Hilbert scheme of points on non-reduced plane curves

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Chapter 1: Abstract

In this thesis, we study plane curves C defined by one polynomial equation $f(x,y) = 0$ on the plane $\mathbb{C}^2$. We can factor the polynomial $f(x,y) = \prod f_i(x,y)^{\beta_i}$ where the polynomials $f_i(x,y)$ are irreducible, and consider the irreducible components of C defined by the equations $f_i = 0$. The curve C might be non-reduced, and the multiplicity of the component C_i equals β_i.

The main object in this thesis is the Hilbert scheme of n points on C defined as the moduli space of ideals I in the quotient ring $\mathbb{C}[x,y]/(f(x,y))$ whose colength is n.

In Chapter 3, we classify the irreducible components of the Hilbert scheme of points on C. We prove that all the components are indexed by multi-partitions λ of n satisfying specific combinatorial conditions. We also prove that all the irreducible components have the same dimension n, and we give an explicit formula for the multiplicity of each component in terms of the multi-partition λ and the multiplicities of the curve components. This part is based on the paper [10].

We further classify the irreducible components of the nested Hilbert scheme of points on the curve $C^{[n,n+1]}$. We prove that all the components have the same dimension n, and are indexed by multi-partitions of n satisfying specific combinatorial conditions.

Chapter 2: Background

In this chapter, we define the Hilbert scheme on the affine plane $\mathbb{C}^2$ and on a plane curve C, and other relevant constructions that we work with. We have a warning that this thesis uses many notations interchangeably. For example, the Hilbert scheme of points on $\mathbb{C}^2$, are denoted interchangeably by both $\mathrm{Hilb}^n(\mathbb{C}^2)$ and $(\mathbb{C}^2)^{[n]}$.

1. Hilbert scheme of points

Definition 1.1. The Hilbert scheme of n points on the affine plane $\mathbb{C}^2$, denoted by $\mathrm{Hilb}^n(\mathbb{C}^2)$ or $(\mathbb{C}^2)^{[n]}$, is the moduli space of ideals I in the polynomial ring of two variables $\mathbb{C}[x, y]$ satisfying the condition that the quotient ring $\mathbb{C}[x, y]/I$ has dimension n as a vector space over $\mathbb{C}$.

The dimension of the quotient ring $\mathbb{C}[x, y]/I$ is also called the colength of the ideal I.

Example 1.2. Consider the Hilbert scheme of 1 point on $\mathbb{C}^2$, denoted by $\mathrm{Hilb}^1(\mathbb{C}^2)$.

An ideal I has colength 1 if and only if it is maximal in the polynomial ring $\mathbb{C}[x, y]$, and by Nullstellensatz the ideal I is equal to $(x - a, y - b)$ for some point $(a, b) \in \mathbb{C}^2$. Therefore I is the defining ideal for the point (a, b) and we can identify $\mathrm{Hilb}^1(\mathbb{C}^2) = \mathbb{C}^2$.

Example 1.3. Let $P_1, ...P_n$ be n distinct points on $\mathbb{C}^2$, and I an ideal defining their union, i.e. intersection of ideals $I = \cap_i(x-x_{P_i}, y-y_{P_i})$, where x_{P_i} and y_{P_i} are the x and y coordinates of the point P_i.

We claim the following: The ideal I has colength n, and thus defines a point in $\mathrm{Hilb}^n(\mathbb{C}^2)$.

Define the map ϕ to be the ring morphism from $\mathbb{C}[x,y]$ to $\mathbb{C}^n$ by sending a polynomial $f(x,y) \in \mathbb{C}[x,y]$ to the tuple $(f(P_1),...,f(P_n))$. The kernel of the map ϕ is the ideal in $\mathbb{C}[x,y]$ such that $(f(P_1),...,f(P_n)) = (0,...,0)$. This kernel is exactly the ideal $I = \cap_i(x - x_{P_i}, y - y_{P_i})$. On the other hand, we know that the image of ϕ is $\mathbb{C}^n$ because ϕ is surjective. By the isomorphism theorem, $\mathbb{C}[x,y]/I \cong \mathbb{C}^n$, so the ideal I has colength n.

Define W to be the set of all possible collections on n unordered distinct points $P_1,...,P_n$ on $\mathbb{C}^2$. By the above argument, this corresponds to a subset of $\mathrm{Hilb}^n(\mathbb{C})$.

Theorem 1.4. [5]*: W is a dense open subset of* $\mathrm{Hilb}^n(\mathbb{C})$.

Example 1.5. We classify all the ideals in the Hilbert scheme of 2 points on $\mathbb{C}^2$, denoted by $\mathrm{Hilb}^n(\mathbb{C}^2)$ in Section 4.

For more examples of monomial ideals see Section 1.3 and Example 1.19.

Theorem 1.6. [4] *The Hilbert scheme of n points on $\mathbb{C}^2$ is smooth of dimension $2n$.*

1.1. **Hilbert-Chow morphism.**

Definition 1.7. The n-th symmetric product of $\mathbb{C}^2$, denoted by $\mathrm{Sym}^n(\mathbb{C}^2)$, is defined to be $(\mathbb{C}^2)^n$, the moduli space of ordered n tuples of points, modulo the action of the n-th symmetric group S_n by permuting the points.

Definition 1.8. The Hilbert-Chow map, denoted by π, sending an ideal $I \in (\mathbb{C}^2)^{[n]}$ to the n-th symmetric product of the affine plane $\mathrm{Sym}^n(\mathbb{C}^2)$, is defined as follows.

For each $I \in (\mathbb{C}^2)^{[n]}$, the scheme $S = \mathrm{Spec}(\mathbb{C}[x,y]/I)$ has a finite number of points $p_1, ..., p_s$. We assign to each point p_i, $i = 1, ..., s$ a multiplicity m_i that is equal to the length of the local ring $O_{p_i,S} = (\mathbb{C}[x,y]/I)_{p_i}$ localized at each p_i. These multiplicities $m_1 + ... + m_s$ sum up to n.

Define the image of an ideal I under the Hilbert-Chow map as $\pi(I) = m_1 \cdot p_1 + ... + m_s \cdot p_s$, the unordered sum of points taking product with their corresponding multiplicities.

Theorem 1.9. [5] *The Hilbert-Chow morphism is projective.*

Theorem 1.10. [4] *The Hilbert scheme of points* $\mathrm{Hilb}^n(\mathbb{C}^2)$ *is a resolution of singularities of the symmetric product* $\mathrm{Sym}^n(\mathbb{C}^2)$.

1.2. **Punctual Hilbert scheme of points.** There is an important subscheme of the Hilbert scheme of points on the plane, being the **punctual** Hilbert scheme of points on the plane.

Definition 1.11. The punctual Hilbert scheme of points on the plane $\mathbb{C}^2$, denoted by $(\mathbb{C}^2)^{[n]}_0$, or $\mathrm{Hilb}^n(\mathbb{C}^2, 0)$, is the subscheme of $(\mathbb{C}^2)^{[n]}$ where each ideal I satisfy $\mathrm{Spec}(\mathbb{C}[x,y]/I) = n \cdot (0,0)$, and $(0,0)$ denotes the origin of the affine plane $\mathbb{C}^2$.

In other words, the punctual Hilbert scheme is a subscheme of the Hilbert scheme of points on the plane that only contains the ideals whose vanishing locus is supported at the point $(0,0)$. Unlike to the dense open subset W of ideals I in the Hilbert scheme of points where $\mathrm{Spec}(\mathbb{C}[x,y]/I)$ is a collection of n **distinct** points, the punctual Hilbert scheme $\mathrm{Hilb}^n(\mathbb{C}^2, 0)$

contains the information of the scheme structure of the Hilbert scheme as those n distinct points collide into one point.

Example 1.12. There is only one maximal ideal (x, y) of the polynomial ring $\mathbb{C}[x, y]$ that vanish at the point $(0, 0)$. So $\mathrm{Hilb}^1(\mathbb{C}^2, 0)$ is a point.

Example 1.13. The punctual Hilbert scheme of 2 points on the plane, $\mathrm{Hilb}^2(\mathbb{C}^2, 0)$ is the collection of ideals $\{(x^2, y - ax)|a \in \mathbb{C}\} \cup \{(x, y^2)\}$, which is isomorphic to $\mathbb{CP}^1$.

We cite a theorem due to Briançon.

Theorem 1.14. [3] *The punctual Hilbert scheme of points on the plane* $\mathrm{Hilb}^n(\mathbb{C}^2, 0)$ *is irreducible of dimension* $n - 1$.

1.3. **Affine charts covering** $\mathrm{Hilb}^n(\mathbb{C}^2)$**.** We first cite Haiman [5] for the classical construction of the coordinate charts on the Hilbert scheme of points $\mathrm{Hilb}^n(\mathbb{C}^2)$, giving it the structure of a smooth and irreducible manifold of dimension $2n$.

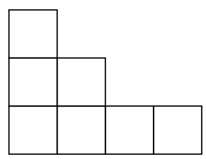
Definition 1.15. For a partition μ of n, we associate to it a Young diagram and a set of monomials in variables x and y.

Say the partition μ has s parts $\mu_1, ..., \mu_s$, and that $\mu_1 \geq \mu_2 \geq ... \geq \mu_s$. Put μ_1 boxes at the bottom row of this Young tableau, μ_2 boxes at the second row of this Young diagram, etc, and put μ_s boxes at the top of this Young diagram.

Next, to each box in the Young diagram we associate a monomial $x^i y^j$ such that (i, j) is the pair of coordinates of each box in the Young diagram. The coordinates (i, j) start at $(0, 0)$ at the South-west corner, and increase by 1 at a time in both the East and North direction.

Call this collection of monomials B_μ.

Example 1.16. For example, for the partition $\mu = (4, 2, 1)$ of 7, we have the following Young diagram.



And we have the following collection of monomials.

y^2			
y	xy		
1	x	x^2	x^3

And $B_{(4,2,1)} = \{1, x, x^2, x^3, y, xy, y^2\}$.

Definition 1.17. For each partition μ, we define U_μ to be the set of ideals $I \in \mathrm{Hilb}^n(\mathbb{C}^2)$ such that the quotient ring $\mathbb{C}[x, y]/I$ as a vector space over $\mathbb{C}$ has basis $B_\mu + I$.

Definition 1.18. Given a partition μ, the monomial ideal I_μ is the ideal generated by monomials not in B_μ. Notice that it's enough to take the monomials at the outside corners of the Young diagram of the partition μ.

Example 1.19. The monomial ideal $I_{(4,2,1)}$ is generated by (x^4, x^2y, xy^2, y^3).

Example 1.20. The ideal (x^4, x^2y, xy^2, y^3) is in $U_{(4,2,1)}$.

Example 1.21. The ideals (x^2, y) and $(x^2 - 4x - 3, y)$ are in $U_{(2)}$, because 1 and x form a basis of $\mathbb{C}[x, y]/I$ when I is either one of the ideals given above.

The ideals (x, y^2) and $(x - 3, y^2 - 6y - 7)$ are in $U_{(1,1)}$, because 1 and y form a basis of $\mathbb{C}[x, y]/I$ when I is either one of the ideals given above.

Both (2) and $(1, 1)$ are partitions of 2.

Example 1.22. All ideals I in the chart $U_{(n)}$, where (n) is the one-row partition of n boxes, are ideals $I \in \mathrm{Hilb}^n(\mathbb{C}^2)$ such that $1, x, .., x^{n-1}$ form a basis of $C[x, y]/I$.

By expanding x^n and y modulo I in this basis, all of the ideals $I \in U_{(n)}$ can be written as $I = (x^n - a_{n-1}x^{n-1} - ... - a_1x - a_0, y - b_{n-1}x^{n-1} - ... - b_1x - b_0)$. Therefore $U_{(n)}$ is an affine space with coordinates $a_0, ..., a_{n-1}, b_0, ..., b_{n-1} \in \mathbb{C}$.

Definition 1.23. Fix a partition μ and an ideal $I \in U_\mu \subset \mathrm{Hilb}^n(\mathbb{C}^2)$, and for some monomial x^ry^s such that (r, s) sits outside of the partition μ, the monomial x^ry^s modulo the ideal I can be uniquely written as a linear combination in the basis B_μ. Denote the coefficients in $\mathbb{C}$ of this linear combination C^{rs}_{hk} by the indexes r, s, h, k.

$$x^ry^s = \sum_{(h,k)\in\mu} C^{rs}_{hk}\; x^hy^k \bmod\; I.$$

Remark 1.24. The coefficients C^{rs}_{hk} depend on the ideal I, and one can check that they determine I completely.

We will consider C_{hk}^{rs} as algebraic functions on the open chart U_μ.

Theorem 1.25. [5] *For all the partitions μ of n, the sets U_μ are open affine subvarieties of $\mathrm{Hilb}^n(\mathbb{C}^2)$ that cover $\mathrm{Hilb}^n(\mathbb{C}^2)$. The affine coordindate ring O_{U_μ} of U_μ is generated by the variables C_{hk}^{rs} for $(h,k) \in \mu$ and all (r,s). The ring O_{U_μ} has extra polynomial relations among variables C_{hk}^{rs}.*

Example 1.26. For the partitions (3), $(1,1,1)$, $(2,1)$ of $n = 3$, we have the following three charts $U_{(3)}$, $U_{(1,1,1)}$ and $U_{(2,1)}$. The 6 coordinates on $U_{(3)}$, $U_{(1,1,1)}$ are as follows.

