Phi_{zz}(\Phi_y + f^2 y)}{Q}$$

$$w = fd\frac{\Phi_{yz}(\Phi_y + f^2 y)}{Q}$$

$$Q = \Phi_{yy}\Phi_{zz} - \Phi_{yz}^2 \tag{A.8}$$

where Q is the leading order potential vorticity of the system. The solutions obtained for v and w also need to satisfy the incompressibility condition. This gives us the following equation for the potential function Φ

$$\left\{\frac{\Phi_{zz}(\Phi_y + f^2 y)}{Q}\right\}_y - \left\{\frac{\Phi_{yz}(\Phi_y + f^2 y)}{Q}\right\}_z = 0 \tag{A.9}$$

The above equation completely solves the system, although it is highly non-linear to be of any use in real scenarios. In case of the subtropical system, there will also be a source term included in the above equation.

APPENDIX B

Slow time evolution of baroclinic fluctuations

The mean fields of section 4.4 evolve at a slower rate than the fluctuations. The slow time evolution of the fluctuations has been described in this section.

To get to the slow time evolution of the leading order fluctuations, we have to go through the fast time evolution of the $O(\epsilon)$ fluctuations.

$$u'_{2t} + ((u'_1 + \overline{u_1})u'_{1x} + Uu'_{2x}) + (v'_2 U_y + (v'_1 u'_{1y})' + v'_1 \overline{u}_{1y}) + (w'_2 U_z + (w'_1 u'_{1z})' + w'_1 \overline{u}_{1z}) - yv'_2 = -p'_{2x}$$

$$v'_{2t} + Uv'_{2x} + (u'_1 v'_{1x})' + \overline{u_1} v'_{1x} + (v'_1 v'_{1y})' + (w'_1 v'_{1z})' + yu'_2 = -p'_{2y}$$

$$\theta'_{2t} + ((u'_1 \theta_{1x})' + \overline{u_1}\theta'_{1x} + U\theta'_{2x}) + (v'_2 \Theta_y + (v'_1 \theta'_{1y})' + v'_1 \overline{\theta}_{1y}) + (w'_2 \Theta_z + (w'_1 \theta'_{1z})' + w'_1 \overline{\theta}_{1z}) - w'_2 = 0$$

$$\theta'_2 = p'_{2z}$$

$$u'_{2x} + v'_{2y} + w'_{2z} = 0$$

(B.1)

The next set of equations is obtained as a mean of the $O(\epsilon^{3/2})$ terms

$$\overline{u}_{1T} + U\overline{u}_{1X} + \overline{u}_1 U_X + \overline{u'_1 u'_{2x}} + \overline{u'_2 u'_{1x}} + \overline{v'_1 u'_{2y}} + \overline{v_2}\overline{u}_{1y} + \overline{v'_2 u'_{1y}} + \overline{v_3} U_y +$$
$$\overline{w'_1 u'_{2z}} + \overline{w_2} u_{1y} + \overline{w'_2 u'_{1z}} + \overline{w_3} U_z - y\overline{v_3} = -\overline{p}_X$$

$$y\overline{u_1} = -\overline{p}_{1y}$$

$$\overline{\theta}_{1T} + U\overline{\theta}_{1X} + \overline{u}_1 \Theta_X + \overline{u'_1 \theta'_{2x}} + \overline{u'_2 \theta'_{1x}} + \overline{v'_1 \theta'_{2y}} + \overline{v_2}\overline{\theta}_{1y} + \overline{v'_2 \theta'_{1y}} + \overline{v_3}\Theta_y +$$
$$\overline{w'_1 \theta'_{2z}} + \overline{w_2}\overline{\theta}_{1y} + \overline{w'_2 \theta'_{1z}} + \overline{w_3}\Theta_z - \overline{w}_3 = 0$$

$$\overline{\theta}_1 = \overline{p}_{1z}$$

(B.2) $$\overline{u}_{1X} + \overline{v}_{3y} + \overline{w}_{3z} = 0$$

Then we have the fluctuations at $O(\epsilon^{3/2})$

$$u'_{1,T} + u'_{3t} + u'_1 U_X + (u'_1 u'_{2x})' + \overline{u}_1 u'_{2x} + (u'_2 u'_{1x})' + \overline{u_2} u'_{1x} + \overline{u_2} u'_{1x} + U u'_{3x} + (v'_1 u'_{2y})' + v'_1 \overline{u}_{2y} +$$

$$\overline{v}_2 u'_{1y} + (v'_2 u'_{1y})' + v'_3 U_y + (w'_1 u'_{2z})' + w'_1 \overline{u}_{2z} + \overline{w}_2 u'_{1z} +$$