$O_{U_{(3)}} = \mathbb{C}[C_{00}^{30}, C_{10}^{30}, C_{20}^{30}, C_{00}^{01}, C_{10}^{01}, C_{20}^{01}]$, such that the monomials $x^3 = C_{00}^{30} \cdot 1 + C_{10}^{30} \cdot x + C_{20}^{30} \cdot x^2 \bmod I$, and $y = C_{00}^{01} \cdot 1 + C_{10}^{01} \cdot x + C_{20}^{01} \cdot +x^2 \bmod I$.

$O_{U_{(1,1,1)}} = \mathbb{C}[C_{00}^{03}, C_{01}^{03}, C_{02}^{03}, C_{00}^{10}, C_{01}^{10}, C_{02}^{10}]$, such that the monomials $x = C_{00}^{10} \cdot 1 + C_{01}^{10} \cdot y + C_{02}^{10} \cdot y^2$, and that $y^3 = C_{00}^{03} \cdot 1 + C_{01}^{03} \cdot y + C_{02}^{03} \cdot y^2$.

There are 9 coordinates on $U_{(2,1)}$ with 3 equations relating them as follows. For simplicity, we omit the notation of C_{hk}^{rs} and give them another name:

$x^2 = a \cdot 1 + b \cdot x + c \cdot y \bmod I$, $xy = d \cdot 1 + e \cdot x + f \cdot y \bmod I$, $y^2 = g \cdot 1 + h \cdot x + k \cdot y \bmod I$.

We can compute the relations among $a, ..., k$ modulo I as follows.

Modulo I: $x^2 y = a \cdot y + b \cdot xy + c \cdot y^2 = a \cdot y + b \cdot (d \cdot 1 + e \cdot x + f \cdot y) + c \cdot (g \cdot 1 + h \cdot x + k \cdot y) = (bd + cg) + (ch + be)x + (a + bf + ck)y$.

$x^2 y = d \cdot x + e \cdot x^2 + f \cdot xy = d \cdot x + e \cdot (a \cdot 1 + b \cdot x + c \cdot y) + f \cdot (d \cdot 1 + e \cdot x + f \cdot y) = (ae + df) + (d + be + ef)x + (f^2 + ce)y$.

$xy^2 = d \cdot y + e \cdot xy + f \cdot y^2 = d \cdot y + e \cdot (d \cdot 1 + e \cdot x + f \cdot y) + f \cdot (g \cdot 1 + h \cdot x + k \cdot y) =$
$(de + fg) + (e^2 + fh)x + (d + ef + fk)y.$

$xy^2 = g \cdot x + h \cdot x^2 + k \cdot xy = g \cdot x + h \cdot (a \cdot 1 + b \cdot x + c \cdot y) + k \cdot (d \cdot 1 + e \cdot x + f \cdot y) =$
$(ah + dk) + (g + bh + ek)x + (ch + kf)y.$

By comparing the coefficients at each basis element $1, x, y$, we obtain 3 independent relations:

$a + bf + ck = f^2 + ce$. $d + ef = ch$. $g + bh + ek = e^2 + fh$. One can check that the rest of the relations follow from these 3.

Therefore, the chart $U_{(2,1)}$ is 6 dimensional with 6 independent coordinates b, c, e, f, h, k, and the coordinates a, d, g can be written as degree 2 polynomials in b, c, e, f, h, k.

1.4. **Torus action on the Hilbert scheme.** The 2-dimensional torus $T^2 = \{(a, b) | a, b \in \mathbb{C}^*\}$ acts on the affine plane $\mathbb{C}^2$ with coordinates (x, y) by $(a, b) \cdot (x, y) = (ax, by)$. This induces an action on any polynomial $f(x, y)$ in the polynomial ring $\mathbb{C}[x, y]$ by $(a, b) \cdot f(x, y) = f(a^{-1}x, b^{-1}y)$. Therefore, we have a well-defined torus action on the Hilbert scheme of points by sending an ideal to another ideal in $\mathrm{Hilb}^n(\mathbb{C}^2)$.

Remark 1.27. The fixed points in $\mathrm{Hilb}^n(\mathbb{C}^2)$ under this torus action are all the monomial ideals I_μ, where μ is a partition of n.

2. Hilbert scheme of points on a non-reduced curve

We now define the Hilbert scheme of points on a non-reduced curve.

Let C denote a plane curve defined by a polynomial equation $f(x,y) = 0$ of two variables x, y. The polynomial $f(x,y)$ could be factored into several irreducible factors f_j with multiplicities k_j, i.e. $f(x,y) = \Pi_j (f_j(x,y))^{k_j}$. In other words, we consider all plane curves that could potentially be reducible (having several components) and non-reduced (having multiplicities ≥ 1) in this paper.

Definition 2.1. The Hilbert scheme of n points on the curve C, denoted by $C^{[n]}$, is the moduli space of ideals I in the polynomial ring of two variables $\mathbb{C}[x,y]$ satisfying the condition that the quotient ring $\mathbb{C}[x,y]/I$ has dimension n as a vector space over $\mathbb{C}$, and also that the polynomial $f(x,y)$ defining the curve C has to be contained in the ideal I as an element.

Remark 2.2. Note that the only difference between definitions of the Hilbert scheme of points on the plane and a curve is that we require the containment of the polynomial $f(x,y) \in I$. Note that equivalently we can define $C^{[n]}$ as the moduli space of ideals I in $O_C = \mathbb{C}[x,y]/(f(x,y))$ such that the quotient ring $\frac{O_C}{I}$ has dimension n.

The Hilbert scheme of points on a curve C, $\mathrm{Hilb}^n(C)$ is a subscheme of $\mathrm{Hilb}^n(\mathbb{C}^2)$ cut out by some equations in local charts U_μ which we write explicitly in the proof of Lemma 6.2.

Example 2.3. The ideal (x^2, y) is in $\mathrm{Hilb}^2(\mathbb{C}^2)$ but not in $\mathrm{Hilb}^2(C)$ when the curve C is $\{x = 0\}$, the y-axis, because the polynomial x is not contained in the ideal (x^2, y).

Example 2.4. When $n = 0$, the Hilbert scheme of 0 points $(\mathbb{C}^2)^{[0]}$ is a point, because the only ideal I of $\mathbb{C}[x,y]$ where $\mathbb{C}[x,y]/I$ has dimension 0 is the entire ring $\mathbb{C}[x,y]$. The Hilbert scheme of 0 points on the curve C, $C^{[0]}$, is also the entire ring $\mathbb{C}[x,y]$, because any polynomial $f(x,y)$ defining the curve C is contained in $\mathbb{C}[x,y]$.

Definition 2.5. Define C^{red} as the reduced subscheme of $\mathbb{C}^2$ corresponding to C. It is the set of points in $\mathbb{C}^2$ satisfying the equation $f(x,y) = 0$, or equivalently given by equation $\prod f_i = 0$.

Example 2.6. The Hilbert scheme of 1 point on a curve C is the the curve C itself as a scheme.

More generically, we have the following lemma.

Lemma 2.7. *If I is the ideal in* $\mathrm{Hilb}^n(C)$ *and p is a point(maximal ideal) in* $\mathrm{Spec}(\mathbb{C}[x,y]/I)$ *then $p \in C^{red}$.*

Proof. For a point p in $\mathrm{Spec}(\mathbb{C}[x,y]/I)$ such that $f(x,y) \in I$, we have that p must satisfy $f(p) = 0$ and be in the vanishing locus of $f(x,y)$, which implies that $p \in C^{red}$. $\square$

Definition 2.8. For a possibly non-reduced plane curve C, we restrict the the Hilbert-Chow map to $C^{[n]}$, which is a subset of $(\mathbb{C}^2)^{[n]}$, and the Hilbert-Chow map π sends an ideal I in $C^{[n]}$ to $\pi(I) = m_1 \cdot p_1 + ... + m_s \cdot p_s$, where $p_1, ..., p_s$ are points on the underlying reduced curve C^{red} of C.

Definition 2.9. Pick a point z on the curve C. The punctual Hilbert scheme of points on the curve C, denoted by $C_z^{[n]}$, is the subscheme of $C^{[n]}$ where each ideal I satisfy $\pi(I) = n \cdot z$, where π denotes the Hilbert-Chow map.

Theorem 2.10. [8] *If the curve C is smooth (and reduced) then* $\mathrm{Hilb}^n(C) = \mathrm{Sym}^n(C)$ *and smooth of dimension n.*

Theorem 2.11. [1][2] *If the curve C is reduced and irreducible then $Hilb^n(C)$ is irreducible.*

Theorem 2.12. [7] *If the curve C is reduced and reducible, the components of* $\mathrm{Hilb}^n(C)$ *are indexed by a composition μ of n.*

The parts of the permutation indicate the number of reduced points on each smooth subset of the curve component, and the components are given by the closure of the products of smooth part of the the curve components, such that the multiplicities of each curve component in the product is the corresponding part of the permutation.

In this thesis, we generalize this Theorem 2.12 to the case of non-reduced curves.

We now introduce the stratification we use throughout the paper.

Definition 2.13. We stratify $\mathrm{Hilb}^n(C)$ by the multiplicities of points in the image of the Hilbert-Chow map π. Let $m_1^1, \ldots, m_i^j, \ldots, m_{t_j}^j$ be a partition of n. Define each stratum $\Sigma_{m_1^1,\ldots,m_{t_j}^j}$ as the preimage under the Hilbert-Chow map of $\{m_1^1 x_1^1 + \ldots m_{t_j}^j x_{t_j}^j\}$, the collections of all possible configuration of s points on the smooth part of the curve C^{red} with multiplicities $m_1^1, \ldots, m_{t_j}^j$.

Define the stratum M as the set of ideals $I \in \mathrm{Hilb}^n(C)$ such that $\pi(I)$ contains singularities of C.

Remark 2.14. In the study of the easier problem of the irreducible components of $\mathrm{Hilb}^n(\{y^\beta = 0\})$, we simplify our notation and denote the multiplicity of points on the reduced smooth line $\{y = 0\}$ as $m_1, \dots, m_s$. The strata are therefore denoted as $\Sigma_{m_1,\dots,m_s}$. The stratum M is empty.

The strata defined above are not to be confused with the following Definition 2.15, which will also be useful in the later section. The distinction is that for any ideal in $\Sigma_{m_1,\dots,m_s}$, we allow the location of the collection of points to vary as long as the multiplicities of the points are unchanged, but all the images of ideals in $\Sigma_{m_1,\dots,m_s}(x_1, \dots, x_s)$ must have exactly the same locations and multiplicities $m_1(x_1, 0) + \dots m_s(x_s, 0)$ under the Hilbert-Chow map. We also have the following Lemma 2.16 computing the dimension of $\Sigma_{m_1,\dots,m_s}(x_1, \dots, x_s)$.

Definition 2.15. Fix s distinct points $(x_1, 0), \dots, (x_s, 0)$ on the line $\{y^\beta = 0\}$ with multiplicities $m_1, \dots, m_s$. We denote their preimage under the Hilbert-Chow map to be $\Sigma_{m_1,\dots,m_s}(x_1, \dots, x_s)$.

Lemma 2.16. *The set* $\Sigma_{m_1,\dots,m_s}(x_1, \dots, x_s)$ *is an irreducible variety, isomorphic to* $\mathrm{Hilb}^{m_1}(\mathbb{C}^2, 0)\times \dots \times \mathrm{Hilb}^{m_s}(\mathbb{C}^2, 0)$*, and has dimension* $n - s$*.*

Proof. The preimage of each point $m_i(x_i, 0)$ under the Hilbert-Chow map is isomorphic to $\mathrm{Hilb}^{m_i}(\mathbb{C}^2, 0)$. The ideals that vanish at all of the points $m_1(x_1, 0) + \dots + m_s(x_s, 0)$ must be in one-to-one correspondence with ideals in the product $\mathrm{Hilb}^{m_1}(\mathbb{C}^2,0)\times\cdots\times\mathrm{Hilb}^{m_s}(\mathbb{C}^2,0)$. And each punctual Hilbert scheme $\mathrm{Hilb}^{m_i}(\mathbb{C}^2,0)$ is irreducible and has dimension m_i-1 (Theorem 1.14), so their product is also irreducible, and has dimension $(m_1-1)+(m_2-1)+\cdots+(m_s-1)=m_1+m_2+\cdots+m_s-s=n-s$.

□

3. Nested Hilbert scheme of points

In this section, we define the nested Hilbert schemes of points on an affine plane and a plane curve, and state some known results about their properties.

3.1. Nested Hilbert schemes on the plane.

Definition 3.1. The nested Hilbert scheme of points on the plane $\mathbb{C}^2$, denoted by $\mathrm{Hilb}^{n,n+i}(\mathbb{C}^2)$, is the moduli space of pairs of ideals (I,J) inside the product space $(\mathbb{C}^2)^{[n+i]}\times(\mathbb{C}^2)^{[n]}$, satisfying the dimension conditions $\dim(\mathbb{C}[x,y]/I)=n+i$, $\dim(\mathbb{C}[x,y]/J)=n$, and that $I\subset J$.

Due to work of Cheah, we have the following classification of the smoothness of the nested Hilbert scheme of points.

Theorem 3.2. *The nested Hilbert scheme of points on $\mathbb{C}^2$, denoted by $\mathrm{Hilb}^{n,n+i}(\mathbb{C}^2)$, is smooth when $i=1$, and is singular when $i\geq 2$.*

The nested punctual Hilbert scheme of points on the plane is an interesting subscheme whose components and smoothness have been studied.

Definition 3.3. The nested punctual Hilbert scheme of points on the plane $\mathbb{C}^2$, denoted by $(\mathbb{C}^2)_0^{[n,n+i]}$, is the moduli space of pairs of ideals (I, J) in the product of punctual Hilbert schemes on the plane $(\mathbb{C}^2)_0^{[n+i]} \times (\mathbb{C}^2)_0^{[n]}$ that satisfies $I \subset J$.

Theorem 3.4. [9] *When $i = 1$, the nested punctual Hilbert scheme of points on the plane $(\mathbb{C}^2)_0^{[n,n+1]}$ is irreducible of dimension n.*

When $i = 2$, the nested punctual Hilbert scheme of points on the plane $(\mathbb{C}^2)_0^{[n,n+2]}$ is equidimensional, and has $\lfloor \frac{n+2}{2} \rfloor$ components of dimension $n+1$.

When $i \geq 3$, the nested punctual Hilbert scheme of points on the plane is reducible.

Remark 3.5. When $i \geq 3$, the components of the nested punctual Hilbert scheme of points on the plane is not known in general. In [9], the authors found the existence of two distinct components, therefore proving that $(\mathbb{C}^2)_0^{[n,n+i]}$ is reducible when $i \geq 3$.

3.2. **Nested Hilbert schemes on a curve.**

Definition 3.6. The nested Hilbert scheme of points on the curve C, denoted by $C^{[n,n+i]}$, is the moduli space of pairs of ideals (I, J) inside the product space $C^{[n+i]} \times C^{[n]}$, satisfying the dimension conditions $\dim(\mathbb{C}[x, y]/I) = n + i$, $\dim(\mathbb{C}[x, y]/J) = n$, and $f(x, y) \in I \subset J$.

Remark 3.7. The nested Hilbert scheme of 0 and 1 points $(\mathbb{C}^2)^{[0,1]}$ is isomorphic to the Hilbert scheme of 1 point on the plane $(\mathbb{C}^2)^{[1]}$, given by the collection of all ideals J of colength 1 in $\mathbb{C}[x, y]$.

The nested Hilbert scheme of 0 and 1 points $C^{[0,1]}$ is isomorphic to the Hilbert scheme of 1 point on the curve $C^{[1]}$, given by the collection of all ideals J of colength 1 in the quotient ring $\mathbb{C}[x,y]/(f(x,y))$.

Definition 3.8. Pick a point z on the curve C. The nested **punctual** Hilbert scheme of points on the curve C, denoted by $C_z^{[n,n+i]}$ is the moduli space of pairs of ideals (I,J) in the product of punctual Hilbert schemes on the curve $C_z^{[n+i]} \times C_z^{[n]}$ that satisfies $I \subset J$.

Now we define a subscheme of the nested Hilbert scheme that will be used in later chapters.

Definition 3.9. We define a subscheme $C^{[n,n+i]}[z]$ of the nested Hilbert scheme $C^{[n,n+i]}$ by requiring that each pair of ideals (I,J) satisfies $\pi(J/I) = i \cdot [z]$, where π denotes the Hilbert-Chow map, and $[z]$ denotes the class of a point z on underlying reduced curve C^{red}. Call this subscheme the special nested Hilbert scheme at point z.

4. Matrix representation of the Hilbert scheme of points

The Hilbert scheme of points $\mathrm{Hilb}^n(\mathbb{C}^2)$ has the following equivalent definition in representation theory.

Theorem 4.1. [8] *The Hilbert scheme of points* $\mathrm{Hilb}^n(\mathbb{C}^2)$ *is isomorphic to the following space*

$$\{(\mathcal{X}, \mathcal{Y}, i)\}/\mathrm{GL}_n(\mathbb{C})$$

such that (1) the entries $\mathcal{X}, \mathcal{Y}$ *are* n *by* n *matrices in* $\mathrm{End}(\mathbb{C}^n)$*, and that the commutator* $[\mathcal{X}, \mathcal{Y}] = 0$*, and that (2) the entry* i *is a vector in* $\mathbb{C}^n$ *whose orbit under the action of* $\mathcal{X}, \mathcal{Y}$ *is the entire space* $\mathbb{C}^n$*. The elements* g *in* $\mathrm{GL}_n(\mathbb{C})$ *act on the tuple* $(\mathcal{X}, \mathcal{Y}, i)$ *by* $g \cdot (\mathcal{X}, \mathcal{Y}, i) = (g\mathcal{X}g^{-1}, g\mathcal{Y}g^{-1}, gi)$*, changing basis for the matrices* $\mathcal{X}, \mathcal{Y}$*.*

Proof. For an given ideal $I \in \mathrm{Hilb}^n(\mathbb{C}^2)$, we want to construct a tuple $(\mathcal{X}, \mathcal{Y}, i)$, and vise versa.

Given an ideal I, consider $\mathbb{C}[x, y]/I$ to be isomorphic to $(\mathbb{C})^n$ as a vector space. Let $\mathcal{X}$ and $\mathcal{Y}$ be the endomorphism that correspond to multiplication by the variables x and y on the vector space $\mathbb{C}[x, y]/I$, and let i be the element $1 \in \mathbb{C}[x, y]/I$. The condition $[\mathcal{X}, \mathcal{Y}] = 0$ is true by construction that multiplication by the variables x and y is commutative. The orbit of the vector i also generate the entire $\mathbb{C}[x, y]/I$ because the any monomial in the ring $\mathbb{C}[x, y]$ can be generated by x and y multiplying with the element 1.

Given a tuple $(\mathcal{X}, \mathcal{Y}, i)$, we construct the ideal I as $I = \{f(x, y) | f(\mathcal{X}, \mathcal{Y}) \cdot i = 0\}$, the collection of polynomials that sends the vector i to 0 when plugging in the matrices $\mathcal{X}, \mathcal{Y}$. This gives a well-defined ideal.

Now, we show that $\mathbb{C}[x, y]/I$ has dimension n as a vector space over $\mathbb{C}$. Let ϕ be the ring morphism that sends a polynomial $f(x, y)$ to $f(\mathcal{X}, Y) \cdot i$. The kernel of ϕ is exactly I by construction, and the image of ϕ is $\mathbb{C}^n$ by the stabilizer condition. By the isomorphism theorem, $\mathbb{C}[x, y]/I \cong \mathbb{C}^n$. $\square$

Example 4.2. We classify all the tuple $(\mathcal{X}, \mathcal{Y}, i)$ in $\mathrm{Hilb}^2(\mathbb{C}^2)$, the Hilbert scheme of 2 points on the plane. Pick the vector i to be $i = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Because of the commutator relation $[\mathcal{X}, \mathcal{Y}] = 0$, both $\mathcal{X}, \mathcal{Y}$ can be made upper triangular simultaneously via conjugation.

Suppose that we denote $\mathcal{X} = \begin{bmatrix} x_1 & z \\ 0 & x_2 \end{bmatrix}$ and $\mathcal{Y} = \begin{bmatrix} y_1 & w \\ 0 & y_2 \end{bmatrix}$. Then $[\mathcal{X}, \mathcal{Y}] = 0$ if and only if $(x_1 - x_2)w = (y_1 - y_2)z$.

If $x_1 \neq x_2$, then this matrix $\mathcal{X}$ has two distinct eiganvalues, and we can diagonalize $\mathcal{X} = \begin{bmatrix} x_1' & 0 \\ 0 & x_2' \end{bmatrix}$ and $\mathcal{Y} = \begin{bmatrix} y_1' & 0 \\ 0 & y_2' \end{bmatrix}$. We have two distinct points (x_1', y_1') and (x_2', y_2') on the plane.

If $y_1 \neq y_2$, we have the same situation. We have two distinct points (x_1', y_1') and (x_2', y_2') on the plane.

If $x_1 = x_2$ and $y_1 = y_2$, then $\mathcal{X} = \begin{bmatrix} x_1 & z \\ 0 & x_1 \end{bmatrix}$ and $\mathcal{Y} = \begin{bmatrix} y_1 & w \\ 0 & y_1 \end{bmatrix}$. We have one point on the plane (x_1, y_1), but the Hilbert scheme of points contains the information of the blowup $\mathbb{P}^1 = \{[z : w] | z, w \in \mathbb{C}, (z, w) \neq (0, 0)\}$ at this point (x_1, y_1).

Remark 4.3. The punctual Hilbert schemes on the plane, $\mathrm{Hilb}^n(\mathbb{C}^2, 0)$ are given by pairs of matrices $(\mathcal{X}, \mathcal{Y})$ such that $\mathcal{X}, \mathcal{Y}$ are both nilpotent. The vector i is given in the same way as the Hilbert scheme of points $\mathrm{Hilb}^n(\mathbb{C}^2)$ in the proof of Theorem 4.1 above. As an example, $\mathrm{Hilb}^2(\mathbb{C}^2, 0) = \mathbb{P}^1$ correspond to the blowup at the point (x_1, y_1) in Example 4.2 above.

Remark 4.4. Given a pair of ideals (I, J) in the nested Hilbert scheme of points on $\mathbb{C}^2$, denoted by $\mathrm{Hilb}^{n,n+i}(\mathbb{C}^2)$ such that $I \subset J \subset \mathbb{C}[x, y]$, they correspond to block triangular matrices of size $n + i$ by $n + i$. i.e. they look like $\begin{bmatrix} T_1 & M \\ 0 & T_2 \end{bmatrix}$ such that T_1 and T_2 are square blocks, and T_1 has size n by n, and T_2 has size i by i. The block M can take any values.

To see this, we look at two operators $\mathcal{X}, \mathcal{Y}$ on the $n + i$ dimensional vector space $\mathbb{C}^{n+i} \cong \mathbb{C}[x, y]/I$, such that both $\mathcal{X}$ and $\mathcal{Y}$ preserve the vector subspace $\mathbb{C}[x, y]/J$ of dimension n inside $\mathbb{C}[x, y]/I$. These matrices are exactly block triangular matrices as above.

Chapter 3: Irreducible components of the Hilbert scheme of points on non-reduced curves

5. Main Theorems

In this section, we list a few theorems and prove them later.

We first define the notation we use.

Definition 5.1. Define the tuple with j parts separated by bars, which looks like the following: $m_1^1, ..., m_1^{a_1}|m_2^1, ..., m_2^{a_2}|...|m_j^1, ..., m_j^{a_j}$.

The numbers m_j^i can be any positive integer, or 0. The numbers $a_1, ..., a_j$ are the number of tuples within each bar, which are required to be finite. In the case that $m_j^i = 0$ for all $i = 1, ..., a_i$, we just write one 0 between the bars, i.e. $|0|$. In the case that $m_j^i = 0$ but

$m_j^{i'} \neq 0$ for some i', we omit writing m_j^i and only write the non-zero numbers between the bars.

This tuple satisfies the following combinatorial rules.

1. The tuple $m_1^1 + ... + m_1^{a_1} + m_2^1 + ... + m_2^{a_2} + ... + m_j^1 + ... + m_j^{a_j} = n$ sums up to n.

2. The numbers within each bar are unordered (swapping the numbers in each bar still indexes the same component), but the bars are ordered (swapping two different bars gives another component).

3. Each number m_j^i satisifies $m_j^i \leq \beta_j$

Theorem 5.2. [10] *Let C denote a plane curve with components C_j, where each C_j has multiplicity β_j. The irreducible components of $C^{[n]}$ are indexed by the tuple of numbers with j parts separated by bars as defined above, i.e. $m_1^1, ..., m_1^{a_1} | m_2^1, ..., m_2^{a_2} | ... | m_j^1, ..., m_j^{a_j}$.*

Theorem 5.3. [10]

Stratify the Hilbert scheme $C^{[n]}$ using the Hilbert-Chow map as follows. Consider a collection of points S on the curve C. For all the points that sit on the curve component C_j, index them as $p_j^1, ..., p_j^i, ..., p_j^{a_j}$, where a_j is the total number of points on the component C_j. Let the numbers m_j^j be the multiplicity of the i-th point on the curve C_j. Fix such multiplicities of the points but allow the location of the points to move along the curve.

Define the strata as $K_{m_1^1, ..., m_1^{a_1} | m_2^1, ..., m_2^{a_2} | ... | m_j^1, ..., m_j^{a_j}} := \pi^{-1}(m_1^1 \cdot p_1^1 + ... + m_1^{a_1} \cdot p_1^{a_1} + m_2^1 \cdot p_2^1 + ... + m_2^{a_2} \cdot p_2^{a_2} + ... + m_j^1 \cdot p_j^1, ..., m_j^{a_j} \cdot p_j^{a_j})$, where π is the Hilbert-Chow map.