$$(w'_2 u'_{1z})' + w'_3 U_z - y v'_3 = -p'_{1X} - p'_{3x}$$

$$v'_{1T} + v'_{3t} + U v'_{3x} + (u'_1 v'_{2x})' + \overline{u}_1 v'_{2x} + (u'_{2x} v'_{1x})' + \overline{u}_2 v'_{1x} + (v'_1 v'_{2y})' + v'_1 \overline{v}_{2y} + \overline{v}_2 v'_{1y} + (v'_2 v'_{1y})'$$

$$+ (w'_1 v'_{2z})' + w'_1 \overline{v}_{2z} + \overline{w}_2 v'_{1z} + (w'_2 v'_{1z})' + y u'_3 = -p'_{3y}$$

$$\theta'_{1,T} + \theta'_{3t} + u'_1 \Theta_X + (u'_1 \theta'_{2x})' + \overline{u}_1 \theta'_{2x} + (u'_2 \theta'_{1x})' + \overline{u_2} \theta'_{1x} + (v'_1 \theta'_{2y})' + v'_1 \overline{\theta}_{2y} + \overline{v}_2 \theta'_{1y} + (v'_2 \theta'_{1y})' + v'_3 \Theta_y$$

$$+ (w'_1 \theta'_{2z})' + w'_1 \overline{\theta}_{2z} + \overline{w}_2 \theta'_{1z} + (w'_2 \theta'_{1z})' + w'_3 \Theta_z + w'_3 = 0$$

$$\theta'_3 = p'_{3z}$$

$$u'_{3x} + u'_{1X} + v'_{3y} + w'_{3z} = 0$$

(B.3)

The slow time derivative of the leading order fluctuations(subscript 1 fluctuations) comes at the same order as the fast time derivative of the O($\epsilon^{3/2}$) fluctuations. This gives a constraint on the amplitudes of the leading order fluctuations so as to prevent secular growth in the $O(\epsilon^{3/2})$ fluctuations.

Bibliography

[1] Joseph A Biello and Andrew J Majda. A new multiscale model for the madden–julian oscillation. *Journal of the atmospheric sciences*, 62(6):1694–1721, 2005.

[2] Joseph A Biello and Andrew J Majda. Intraseasonal multi-scale moist dynamics of the tropical atmosphere. *Communications in Mathematical Sciences*, 8(2):519–540, 2010.

[3] Joseph A Biello and Andrew J Majda. A multiscale model for the modulation and rectification of the itcz. *Journal of the atmospheric sciences*, 70(4):1053–1070, 2013.

[4] Jule G Charney. The dynamics of long waves in a baroclinic westerly current. *Journal of the Atmospheric Sciences*, 4(5):136–162, 1947.

[5] Timothy J Dunkerton. On the inertial stability f the equatorial middle atmosphere. *Journal of Atmospheric Sciences*, 38(11):2354–2364, 1981.

[6] Timothy J Dunkerton. Body force circulations in a compressible atmosphere: Key concepts. *pure and applied geophysics*, 130:243–262, 1989.

[7] Timothy J Dunkerton. Nonlinear hadley circulation driven by asymmetric differential heating. *Journal of the atmospheric sciences*, 46(7):956–974, 1989.

[8] Eric T Eady. Long waves and cyclone waves. *Tellus*, 1(3):33–52, 1949.

[9] Arnt Eliassen. On the vertical circulation in frontal zones. *Geofys. publ*, 24(4):147–160, 1962.

[10] Adrian E Gill. Some simple solutions for heat-induced tropical circulation. *Quarterly Journal of the Royal Meteorological Society*, 106(449):447–462, 1980.

[11] Isaac M Held and Brian J Hoskins. Large-scale eddies and the general circulation of the troposphere. In *Advances in geophysics*, volume 28, pages 3–31. Elsevier, 1985.

[12] Isaac M Held and Arthur Y Hou. Nonlinear axially symmetric circulations in a nearly inviscid atmosphere. *Journal of the Atmospheric Sciences*, 37(3):515–533, 1980.

[13] Isaac M Held and Peter J Phillips. A barotropic model of the interaction between the hadley cell and a rossby wave. *Journal of Atmospheric Sciences*, 47(7):856–869, 1990.

[14] Brian J Hoskins and Francis P Bretherton. Atmospheric frontogenesis models: Mathematical formulation and solution. *Journal of the atmospheric sciences*, 29(1):11–37, 1972.

[15] Jirair K Kevorkian and Julian D Cole. *Multiple scale and singular perturbation methods*, volume 114. Springer Science & Business Media, 2012.

[16] Jirayr Kevorkian and Julian D Cole. Perturbation methods in applied mathematics. 34, 2013.

[17] Rupert Klein. Asymptotic analyses for atmospheric flows and the construction of asymptotically adaptive numerical methods. *ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik: Applied Mathematics and Mechanics*, 80(11-12):765–777, 2000.