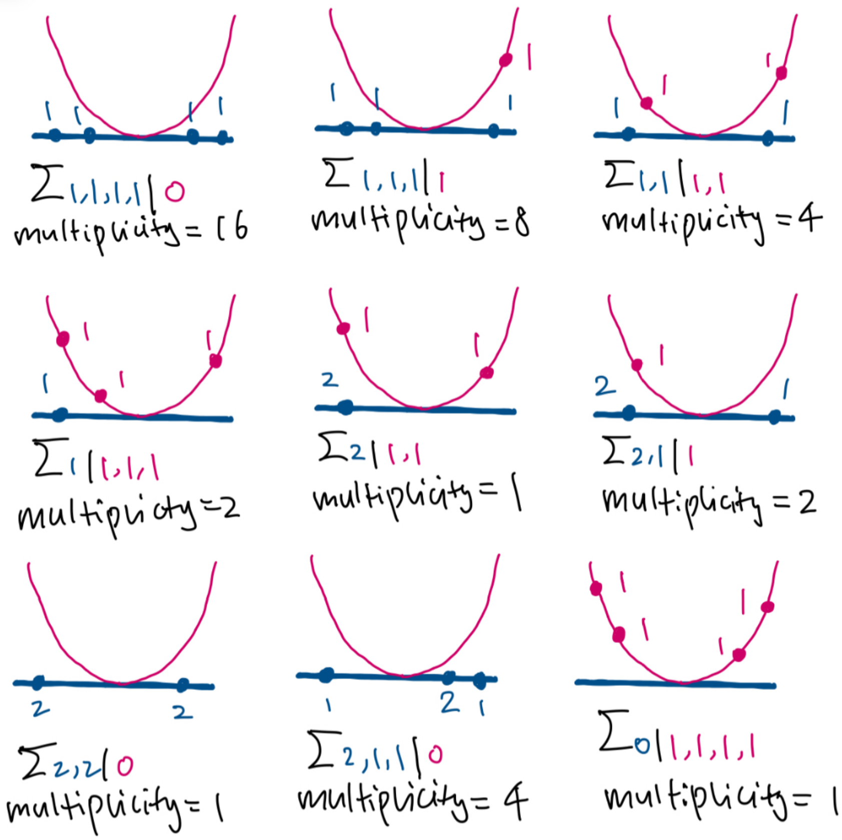
In other words, define the stratum, which is indexed by such fixed multiplicities, to be the collection of ideals I whose image under the Hilbert-Chow map $\pi(I)$ are exactly this collection of points S with the desired multiplicities m_j^i.

The irreducible components $\Sigma_{m_1^1,...,m_1^{a_1}|m_2^1,...,m_2^{a_2}|...|m_j^1,...,m_j^{a_j}} := \pi^{-1}(m_1^1 \cdot p_1^1 + ... + m_1^{a_1} \cdot p_1^{a_1} + m_2^1 \cdot p_2^1 + ... + m_2^{a_2} \cdot p_2^{a_2} + ... + m_j^1 \cdot p_j^1, ..., m_j^{a_j} \cdot p_j^{a_j})$ *of* $C^{[n]}$ *are the closure of the strata* $K_{m_1^1,...,m_1^{a_1}|m_2^1,...,m_2^{a_2}|...|m_j^1,...,m_j^{a_j}}$ *satisfying that* $m_j^i \leq \beta_j$.

Theorem 5.4. [10] *All the irreducible components of* $C^{[n]}$ *have dimension* n.

To make the notations easier to understand, we give an example as an application of the theorems above.

Example 5.5. We classify all the 9 components of the Hilbert scheme of 4 points on the curve $y^2(y - x^2) = 0$ and compute their multiplicities.



6. Easier problem: studying the components of $\mathrm{Hilb}^n(\{y^\beta = 0\})$.

6.1. **Charts and the lower bound.**

We first define and state a theorem about the affine charts of $\mathrm{Hilb}^n(\mathbb{C}^2)$ following Haiman's paper [5].

Theorem 6.1. [5] *The collection of all U_μ, where μ is a partition of n, forms an open cover of $\mathrm{Hilb}^n(\mathbb{C}^2)$. Each chart U_μ is open, irreducible, smooth, and affine of dimension $2n$.*

Now, we put a lower bound on the dimension of the irreducible components of $\mathrm{Hilb}^n(C)$.

Lemma 6.2. *The irreducible components of* $\mathrm{Hilb}^n(C)$ *have dimension at least* n*.*

Proof. Fix a chart U_μ which has non-empty intersection with $\mathrm{Hilb}^n(C)$. Let f denote the defining polynomial of curve C. Because $f \in I$, we can write f as a linear combination of the monomial basis B_μ mod I, and the coefficients in this linear combination should all be 0. There are n basis elements in B_μ, so there are n conditions imposed on the $2n$ coordinates of U_μ, making the dimension of the irreducible components of $\mathrm{Hilb}^n(C) \cap U_\mu$ at least n. And the irreducible components of $\mathrm{Hilb}^n(C)$ should have the same dimension as the irreducible components of $\mathrm{Hilb}^n(C) \cap U_\mu$, because intersecting an irreducible component with an open set U_μ does not change its dimension. So each irreducible component of $\mathrm{Hilb}^n(C)$ has dimension at least n. □

6.2. **The special affine chart where everything happens:** $U_{(n)}$**.**

From now on through the end of this section, we focus on studying the easier problem, the irreducible components of $\mathrm{Hilb}^n(\{y^\beta = 0\})$.

The only chart relevant to our computation is $U_{(n)}$, the chart indexed by the partition (n). In the notation of the Young diagram, this partition corresponds to a row of n boxes, and the corresponding $B_{(n)} = \{1, x, x^2, \ldots, x^{n-1}\}$.

1	x	x^2	$\cdots$	x^{n-1}

Now, we reveal to the reader why we only need this one specific chart $U_{(n)}$. It is given as the corollary of the following Lemma.

Lemma 6.3. *(a) The space* $\Sigma_{m_1,\ldots,m_s}(x_1, \ldots, x_s) - U_{(n)}$ *has dimension strictly less than* $n - s$.

(b)The complement of $U_{(n)}$ *in each stratum* $\Sigma_{m_1,\ldots,m_s}$ *has dimension strictly less than* n. *Equivalently,* $dim(\Sigma_{m_1,\ldots,m_s} - U_{(n)}) < n$.

Proof. (a) The chart $U_{(n)}$ is open in each $\Sigma_{m_1,\ldots,m_s}(x_1, \ldots, x_s)$. The intersection $U_{(n)} \cap \Sigma_{m_1,\ldots,m_s}(x_1, \ldots, x_s)$ is nonempty because the ideal $I = ((x - x_1)^{m_1} \cdot \cdots \cdot (x - x_s)^{m_s}, y)$ is in the intersection. Therefore, $\Sigma_{m_1,\ldots,m_s}(x_1, \ldots, x_s) - U_{(n)}$ is a closed and proper subset of $\Sigma_{m_1,\ldots,m_s}(x_1, \ldots, x_s)$, and $\Sigma_{m_1,\ldots,m_s}(x_1, \ldots, x_s)$ is irreducible by Lemma 2.16. So $\Sigma_{m_1,\ldots,m_s}(x_1, \ldots, x_s) - U_{(n)}$ has dimension strictly less than $n - s$.

(b) Varying each x_i adds 1 degree of freedom, and varying all the $x_1, \ldots, x_s$ adds s degrees of freedom in total. By part (a), $\Sigma_{m_1,\ldots,m_s}(x_1, \ldots, x_s) - U_{(n)}$ has dimension strictly less than $n - s$, and therefore $\Sigma_{m_1,\ldots,m_s} - U_{(n)}$ has dimension strictly less than $n - s + s = n$. $\square$

Corollary 6.4. *All irreducible components of* $\mathrm{Hilb}^n(\{y^\beta = 0\})$ *intersect the chart* $U_{(n)}$.

Proof. We want to show that $\mathrm{Hilb}^n(\{y^\beta = 0\}) - U_{(n)}$ does not fully contain any irreducible components of $\mathrm{Hilb}^n(\{y^\beta = 0\})$.

Suppose for contradiction that $\mathrm{Hilb}^n(\{y^\beta = 0\}) - U_{(n)}$ contains some irreducible component A of $\mathrm{Hilb}^n(\{y^\beta = 0\})$, then the dimension of $\mathrm{Hilb}^n(\{y^\beta = 0\}) - U_{(n)}$ must be greater than or equal to n by Lemma 6.2.

Now we consider the union of complements of $U_{(n)}$ in every stratum $U := \cup_{m_1,\ldots,m_s}(\Sigma_{m_1,\ldots,m_s} - U_{(n)})$. Because the dimension of each $\Sigma_{m_1,\ldots,m_s} - U_{(n)}$ is strictly less than n by Lemma 6.3, the union U also has dimension strictly less than n. But $\mathrm{Hilb}^n(\{y^\beta = 0\}) - U_{(n)}$ is contained in U, so we have a contradiction.

□

Now we describe the coordinate system on $U_{(n)}$. This also follows from the discussion in Haiman's paper [5].

Write x^n and y as a linear combination of the basis $B_{(n)} = \{1, x, x^2, \ldots, x^{n-1}\}$ of $\mathbb{C}[x, y]/I$, and denote the coefficients as follows:

$$x^n = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} \mod I$$

$$y = b_0 + b_1 x + \ldots + b_{n-1}x^{n-1} \mod I.$$

Define polynomials $a(x) = x^n - a_{n-1}x^{n-1} - \ldots - a_1 x - a_0$ and $b(x) = b_{n-1}x^{n-1} + \ldots + b_1 x + b_0$, then any ideal I in $U_{(n)}$ is generated as $I = (a(x), y - b(x))$. Throughout this paper, we will use $a(x)$ and $b(x)$ to denote the polynomials defined above. As a remark, $a(x)$ has degree exactly n, but $b(x)$ can have any degree less than or equal to $n-1$.

6.3. **Stratification inside $U_{(n)}$ and classifying the ideals in each stratum.**

The stratification $\Sigma_{m_1,\ldots,m_s}$ of $\mathrm{Hilb}^n(\{y^\beta = 0\})$ induces a stratification

$$C_{m_1,\ldots,m_s} := \Sigma_{m_1,\ldots,m_s} \cap U_{(n)}$$

on $\mathrm{Hilb}^n(\{y^\beta = 0\}) \cap U_{(n)}$. And we know from Corollary 6.4 that all irreducible components of $\mathrm{Hilb}^n(\{y^\beta = 0\})$ intersect $U_{(n)}$. Later in Lemma 6.11 we prove that the strata $C_{m_1,\ldots,m_s}$ are irreducible. And as we show in this section, because of the nice coordinate system on $U_{(n)}$, we are able to write out specifically the ideals in each stratum of $C_{m_1,\ldots,m_s}$ as in Proposition 6.9.

Proposition 6.5. *The condition that y^β is contained in $I = (a(x), y - b(x)) \in U_{(n)}$ is equivalent to the condition that the polynomial $a(x)$ divides $b^\beta(x)$.*

To prove one direction of the proposition that $y^\beta \in I$ implies $a(x)|b^\beta(x)$, we first prove the following lemma:

Lemma 6.6. *Let $f(x)$ be a polynomial in $I = (a(x), y - b(x))$ which does not depend on the variable y. Then $f(x)$ is divisible by $a(x)$.*

Proof. Perform polynomial long division of $f(x)$ by $a(x)$ and we get that $f(x) = a(x) \cdot q(x) + r(x)$ for some polynomial $r(x)$ and $q(x)$. Suppose for the purpose of contradiction that $r(x)$ is not 0, and denote the degree of $r(x)$ by r, $r < n$. We can explicitly write out $r(x) = l_0 + l_1 x + \cdots + l_r x^r$ for some $l_i \in \mathbb{C}$ and $l_r \neq 0$.

Because both $f(x)$ and $a(x)$ are in I, we have that $r(x) = f(x) - a(x) \cdot q(x)$ must also be in I, so $r(x) = 0$ mod I, and therefore $r(x)$ is a nonzero linear combination of $1, x, \ldots, x^r$. But we also know that $B_\mu = \{1, x, x^2, \ldots, x^{n-1}\}$ is a basis of $\mathbb{C}[x, y]/I$, contradiction. So $r(x)$ must be 0 and $a(x)$ divides $f(x)$. $\square$

Now, we are ready to prove Proposition 6.5.

Proof of Proposition 6.5. Assume $y^\beta \in I$. Because $y - b(x)$ is a generator of I, $y = b(x)$ mod I, which implies that $b(x)^\beta \in I$ and by Lemma 6.6, $b^\beta(x)$ is divisible by $a(x)$.

Suppose $a(x)|b^\beta(x)$, then because $a(x) \in I$, we have $b^\beta(x) \in I$. Again because $y = b(x)$ mod I, and $b^\beta(x) \in I$, we must have $y^\beta \in I$.

$\square$

We factor $a(x)$ and $b(x)$ into linear factors in terms of its roots, and study the possible multiplicities of roots that $a(x)$ and $b(x)$ can have.

Lemma 6.7. *Let $x_1 \dots x_s$ denote the distinct roots of $a(x)$, and m_i the multiplicity of each root x_i, then we can explicitly make the factorization:*

$$a(x) = (x - x_1)^{m_1} \cdot (x - x_2)^{m_2} \cdot \ldots \cdot (x - x_s)^{m_s} \text{ where } \sum_i m_i = n.$$

Then condition $a(x)|b^\beta(x)$ splits into 2 cases depending on the multiplicities m_i.

(1) *General case: If $\left\lceil \frac{m_1}{\beta} \right\rceil + \cdots + \left\lceil \frac{m_s}{\beta} \right\rceil \leq n-1$,*

then $a(x)|b^\beta(x)$ if and only if

$$b(x) = (x - x_1)^{\left\lceil \frac{m_1}{\beta} \right\rceil} \cdot (x - x_2)^{\left\lceil \frac{m_2}{\beta} \right\rceil} \cdot \ldots \cdot (x - x_s)^{\left\lceil \frac{m_s}{\beta} \right\rceil} \cdot \alpha(x)$$

for some polynomial $\alpha(x)$ of degree at most $t = n - 1 - \sum \left\lceil \frac{m_i}{\beta} \right\rceil$ $(**)$.

(2) *Special case: If $\left\lceil \frac{m_1}{\beta} \right\rceil + \cdots + \left\lceil \frac{m_s}{\beta} \right\rceil > n-1$,*

then $a(x)|b^\beta(x)$ if and only if $b(x) = 0$.

Proof. Denote the multiplicity of $(x - x_i)$ in $b(x)$ by q_i.

Because $a(x)$ divides $b^\beta(x)$, each factor $(x - x_i)$ in $b^\beta(x)$ must have multiplicity higher than m_i, or $b(x)$ has to be 0. That is to say, the multiplicity q_i must satisfy $\beta \cdot q_i \geq m_i$.

Because q_i are integers, the smallest possible value of q_i is $\left\lceil \frac{m_i}{\beta} \right\rceil$. So $b(x)$ must have the factor $(x-x_1)^{\lceil \frac{m_1}{\beta} \rceil} \cdot (x-x_2)^{\lceil \frac{m_2}{\beta} \rceil} \ldots \cdot (x-x_s)^{\lceil \frac{m_s}{\beta} \rceil}$, or is equal to 0.

Recall that $b(x)$ is constructed to have degree at most $n-1$, so if $\left\lceil \frac{m_1}{\beta} \right\rceil + \cdots + \left\lceil \frac{m_s}{\beta} \right\rceil \leq n-1$, then $\alpha(x)$ is some polynomial of degree at most $n-1-(\left\lceil \frac{m_1}{\beta} \right\rceil + \cdots + \left\lceil \frac{m_s}{\beta} \right\rceil)$. The special case happens when $\left\lceil \frac{m_1}{\beta} \right\rceil + \cdots + \left\lceil \frac{m_s}{\beta} \right\rceil > n-1$, then $b(x)$ has to be 0.

$\square$

Remark 6.8. The special case $\left\lceil \frac{m_1}{\beta} \right\rceil + \cdots + \left\lceil \frac{m_s}{\beta} \right\rceil > n-1$ happens exactly when either (a) $\beta \geq 2$, all $m_i = 1$ and $s = n$, or (b) $\beta = 1$ and m_i can be any positive integers.

Proof. Recall that we assumed $m_i \geq 1$. (a) When $\beta \geq 2$, we have $1 \leq \left\lceil \frac{m_i}{\beta} \right\rceil \leq \lceil m_i \rceil = m_i$. We also have $\sum_{i=1}^{s} m_i = n$ as the total degree of $a(x)$. So $\left\lceil \frac{m_1}{\beta} \right\rceil + \cdots + \left\lceil \frac{m_s}{\beta} \right\rceil \leq n$, and $\left\lceil \frac{m_1}{\beta} \right\rceil + \cdots + \left\lceil \frac{m_s}{\beta} \right\rceil = n$ only if $\left\lceil \frac{m_i}{\beta} \right\rceil = m_i$. Finding the possible values of m_i so that $\left\lceil \frac{m_i}{\beta} \right\rceil = m_i$ is equivalent to finding m_i such that $\frac{m_i}{\beta} \leq m_i < \frac{m_i}{\beta} + 1$. For any $\beta \geq 2$, $\frac{m_i}{\beta} \leq m_i$ is always true, and $m_i < \frac{m_i}{\beta} + 1$ is equivalent to $m_i < \frac{\beta}{\beta - 1}$.

Observe that $1 < \frac{\beta}{\beta-1} \leq 2$ for all $\beta \geq 2$, so the only possible value that m_i can take is 1.

(b) When $\beta = 1$, $\left\lceil \frac{m_1}{\beta} \right\rceil + \cdots + \left\lceil \frac{m_s}{\beta} \right\rceil = \lceil m_1 \rceil + \cdots + \lceil m_s \rceil = m_1 + \cdots + m_s = n$. So $\left\lceil \frac{m_1}{\beta} \right\rceil + \cdots + \left\lceil \frac{m_s}{\beta} \right\rceil \geq n-1$ is always satisfied for arbitrary m_i. $\square$

Now, we conclude our results from above and explicitly write out the ideals in each stratum $C_{m_1,\ldots,m_s}$.

Proposition 6.9. *Each stratum $C_{m_1,\dots,m_s}$ contains exactly the ideals I of the form $I = (a(x), y - b(x))$, where $a(x) = (x-x_1)^{m_1} \cdot (x-x_2)^{m_2} \dots . (x-x_s)^{m_s}$, and $b(x) = (x-x_1)^{\lceil \frac{m_1}{\beta} \rceil} \cdot (x-x_2)^{\lceil \frac{m_2}{\beta} \rceil} \dots \cdot (x-x_s)^{\lceil \frac{m_s}{\beta} \rceil} \cdot \alpha(x)$ when $\left\lceil \frac{m_1}{\beta} \right\rceil + \dots + \left\lceil \frac{m_s}{\beta} \right\rceil \leq n-1$ (general case); $b(x) = 0$ when $\left\lceil \frac{m_1}{\beta} \right\rceil + \dots + \left\lceil \frac{m_s}{\beta} \right\rceil > n-1$ (special case).*

Proof. The ideals I in each stratum $\Sigma_{m_1,\dots,m_s} \cap U_{(n)}$ have the form $(a(x), y - b(x))$, where $a(x) = (x-x_1)^{m_1} \dots (x-x_s)^{m_s}$. By Proposition 6.5, finding the ideals I in the intersection of $\mathrm{Hilb}^n(\{y^\beta = 0\})$ with $U_{(n)}$ is equivalent to imposing the condition that $b(x) | a^\beta(x)$ for ideals $I = (a(x), y - b(x)) \in U_{(n)}$. By Lemma 6.7, $b(x)|a^\beta(x)$ is equivalent to the condition that $b(x) = (x - x_1)^{\lceil \frac{m_1}{\beta} \rceil} \cdot (x - x_2)^{\lceil \frac{m_2}{\beta} \rceil} \cdot \dots \cdot (x - x_s)^{\lceil \frac{m_s}{\beta} \rceil} \cdot \alpha(x)$ or $b(x) = 0$, depending on the multiplicities m_i. □

6.4. **Counting dimension and finding irreducible components.**

For each stratum $C_{m_1,\dots,m_s}$, we compute its dimension by counting the degrees of freedom given by polynomials $a(x)$, $b(x)$, and $\alpha(x)$.

Lemma 6.10. *If $\left\lceil \frac{m_1}{\beta} \right\rceil + \dots + \left\lceil \frac{m_s}{\beta} \right\rceil \leq n-1$, then $\dim(C_{m_1,\dots,m_s}) = t + s + 1$, where t is the maximum degree that $\alpha(x)$ can have as in (**).*

For $\beta \geq 2$, there is exactly one stratum $C_{1,\ldots,1}$ satisfying special condition for Lemma 6.7 (2), and this stratum $C_{1,\ldots,1}$ has dimension n. For $\beta = 1$, all strata $C_{m_1,\ldots,m_s}$ have dimension s.

Proof. Let's first look at the *general case*: $\left\lceil \frac{m_1}{\beta} \right\rceil + \cdots + \left\lceil \frac{m_s}{\beta} \right\rceil \leq n-1$.

Each distinct root x_i of $a(x)$ gives a degree of freedom, so $a(x)$ has s degrees of freedom. Denote the maximum degree of $\alpha(x)$ by t, then we can explicitly write out $\alpha(x)$ as $\alpha(x) = \alpha_0 + \ldots + \alpha_t x^t$ for coefficients $\alpha_i \in \mathbb{C}$, and each α_i gives a degree of freedom. So $\alpha(x)$ gives $t+1$ degrees of freedom. Note that $b(x)$ is completely determined by $a(x)$ and $\alpha(x)$ so it does not contribute to any degree of freedom. The dimension of stratum $C_{m_1,\ldots,m_s}$ therefore is $t+1+s$.

Special case: $\left\lceil \frac{m_1}{\beta} \right\rceil + \cdots + \left\lceil \frac{m_s}{\beta} \right\rceil > n-1$. As discussed in Lemma 6.8, in the case of $\beta \geq 2$, we need to have $s = n$ and all the $m_i = 1$, which gives us the stratum $C_{m_1=1,\ldots,m_n=1}$.

Recall from Proposition 6.9 that in the special case, $b(x) = 0$, so only $a(x)$ contributes to degrees of freedom, which are given by the n variables $x_1, \ldots, x_n$. Therefore the dimension of $C_{1,\ldots,1}$ is n.

We can also have $\beta = 1$. In this case, no matter which stratum we look at, the ideals $I = (a(x), y - b(x))$ in it must satisfy $b(x) = 0$ by Proposition 6.9. All the degrees of freedom are given by $a(x)$, so the dimension of $C_{m_1,\ldots,m_s}$ is s. $\square$

Lemma 6.11. *All the strata $C_{m_1,\dots,m_s}$ are irreducible.*

Proof. By Proposition 6.9 and Lemma 6.10, a stratum $C_{m_1,\dots,m_s}$ is isomorphic to

$(\mathbb{C}^{t+s+1} - \{(x_1,\dots,x_s,\alpha_0,\dots,\alpha_t)|x_i = x_j \text{ for some } 1 \le i,j \le s\})/Stab(\{m_1,\dots,m_s\})$. We remove the set of all ideals where some $x_i = x_j$ because the roots should be distinct by construction. We mod out by the action of the stabilizer of the multiplicities to eliminate the over-counting of swapping x_i and x_j when $m_i = m_j$. An affine space with a closed subvariety removed is irreducible, and the quotient by action of a finite group again keeps the space irreducible. □

Now we can conclude that all the closures $\overline{C_{m_1,\dots,m_s}}$ are irreducible.

We conclude all the previous results and classify all the irreducible components of $\mathrm{Hilb}^n(\{y^\beta = 0\})$ as the following theorem.

Theorem 6.12. *All the irreducible components of $\mathrm{Hilb}^n(\{y^\beta = 0\})$ have dimension n and are closures of the strata $C_{m_1,\dots,m_s}$ where $1 \le m_i \le \beta$ for all i. Given $m_1,\dots,m_s \le \beta$, the generic point of this component $\overline{C_{m_1,\dots,m_s}}$ consists of s distinct points on $\{y = 0\}$ with multiplicities $m_1,\dots,m_s$.*

Proof. We remind the reader that in our previous Lemma 6.2, we show that the dimensions of the irreducible components are at least n. Now we need to find the strata whose dimensions are n or more, and their closures are candidates of the irreducible components.

We first discuss the special case.

When $\beta = 1$, a stratum $C_{m_1,\dots,m_s}$ has dimension n only when $s = n$, so m_i must all be 1. Therefore, $\mathrm{Hilb}^n(\{y = 0\})$ has only one irreducible component $\overline{C_{1,\dots,1}}$ of dimension n. This recovers the previous results of [1] and [2] that $\mathrm{Hilb}^n(\{y = 0\})$ is irreducible of dimension n.

When $\beta \geq 2$, the stratum $C_{1,\dots,1}$ has dimension n and it is not a subset of the closure of any other strata. So its closure is an irreducible component of $\mathrm{Hilb}^n(\{y^\beta = 0\})$.

Now we look at the general case and want to find the conditions on t and s such that $\dim(C_{m_1,\dots,m_s}) = t + 1 + s \geq n$. Recall that we use t to denote the maximum degree that $\alpha(x)$ can have, and s the number of distinct roots of $a(x)$.

Here are all the equations relating the dimension of a stratum $C_{m_1,\dots,m_s}$, t, s and n:

$$\dim(C_{m_1,\dots,m_s}) = t + 1 + s. \text{ (Lemma 6.10)}$$

Because $b(x)$ has degree at most $n - 1$, we have

$$t + \left\lceil \frac{m_1}{\beta} \right\rceil + \ldots + \left\lceil \frac{m_s}{\beta} \right\rceil = n - 1.$$

Because $m_i \geq 1$, we must have $\left\lceil \frac{m_i}{\beta} \right\rceil \geq 1$. So

$$\left\lceil \frac{m_1}{\beta} \right\rceil + \left\lceil \frac{m_2}{\beta} \right\rceil + \ldots + \left\lceil \frac{m_s}{\beta} \right\rceil \geq s.$$

This implies

$$\dim(C_{m_1,\ldots,m_s}) = 1 + s + \left(n - 1 - \left(\left\lceil\frac{m_1}{\beta}\right\rceil + \ldots + \left\lceil\frac{m_s}{\beta}\right\rceil\right)\right)$$

$$= n + s - \left(\left\lceil\frac{m_1}{\beta}\right\rceil + \ldots + \left\lceil\frac{m_s}{\beta}\right\rceil\right) \leq n. \quad (***) \tag{1}$$

In particular, this implies that the closure of a stratum $\overline{C_{m_1,\ldots,m_s}}$ of dimension n are exactly the irreducible components, because there are no other higher dimensional strata.