[18] Rupert Klein. Scale-dependent models for atmospheric flows. *Annual review of fluid mechanics*, 42:249–274, 2010.

[19] PA Lagerstrom and JD Cole. Examples illustrating expansion procedures for the navier-stokes equations. *Journal of Rational Mechanics and Analysis*, 4:817–882, 1955.

[20] Paco A Lagerstrom and RG Casten. Basic concepts underlying singular perturbation techniques. *SIAM review*, 14(1):63–120, 1972.

[21] Conway Leovy. Simple models of thermally driven mesopheric circulation. *Journal of Atmospheric Sciences*, 21(4):327–341, 1964.

[22] Andrew Majda. *Introduction to PDEs and Waves for the Atmosphere and Ocean*, volume 9. American Mathematical Soc., 2003.

[23] Andrew J Majda and Joseph A Biello. A multiscale model for tropical intraseasonal oscillations. *Proceedings of the National Academy of Sciences*, 101(14):4736–4741, 2004.

[24] Andrew J Majda and Rupert Klein. Systematic multiscale models for the tropics. *Journal of the Atmospheric Sciences*, 60(2):393–408, 2003.

[25] Taroh Matsuno. Quasi-geostrophic motions in the equatorial area. *Journal of the Meteorological Society of Japan. Ser. II*, 44(1):25–43, 1966.

[26] J David Neelin and Isaac M Held. Modeling tropical convergence based on the moist static energy budget. *Monthly Weather Review*, 115(1):3–12, 1987.

[27] Joseph Pedlosky. *Geophysical fluid dynamics*. Springer Science & Business Media, 2013.

[28] Norman A Phillips. Energy transformations and meridional circulations associated with simple baroclinic waves in a two-level, quasi-geostrophic model. *Tellus*, 6(3):274–286, 1954.

[29] R Alan Plumb and Arthur Y Hou. The response of a zonally symmetric atmosphere to subtropical thermal forcing: Threshold behavior. *Journal of Atmospheric Sciences*, 49(19):1790–1799, 1992.

[30] Lorenzo M Polvani and Adam H Sobel. The hadley circulation and the weak temperature gradient approximation. *Journal of the atmospheric sciences*, 59(10):1744–1752, 2002.

[31] Rick Salmon, Issac M Held, Janet Fields, and Jean-Luc Thiffeault. The general circulation of the atmosphere: 2000 program in geophysical fluid dynamics. 2001.

[32] John Stanley Sawyer. *The Free Atmosphere in the Vicinity of Fronts-Analysis of Observations by the Meteorological Research Flight (1950-52)*. Air Ministry, 1953.

[33] Edwin K Schneider. Axially symmetric steady-state models of the basic state for instability and climate studies. part ii. nonlinear calculations. *Journal of Atmospheric Sciences*, 34(2):280–296, 1977.

[34] Edwin K Schneider. A simplified model of the modified hadley circulation. *Journal of Atmospheric Sciences*, 44(22):3311–3328, 1987.

[35] Edwin K Schneider and Richard S Lindzen. The influence of stable stratification on the thermally driven tropical boundary layer. *Journal of Atmospheric Sciences*, 33(7):1301–1307, 1976.

[36] Edwin K Schneider and Richard S Lindzen. Axially symmetric steady-state models of the basic state for instability and climate studies. part i. linearized calculations. *Journal of Atmospheric Sciences*, 34(2):263–279, 1977.

[37] Tapio Schneider. The general circulation of the atmosphere. *Annu. Rev. Earth Planet. Sci.*, 34:655–688, 2006.

[38] Tapio Schneider and Christopher C Walker. Self-organization of atmospheric macroturbulence into critical states of weak nonlinear eddy–eddy interactions. *Journal of the atmospheric sciences*, 63(6):1569–1586, 2006.

[39] Adam H Sobel and Christopher S Bretherton. Modeling tropical precipitation in a single column. *Journal of climate*, 13(24):4378–4392, 2000.

[40] Adam H Sobel, Johan Nilsson, and Lorenzo M Polvani. The weak temperature gradient approximation and balanced tropical moisture waves. *Journal of the atmospheric sciences*, 58(23):3650–3665, 2001.

[41] Duane E Stevens. On symmetric stability and instability of zonal mean flows near the equator. *Journal of the Atmospheric Sciences*, 40(4):882–893, 1983.

[42] Peter H Stone. Baroclinic adjustment. *Journal of Atmospheric Sciences*, 35(4):561–571, 1978.

[43] Geoffrey K Vallis. *Atmospheric and oceanic fluid dynamics*. Cambridge University Press, 2017.

[44] Christopher C Walker and Tapio Schneider. Eddy influences on hadley circulations: Simulations with an idealized gcm. *Journal of the atmospheric sciences*, 63(12):3333–3350, 2006.