The equality $(***)$ of equation (1) holds when $\left\lceil\frac{m_1}{\beta}\right\rceil + \ldots + \left\lceil\frac{m_s}{\beta}\right\rceil = s$, which is equivalent to $\left\lceil\frac{m_i}{\beta}\right\rceil = 1$, and this happens precisely when $1 \leq m_i \leq \beta$ for all i.

In conclusion, all the strata have dimension $\leq n$, and the closure of a stratum $C_{m_1,\ldots,m_s}$ is an irreducible component if and only if $1 \leq m_i \leq \beta$.

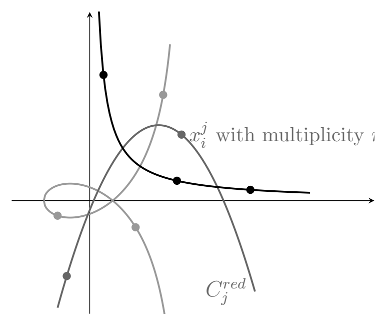
□

7. Generalization: studying components of $\mathrm{Hilb}^n(C)$

Now, we generalize our results to $\mathrm{Hilb}^n(C)$, where C is any non-reduced plane curve.

We remind the reader that we have defined the stratification in Definition 2.13. To briefly restate the definition, the stratum $\Sigma_{m_1^1,\ldots,m_i^j,\ldots,m_{s_r}^r}$ is the set of all ideals I such that the of multiplicity is exactly m_i^j for a point x_i^j on the smooth part of underlying reduced curve C_j^{red} of C_j.

The stratum M is the collection of all ideals whose images contain some singularities of C.



Similar to the approach to the easier problem, we also want to show that the closure of each stratum $\overline{\Sigma_{m_1^1,\ldots,m_i^j,\ldots,m_{s_r}^r}}$ is irreducible. In order to do this, we embed each stratum $\Sigma_{m_1^1,\ldots,m_i^j,\ldots,m_{s_r}^r}$ into another irreducible stratum of a bigger space, and show that they have the same closure.

Definition 7.1. Consider the space $\mathrm{Hilb}^n(\mathbb{C}^2, C^{red,sm})$, the Hilbert scheme of points on $\mathbb{C}^2$ supported on the smooth subset of the reduced curve $C^{red,sm}$. We similarly stratify it using the preimage of the Hilbert-Chow map. Denote each stratum by $L_{m_1^1,\ldots,m_i^j,\ldots,m_{s_r}^r}$. Define $L_{m_1^1,\ldots,m_i^j,\ldots,m_{s_r}^r} := \pi^{-1}(\sum_{i,j} m_i^j x_i^j)$ where each point x_i^j of multiplicity m_i^j is on the open subset $C_j^{red,sm}$ of the smooth points of the reduced curve C_j^{red}.

It's an easy check that $\Sigma_{m_1^1,\dots,m_i^j,\dots,m_{s_r}^r} \subset L_{m_1^1,\dots,m_i^j,\dots,m_{s_r}^r}$ directly from their definitions. However, the condition of being an ideal in the stratum $\Sigma_{m_1^1,\dots,m_i^j,\dots,m_{s_r}^r}$, namely containing the equation $f(x,y)$ of the curve C, is stronger than the condition of being an ideal in the stratum $L_{m_1^1,\dots,m_i^j,\dots,m_{s_r}^r}$, namely vanishing at points on the curve C. So the set containment $\Sigma_{m_1^1,\dots,m_i^j,\dots,m_{s_r}^r} \subset L_{m_1^1,\dots,m_i^j,\dots,m_{s_r}^r}$ is proper.

Now we continue with the irreducibility argument.

Lemma 7.2. *Each stratum $L_{m_1^1,\dots,m_i^j,\dots,m_{s_r}^r}$ is irreducible of dimension n.*

Proof. Each stratum $L_{m_1^1,\dots,m_i^j,\dots,m_{s_r}^r}$ is isomorphic to

$$\Big(\prod_{1\le i\le s_j, 1\le j\le r} \mathrm{Hilb}^{m_i^j}(\mathbb{C}^2,0)\times\Big(\prod_{j=1\dots r}(C_j^{sm})^{s_j}-\{(x_1^1,\dots,x_{s_r}^r)|x_a^j=x_b^j \text{ for some } 1\le a,b\le s_j\}\Big)\Big)$$
$$\Big/\prod_{j=1\dots r} Stab(m_1^j,\dots,m_{s_j}^j).$$

The preimage of a point with multiplicity m_i^j under the Hilbert-Chow map is isomorphic to $\mathrm{Hilb}^{m_i^j}(\mathbb{C}^2,0)$. Because the points in the image of $L_{m_1^1,\dots,m_i^j,\dots,m_{s_r}^r}$ can land anywhere on $C_j^{red,sm}$, as long as they don't collide, we multiply by the factor $\big(\prod_{j=1\dots r}(C_j^{sm})^{s_j} - \{(x_1^1,\dots,x_{s_r}^r)|x_a^j = x_b^j \text{ for some } 1 \le a,b \le s_j\}\big)$. We also need to mod out by the stabilizer of the multiplicities to account for the over-counting when $m_i^j = m_{i'}^j$, and x_i^j and $x_{i'}^j$ are interchanged.

The curve C_j^{red} is irreducible, and its points of singularities form a closed set, so the smooth part of the curve $C_j^{red,sm}$ is irreducible. Removing a closed set from the product of all $C_j^{red,sm}$ leaves the product irreducible. By Theorem 1.14, the punctual Hilbert scheme is irreducible, so the product of all these factors is also irreducible. This irreducible product taking the quotient by a finite group is again irreducible.

Now, we want to show that $L_{m_1^1,\ldots,m_i^j,\ldots,m_{s_r}^r}$ has dimension n. By Theorem 1.14, the product of the punctual Hilbert schemes has dimension $m_1^1 - 1 + \ldots + m_{s_r}^r - 1 = m_1^1 + \ldots + m_{s_r}^r - r$. The product of r planar curves with some closed subsets removed gives r more degrees of freedom. Quotienting out by the action of a finite group does not change the dimension. So $L_{m_1^1,\ldots,m_i^j,\ldots,m_{s_r}^r}$ has dimension $n - r + r = n$.

$\square$

Lemma 7.3. *When $1 \leq m_i^j \leq \beta$ for all i, j, the closures of the two types of strata are the same: $\overline{\Sigma_{m_1^1,\ldots,m_i^j,\ldots,m_{s_r}^r}} = \overline{L_{m_1^1,\ldots,m_i^j,\ldots,m_{s_r}^r}}$. Therefore $\overline{\Sigma_{m_1^1,\ldots,m_i^j,\ldots,m_{s_r}^r}}$ are irreducible and have dimension n.*

Proof. For the proof we use the following fact: If Y is a closed subset of an irreducible finite-dimensional topological space X, and if $\dim Y = \dim X$, then $Y = X$. Here, we want $Y = \overline{\Sigma_{m_1^1,\ldots,m_i^j,\ldots,m_{s_r}^r}}$ and $X = \overline{L_{m_1^1,\ldots,m_i^j,\ldots,m_{s_r}^r}}$, and we want to show that they satisfy the conditions on X and Y.

Assume $m_i^j \leq \beta$. We know from Lemma 7.2 that the closures of the strata $L_{m_1^1,\ldots,m_i^j,\ldots,m_{s_r}^r}$ are closed, irreducible, and have dimension n.

A collection of points moving along the smooth part of C are locally the same as the points moving along $\{y_j^\beta = 0\}$, because the local ring at any point on $\{y_j^\beta = 0\}$ is isomorphic to the local ring at any smooth point on $\{f_j^{\beta_j}(x,y)=0\}$ by locally changing coordinates between y and $f_j(x,y)$. Therefore the dimension of the stratum $\Sigma_{m_1^1,\dots,m_i^j,\dots,m_{s_r}^r}$ is the sum of the dimension of each stratum $\Sigma_{m_1^j,\dots,m_{s_j}^j}$ of $\mathrm{Hilb}^n(\{y^\beta=0\})$. When $1\le m_i^j\le\beta_j$, the dimension of each stratum $\Sigma_{m_1^j,\dots,m_{s_j}^j}$ is $m_1^j+\ldots+m_{s_j}^j$ by Theorem 6.12. So the dimension of the stratum $\Sigma_{m_1^1,\dots,m_i^j,\dots,m_{s_r}^r}$ is $m_1^1+\cdots+m_i^j+\ldots+m_{s_r}^r=n$.

So the closures of strata $\overline{\Sigma_{m_1^1,\dots,m_i^j,\dots,m_{s_r}^r}}$ also have dimension n. So the closures of the two types of strata $Y=\overline{\Sigma_{m_1^1,\dots,m_i^j,\dots,m_{s_r}^r}}$ and $X=\overline{L_{m_1^1,\dots,m_i^j,\dots,m_{s_r}^r}}$ satisfy the conditions of being X and Y, and therefore they are equal. So $\overline{\Sigma_{m_1^1,\dots,m_i^j,\dots,m_{s_r}^r}}$ is irreducible of dimension n.

□

Finally, we have the theorem that classifies the irreducible components of $\mathrm{Hilb}^n(C)$. The reader might notice that we have not discussed if the stratum M is irreducible or not. As it turns out in the proof of the following theorem, $\overline{M}$ is never an irreducible component because its dimension is too small.

Theorem 7.4. *The irreducible components of* $\mathrm{Hilb}^n(C)$ *are the closures of the strata* $\overline{\Sigma_{m_1^1,\dots,m_i^j,\dots,m_{s_r}^r}}$ *where* $1\le m_i^j\le\beta_j$.

Proof. We first remind the reader of Lemma 6.2 that we proved: the irreducible components all have dimension n or more.

We look at the two types of strata separately. Case 1: The points in the image are all contained in C_j^{sm}. The strata are $\Sigma_{m_1^1,\dots,m_i^j,\dots,m_{s_r}^r}$.

When $m_i^j > \beta_j$, such strata have dimension strictly less than n by Theorem 6.12 and a similar argument of locally changing coordinates between y and $f_j(x,y)$ as in the proof of Lemma 7.3, and their closures are not the irreducible components. When $1 \le m_i^j \le \beta_j$, we have argued in Lemma 7.3 that such strata are irreducible and have dimension n. So their closures must be the irreducible components.

Case 2: Some of the points in the image $\pi(I)$ are singularities or the intersection points of the curves C. We have one stratum M of such ideals.

The preimage of s points with multiplicities m_i^j on C_j is a subset of the product of the punctual Hilbert scheme $\mathrm{Hilb}^{m_1^1}(\mathbb{C}^2, 0) \times \cdots \times \mathrm{Hilb}^{m_{s_r}^r}(\mathbb{C}^2, 0)$. By Theorem 1.14, the product of the punctual Hilbert schemes has dimension $m_1^1 - 1 + \ldots + m_{s_r}^r - 1 = m_1^1 + \ldots + m_{s_r}^r - r$. When we allow the points on the smooth part to move, the points at the singularities or the intersections do not move. So the dimension added by moving the points are strictly less than r. So the preimage of a collection of points containing some singularity has dimension strictly less than n. M is contained in such preimage so has dimension strictly less than n and cannot be irreducible components by Lemma 6.2.

□

8. COMPUTATION OF MULTIPLICITIES OF COMPONENTS

We intersect each stratum Σ_μ with $U_{(n)}$ and the intersection is an open dense subset of each stratum. We study the multiplicities of points in this open dense subset of the intersection.

8.1. **The stratum $\Sigma_{(n)}$ of $\mathrm{Hilb}^n(\{y^\beta = 0\})$.**

We begin by studying the stratum $\Sigma_{(n)}$ corresponding to the 1-part partition (n) of n. Because we want the closure of this stratum to be an irreducible component, we assume $n \leq \beta$ in this subsection.

We want to study the stratum $\Sigma_{(n)}$ inside the chart $U_{(n)}$, so we pick an ideal $I \in U_{(n)}$, and I is necessarily generated as $I = (a(x), y - b(x))$, where $a(x) = x^n + a_{n-1}x^{n-1} + \dots a_0$ and $b(x) = b_{n-1}x^{n-1} + \dots + b_0$. We also want that $I \in \Sigma_n$, so $a(x) = (x - x_1)^n$ for some $x_1 \in \mathbb{C}$ and $b^\beta(x) = 0 \mod a(x)$. Those are all the conditions we have to consider to compute the coordinate ring of $\Sigma_{(n)}$.

Remark 8.1. We notice that $b^\beta(x) = 0 \mod (x - x_1)^n$ is equivalent to $b^\beta(x + x_1) = 0 \mod x^n$. So we expand the polynomial $b^\beta(x + x_1)$ and set each polynomial coefficient in variables $b_0, \dots, b_{n-1}$ of the term x^i to be 0 for $0 \leq i \leq n-1$.

We define the coefficients of $b(x + x_1)$ first before taking its β-th power.

Definition 8.2. We define the coefficients B_i of $b(x + x_1) = b_0 + b_1(x + x_1) + b_2(x + x_1)^2 + \dots + b_{n-1}(x + x_1)^{n-1} := B_0 + B_1 x + B_2 x^2 + \dots + B_{n-1}x^{n-1}$.

Each B_i is a polynomial of variables $b_i, \dots, b_{n-1}$ and x_i: For $0 \leq i \leq n-1$,

$$B_i := b_i + \binom{i+1}{i} b_{i+1} x_1 + \binom{i+2}{i} b_{i+2} x_1^2 + \dots + \binom{n-1}{i} b_{n-1} x_1^{n-i-1}.$$

Now we take the β-th power of $b(x + x_1)$ and find the coefficients of x^i in terms of B_i.

Definition 8.3. Define E_i as a function of B_i's. Set

$$E_i := \sum_{k_0,\ldots,k_{n-1}} \binom{\beta}{k_0, \ldots, k_{n-1}} B_0^{k_0} \ldots B_{n-1}^{k_{n-1}},$$

where $k_0, \ldots, k_{n-1}$ satisfy $0 \cdot k_0 + \cdots + (n-1) \cdot k_{n-1} = i$ and $k_0 + \cdots + k_{n-1} = \beta$.

Lemma 8.4. *The function $b^\beta(x + x_1)$ can be written as $b^\beta(x + x_1) = \sum_{i=0,\ldots,n-1} E_i x^i$.*

Proof. By the multinomial theorem,

$$\begin{aligned} b^\beta(x + x_1) &= (B_0 + B_1 x + B_2 x^2 + \cdots + B_{n-1} x^{n-1})^\beta \\ &= \sum_{k_0+\cdots+k_{n-1}=\beta} \binom{\beta}{k_0, \ldots, k_{n-1}} B_0^{k_0} (B_1 x)^{k_1} \ldots (B_{n-1} x^{n-1})^{k_{n-1}} \\ &= \sum_{k_0+\cdots+k_{n-1}=\beta} \binom{\beta}{k_0, \ldots, k_{n-1}} (B_0^{k_0} B_1^{k_1} \ldots B_{n-1}^{k_{n-1}}) x^{(0\cdot k_0 + 1\cdot k_1 + \ldots (n-1)\cdot k_{n-1})}. \end{aligned}$$

We denote the power of x as i, and therefore for each term x^i, we have $i = 0 \cdot k_0 + 1 \cdot k_1 + \ldots (n-1) \cdot k_{n-1}$ and $k_0 + \cdots + k_{n-1} = \beta$. The coefficient of x^i is E_i. $\square$

Corollary 8.5. *From the computation above, $b^\beta(x + x_1)$ mod $x^n = 0$ if and only if $E_i = 0$ for $0 \le i \le n - 1$.*

Corollary 8.6. *Denote the coordinate ring of the component $\Sigma_{(n)}$ as $R_{(n)}$, then $R_{(n)}$ is isomorphic to*

$$R_{(n)} := \mathbb{C}[b_0, \ldots, b_{n-1}, x_1]/(E_0, \ldots, E_{n-1}).$$

We also make the following observations about the functions E_i.

Lemma 8.7. *For $0 \leq i \leq n-1$, the function E_i is divisible by $B_0^{\beta-i}$ but not divisible by $B_0^{\beta-i+1}$*

Proof. Let's look at a term $\binom{\beta}{k_0,\ldots,k_{n-1}} B_0^{k_0} \ldots B_{n-1}^{k_{n-1}}$ in E_i. In order that $k_0, \ldots, k_{n-1}$ satisfy $0 \cdot k_0 + \cdots + (n-1) \cdot k_{n-1} = i$ and $k_0 + \cdots + k_{n-1} = \beta$ for $i \leq n-1 < \beta$, we must have $k_0 \geq \beta - i$.

When $i = 0$, we have $E_0 = B_0^{\beta}$ and therefore E_0 is not divisible by $B_0^{\beta+1}$.

For every i such that $1 \leq i \leq n-1$, the term $\binom{\beta}{k_0,k_1} B_0^{k_0} B_1^{k_1}$ where $k_0 + k_1 = \beta$ and $k_1 = i$ is in E_i. This term is not divisible by $B_0^{\beta-i+1}$. Each term of E_i is a positive constant times a monomial of $B_0, \ldots, B_{n-1}$ and each monomial is different, so if one monomial term is not divisible by $B_0^{\beta-i+1}$, the entire function E_i is not divisible by $B_0^{\beta-i+1}$. □

Corollary 8.8. *Given $n \leq \beta$, E_i must be divisible by B_0. So $B_0 = 0$ implies $E_i = 0$ for all i.*

Proof. When $n \leq \beta$, we have that $i \leq n-1 \leq \beta - 1$. So $\beta - i \geq 1$, and we must have that E_i is divisible by B_0. □

Definition 8.9. Define $\Sigma_{(n)}^{red}$ as the reduced variety corresponding to $\Sigma_{(n)}$.

Lemma 8.10. *The reduced variety $\Sigma_{(n)}^{red}$ is cut out in $U_{(n)}$ by the equations*

$$B_0 = b(x_1) = b_0 + b_1 x_1 + b_2 x_1^2 + \cdots + b_{n-1} x_1^{n-1} = 0.$$

and $a(x) = (x - x_1)^n = 0$. In other words,

$$\Sigma_{(n)}^{red} := \{(b_0, \ldots, b_{n-1}, x_1) | b(x_1) = 0, (x - x_1)^n = 0\}.$$

Proof. The equation $E_0 = B_0^\beta = 0$ holds true if and only of $B_0 = 0$. Additionally, because B_0 is a factor of all the E_i's, $B_0 = 0$ implies that all the E_i's are equal to 0.

□

To compute the multiplicity of the component $\Sigma_{(n)}$, we localize at a generic point p of V. Note that p is the prime ideal corresponding to $\Sigma_{(n)}^{red}$.

Corollary 8.11. *The local ring $(R_{(n)})_p$ is isomorphic to*

$$\mathbb{C}[b_0, \ldots, b_{n-1}, x_1]_p / (B_0^\beta, B_0^{\beta-1}, B_0^{\beta-2}, \ldots B_0^{\beta-n+1})_p,$$

and it has dimension $\beta - n + 1$.

We conclude that the multiplicity of $\Sigma_{(n)}$ is $\beta - n + 1$.

Proof. By Lemma 8.7, we can factor E_i into $E_i = B_0^{\beta-i}F_i$ where F_i is a factor not divisible by B_0. Therefore, we can rewrite the local ring $(R_{(n)})_p$ as

$$\mathbb{C}[b_0, \ldots, b_{n-1}, x_1]_p/(B_0^{\beta}, \ldots, B_0^{\beta-i}, \ldots, B_0^{\beta-n+1})_p.$$

Recall that $B_0 = b_0 + b_1x_1 + b_2x_1^2 + \cdots + b_{n-1}x_1^{n-1}$. So the local ring $(R_{(n)})_p$ has basis $1, B_0, \ldots, B_0^{\beta-n}$, and the dimension of the local ring $(R_{(n)})_p$ is $\beta - n + 1$. □

8.2. **The multiplicity of a general stratum $\Sigma_{m_1,\ldots,m_s}$.**

Now we consider all the strata $\Sigma_{m_1,\ldots,m_s}$ whose closures are the irreducible components of $\mathrm{Hilb}^n(\{y^\beta = 0\})$, so $\sum_i m_i = n$ and $m_s \leq \beta$. Recall that every irreducible component intersects $U_{(n)}$ and the intersection is open and dense in $\Sigma_{m_1,\ldots,m_s}$, so we study the multiplicities of the points of each irreducible components in the chart $U_{(n)}$.

The strategy of this section is to show that the coordinate ring of the stratum $\Sigma_{m_1,\ldots,m_s}$ in $U_{(n)}$ is isomorphic as a $\mathbb{C}$-vector space to a tensor product of the coordinate rings of the strata $\Sigma_{(m_i)}$ of $\mathrm{Hilb}^{m_i}(\{y^\beta = 0\})$, where $1 \leq i \leq n$, which we computed in the last section. Therefore the multiplicity of $\Sigma_{m_1,\ldots,m_s}$ is the product of the multiplicities of Σ_{m_i}.

We define the constructions to show this isomorphism.

We first recall our coordinate systems. Denote an ideal in $\Sigma_{m_1,\ldots,m_s} \cap U_{(n)}$ as $((x - x_1)^{m_1} \ldots (x - x_s)^{m_s}, y - b(x))$ for some $x_1, \ldots, x_s \in \mathbb{C}$ satisfying $x_i \neq x_j$ for all $1 \leq i, j \leq s$.

Denote an ideal in $\Sigma_{(m_i)} \cap U_{(m_i)} \subset \mathrm{Hilb}^{m_i}(\{y^\beta = 0\})$, where (m_i) is the one-part partition of the number m_i, as $((x-x_i)^{m_i}, y - b^i(x))$ where $b^i(x) = b^i_0 + b^i_1 x + \cdots + b^i_{m_i-1}x^{m_i-1}$, and the coordinates are $b^i_0, \ldots, b^i_{m_i-1}, x_i$.

Definition 8.12. (1)Work in $U_{(n)} \subset \mathrm{Hilb}^n(\{y^\beta = 0\})$.

Define $r(x) \coloneqq b^\beta(x) \bmod a(x)$, the remainder of polynomial long division.

Denote $r(x) \coloneqq r_0 + r_1 x + \ldots r^{n-1}x$.

(2)Work in $U_{(m_i)} \subset \mathrm{Hilb}^{m_i}(\{y^\beta = 0\})$.

Define $r^i(x) \coloneqq (b^i(x))^\beta \bmod (x-x_i)^{m_i}$.

Denote $r^i(x) = r^i_0 + r^i_1 x + \ldots r^i_{m_i-1}x^{m_i-1}$.

We remark that $b_0, \ldots, b_{n-1}$ and $b^i_0, \ldots, b^i_{m_i-1}$ are formal variables as generators of coordinate rings of the corresponding strata. But $r_0, \ldots, r_{n-1}$ are polynomials of variables $b_0, \ldots, b_{n-1}, x_1, \ldots, x_s$, and $r^i_0, \ldots, r^i_{m_i-1}$ are polynomials of variables $b^i_0, \ldots, b^i_{m_i-1}, x_i$.

Lemma 8.13. *Immediately following the definitions, the coordinate ring R_μ of the scheme $\Sigma_{m_1,\ldots,m_s}$ in $U_{(n)}$ of $\mathrm{Hilb}^n(\{y^\beta = 0\})$ is given by*

$$R_\mu = \mathbb{C}[b_0, \ldots, b_{n-1}, x_1, \ldots, x_s]/(r_0, \ldots, r_{n-1}).$$

Recall from the last subsection that the coordinate ring $R_{(m_i)}$ of the scheme $\Sigma_{(m_i)}$ in $U_{(m_i)}$ of $\mathrm{Hilb}^{m_i}(\{y^\beta = 0\})$ is given by

$$R_{(m_i)} = \mathbb{C}[b_0^i, \dots, b_{m_i-1}^i, x_i]/(r_0^i, \dots, r_{m_i-1}^i).$$

Let p be a generic point of R_μ, such that $x_i \neq x_j$ for all $i \neq j$. We localize R_μ at p and compute the $\mathbb{C}$-dimension of the ring $(R_\mu)_p$ as the multiplicity of the stratum $\Sigma_{m_1,\dots,m_s}$.

We now state the proposition below, that allows us to compute the dimension of $(R_\mu)_p$ by the dimension of $(R_{(m_i)})_p$. Recall that the dimension of $(R_{(m_i)})_p$ is $\beta - m_i + 1$ as computed in Corollary 8.11.

Proposition 8.14. *The following two local algebras are isomorphic:* $(R_\mu)_p \cong \bigotimes_{i=1,\dots,s} (R_{(m_i)})_p$. *Specifically,*

$$\frac{\mathbb{C}[x_0, \dots, x_s, b_0, \dots, b_{n-1}]_p}{(r_0, \dots, r_{n-1})_p} \cong \bigotimes_{i=1,\dots,s} \frac{\mathbb{C}[x_i, b_0^i, \dots, b_{m_i-1}^i]_p}{(r_0^i, \dots, r_{m_i-1}^i)_p}.$$

Definition 8.15. We define the ring homomorphism ϕ as follows.

$$\phi : \bigotimes_{i=1,\dots,s} \frac{\mathbb{C}[x_i, b_0^i, \dots, b_{m_i-1}^i]_p}{(r_0^i, \dots, r_{m_i-1}^i)_p} \to \frac{\mathbb{C}[x_0, \dots, x_s, b_0, \dots, b_{n-1}]_p}{(r_0, \dots, r_{n-1})_p}.$$

Define ϕ to be identity on the variables x_i, $\phi(x_i) = x_i$. And define $\phi(b_j^i)$ to be the coefficient of the term x^j in the polynomial long division $b(x)$ mod $(x - x_i)^{m_i}$.

We define the image of ϕ on the generators, and extend the map ϕ to the entire ring by declaring that ϕ is a ring isomorphism. i.e. for any element f in the ring $\bigotimes_{i=1,\dots,s}(R_{(m_i)})_p$, define

$$\phi(f(x_1,\dots,x_s,b_0^1,\dots,b_j^i,\dots,b_{m_s-1}^s)) = f(x_1,\dots,x_s,\phi(b_0^1),\dots,\phi(b_j^i),\dots,\phi(b_{m_s-1}^s)))$$

Remark 8.16. We can add a variable x and extend ϕ to a ring homomorphism from $\bigotimes_{i=1,\dots,s}(R_{(m_i)})_p[x]$ to $(R_\mu)_p[x]$ by sending $\phi(x) = x$. This homomorphism satisfies that $\phi(b^i(x)) = b(x) \bmod (x-x_i)^{m_i}$ by construction. Due to the construction that ϕ is a ring homomorphism, we also have $\phi((b^i(x)^\beta) = \phi((b^i(x))^\beta = b^\beta(x) \bmod (x-x_i)^{m_i}$.

Lemma 8.17. *We have that* $\phi(r^i(x)) = r(x)$ *mod* $(x-x_i)^{m_i}$.

Proof. Because $(x-x_i)^{m_i}$ is a factor of $a(x)$, we have that

$\phi(r^i(x)) = r^i(x,x_1,\dots,x_s,\phi(b_j^i)) = (b^i(x,x_1,\dots,x_s,\phi(b_j^i)))^\beta \bmod (x-x_i)^{m_i} = (b(x))^\beta \bmod (x-x_i)^{m_i} = r(x) \bmod (x-x_i)^{m_i}$. □

Proof of Proposition 8.14. We first want to show that the rings $\bigotimes\limits_{i=1,\dots,s} \mathbb{C}[x_i,b_0^i,\dots,b_{m_i-1}^i]_p$ and $\mathbb{C}[x_0,\dots,x_s,b_0,\dots,b_{n-1}]_p$ are isomorphic by proving that ϕ is a linear change of variables between the ring generators.

We write ϕ as a change-of-basis matrix with polynomial entries of variables x_i that changes basis from the ring generators $b_0^1,\dots,b_j^i,\dots,b_{m_s-1}^s$ to the ring generators $b_0,\dots,b_{n-1}$. By construction, each b^i_j is sent to a linear combination of $b_0, \ldots, b_{n-1}$ with polynomial coefficients of variables x_i.

Now we show the other direction that each b_j can be written as a linear combination of b^i_j with coefficients being rational functions in $x_1, \ldots, x_s$, such that the denominator of each rational function is a product of $x_i - x_l$ where $i \neq l$.

First, we want to show that the determinant of the matrix ϕ is a product of $x_i - x_l$ where $i \neq l$. Consider the x_i's not as variables, but fixed numbers in $\mathbb{C}$, and assume all $x_i \neq x_l$ for all $i, l \in \{1, \ldots, s\}$. Similarly, consider $b_0, \ldots, b_{n-1}$ not as variables but fixed numbers in $\mathbb{C}$. So $b(x), b^i(x), r(x), r^i(x)$, $(x - x_1)^{m_1} \ldots (x - x_s)^{m_s}$ and $(x - x_i)^{m_i}$ are all polynomials with complex coefficients in one variable x.

We have that $\frac{\mathbb{C}[x]}{((x-x_1)^{m_1} \ldots (x-x_s)^{m_s})} \cong \bigotimes_i \frac{\mathbb{C}[x]}{((x-x_i)^{m_i})}$ by the Chinese remainder theorem. We can vary the values of $b_0, \ldots, b_{n-1}$ such that $b(x)$ gives an arbitrary element of $\mathbb{C}[x]$. By the Chinese remainder theorem, ϕ is an isomorphism for all $x_i \neq x_l$.

Now we let x_i be variables and consider $\det(\phi)$ as a non-constant function in $\mathbb{C}[x_1, \ldots, x_s]$. For any set of values of $b_0, \ldots, b_{n-1}$ and $x_1, \ldots, x_s$ such that $i \neq l$, we have that ϕ is an invertible linear map and $\det(\phi) \neq 0$ for any $x_i \neq x_l$. Then $\det(\phi)$ does not vanish on the quasi-affine space defined by equations $\{x_i \neq x_l | \ \forall i \neq l\}$. So $\det(\phi)$ can only have factors that are $x_i - x_l$ for some $i \neq l$.

When we assume that $x_i \neq x_l$, the determinant $\det(\phi)$ never vanishes, and ϕ is invertible. By construction, ϕ^{-1} maps each b_j to a linear combination of b^i_j with coefficients being rational functions in $x_1, \ldots, x_s$, such that the denominator of each rational function is $\det(\phi)$, a product of $x_i - x_l$ where $i \neq l$.

In the localized ring, $x_i \neq x_l$, so we have the ring isomorphism

$$\bigotimes_{i=1,\ldots,s} \mathbb{C}[x_i, b_0^i, \ldots, b_{m_i-1}^i]_p \cong \mathbb{C}[x_0, \ldots, x_s, b_0, \ldots, b_{n-1}]_p.$$

By Lemma 8.17, $\phi(r_j^i)$ is the coefficient of the term x^j in the polynomial long division $r(x)$ mod $(x-x_i)^{m_i}$. So by the property of ϕ we just showed, ϕ linearly changes variables between the collection r_j^i and the collection r_j. So the ideals in the corresponding polynomial rings satisfy $(r_0, \ldots, r_{m_i-1})_p \cong (r_0^1, \ldots, r_j^i, \ldots, r_{m_s-1}^s)_p$.

The isomorphism of the quotient rings follows from the isomorphisms of the polynomial rings and the ideals we're quotienting them out with. $\square$

Corollary 8.18. *The multiplicity at a generic point p of $\Sigma_{m_1,\ldots,m_s}$ is $\prod_i(\beta - m_i + 1)$.*

Proof. The multiplicity of $\Sigma_{m_1,\ldots,m_s}$ is the $\mathbb{C}$-dimension of the algebra $\frac{\mathbb{C}[x_0,\ldots,x_s,b_0,\ldots,b_{n-1}]_p}{(r_0,\ldots,r_{n-1})_p}$. By the isomorphism in the theorem above, its dimension is equal to the product of the dimensions of $\frac{\mathbb{C}[x_i,b_0^i,\ldots,b_{m_i-1}^i]_p}{(r_0^i,\ldots,r_{m_i-1}^i)_p}$, which are $\beta - m_i + 1$ as we computed in the last section. $\square$

8.3. **Generalization: multiplicity of the irreducible components of any curve C.**

Theorem 8.19. *Recall that the irreducible components of* $\mathrm{Hilb}^n(C)$ *are indexed by partitions $m_1^1, \ldots, m_i^j, \ldots, m_{s_r}^r$ and each $m_i^j \leq \beta_j$ for all i and j. The multiplicity of the component indexed by $m_1^1, \ldots, m_i^j, \ldots, m_{s_r}^r$ is $\Pi_{i,j}(\beta - m_i^j + 1)$.*

Proof. Locally, the component $\Sigma_{m_1^1,\dots,m_{s_1}^1,\dots,m_1^r,\dots,m_{s_r}^r}$ have the same multiplicity as the product of the components: $\Sigma_{m_1^1,\dots,m_{s_1}^1} \times \cdots \times \Sigma_{m_1^r,\dots,m_{s_r}^r}$, where each $\Sigma_{m_1^j,\dots,m_{s_r}^j}$ is a stratum of $\mathrm{Hilb}^n(\{y^{\beta_j} = 0\})$. Therefore the multiplicity of the component indexed by

$m_1^1, \dots, m_{s_1}^1, \dots, m_1^r, \dots, m_{s_r}^r$ is $\Pi_{i,j}(\beta_j - m_i^j + 1)$. □

Chapter 4: The nested Hilbert scheme of points

9. Stratification of the nested punctual Hilbert scheme of points on the affine plane and on a plane curve

Lemma 9.1. *The special nested Hilbert scheme of points $C^{[n,n+i]}[z]$ can be decomposed into strata $C^{[n,n+i]}[z] \cong \bigcup_{k=0}^{n}(C \setminus \{z\})^{[n-k]} \times C_z^{[k,k+i]}$. Here $(C \setminus \{z\})^{[n-k]}$ denotes the (non-nested) Hilbert scheme of points on the subset of a curve C minus a point z on the smooth part of C.*

Proof. For every pair of ideals $(I, J) \in C^{[n,n+i]}[z]$, it can be uniquely written as the intersection of ideals as follows $I = I_1 \cap I_2$, $J = J_1 \cap J_2$, such that $I_1 = J_1$ is supported away from the point z, and $\pi(J_2) = k[z]$, $\pi(I_2) = (k + i)[z]$, where π denotes the Hilbert-Chow map, and $k \in \{0, ..., n\}$.

Therefore, there's an isomorphism from $C^{[n,n+i]}[z]$ to $\bigcup_{k=0}^{n}(C \setminus \{z\})^{[n-k]} \times C_z^{[k,k+i]}$ by sending (I, J) to (I_1, I_2, J_2), where $I_1 \in (C \setminus \{z\})^{[n-k]}$, and $(I_2, J_2) \in C_z^{[k,k+i]}$. □

10. Special nested Hilbert schemes of points

When $i = 1$, fix a point z on the **smooth** part of the (underlying reduced) plane curve C. We classify the top-dimensional irreducible components of the special nested Hilbert scheme of points $C^{[n,n+1]}[z]$.

Because many properties of the Hilbert scheme are local, we start by working with the curve $Y = \{y^\beta = 0\}$, and generalize the results to any plane curve C.

Lemma 10.1. *When $k \le \beta-1$, $Y_0^{[k,k+1]} \cong (\mathbb{C}^2)_0^{[k,k+1]}$. When $k > \beta-1$, $Y_0^{[k,k+1]} \subsetneq (\mathbb{C}^2)_0^{[k,k+1]}$.*

Proof. (1) When $k \le \beta - 1$, one side of the containment $Y_0^{[k,k+1]} \subset (\mathbb{C}^2)_0^{[k,k+1]}$ is obvious. Suppose a pair of ideals $(I, J) \in Y_0^{[k,k+1]}$, then it must be true that $I \subset (\mathbb{C}^2)_0^{[k]}$ and $J \subset (\mathbb{C}^2)_0^{[k+1]}$, and that $I \subset J$ by definition. Therefore $(I, J) \in (\mathbb{C}^2)_0^{[k,k+1]}$

To prove the other containment, $(\mathbb{C}^2)_0^{[k,k+1]} \subset Y_0^{[k,k+1]}$, consider a pair of ideals $(I, J) \in (\mathbb{C}^2)_0^{[k,k+1]}$. We need to show that the polynomial y^β is contained in $I \subset J$ in order to show that $(I, J) \in Y_0^{[k,k+1]}$.

As explained in Nakajima's book [8], an ideal A in $(\mathbb{C}^2)^{[k]}$ the Hilbert scheme of k points on the plane can be represented by a k dimensional vector space $V_A = \mathbb{C}[x, y]/A$ with an action of n by n square matrices $\mathbb{X}_A$ and $\mathbb{Y}_A$ such that the matrix $\mathbb{X}_A$ and the matrix $\mathbb{Y}_A$ each correspond to multiplication of elements in the vector space V_A by the variable x and y. It's also required that there exists a vector $1 \in V_A$ such that $\mathbb{X}^i\mathbb{Y}^j \cdot 1$ span V_A.

When we require that the ideal I is contained in $(\mathbb{C}^2)_0^{[k+1]}$ the **punctual** Hilbert scheme of points on the plane, the matrices $\mathbb{X}_I$ and $\mathbb{Y}_I$ acting on the vector space V_I are both nilpotent and of dimension $k+1$ by $k+1$.

Because $\mathbb{Y}_I$ is nilpotent of dimension $k+1$ by $k+1$, write it in the Jordan block form, and $\mathbb{Y}_I$ is a matrix with 0's on the diagonal and some 1's and 0's on the upper skew diagonal. So taking the $k+1$-th power gives $(\mathbb{Y}_I)^{k+1} = 0$. The polynomial y^{k+1} corresponding to the action by the element 1 by the matrix $\mathbb{Y}_I$ for $k+1$ times is equal to 0, i.e. $y^{k+1} \cdot 1 = (\mathbb{Y}_I)^{k+1} \cdot 1 = 0$, and therefore contained in the ideal I. Because of the assumption $k \leq \beta - 1$, y^β is also contained in the ideal I as an element. So $y^\beta \in I \subset J$ and $(I, J) \in (\mathbb{C}^2)_0^{[k,k+1]}$.

(2) When $k > \beta - 1$, take the pair of ideals $(J, I) = ((x, y^k), (x, y^{k+1})) \in (\mathbb{C}^2)_0^{[k,k+1]}$. Because $\beta < k+1$, (x, y^{k+1}) is not in $(\{y^\beta = 0\})^{[k+1]}$, so the pair $((x, y^k), (x, y^{k+1}))$ is not in $C_0^{[k,k+1]}$.

□

Now consider any plane curve C. Recall that C can be decomposed into components C_j with corresponding multiplicities β_j. Fix a point z_j on the **smooth** part of the component C_j of the curve C, and consider the special nested punctual Hilbert scheme of points on the curve $C_{z_j}^{[k,k+1]}$. We have a similar lemma as above.

Lemma 10.2. *Fix a point z_j on the **smooth** part of the component C_j of the curve C, and consider the nested punctual Hilbert scheme of points on the curve $C_{z_j}^{[k,k+1]}$. When $k \leq \beta_j - 1$, $C_{z_j}^{[k,k+1]} \cong (\mathbb{C}^2)_0^{[k,k+1]}$. When $k > \beta_j - 1$, $C_{z_j}^{[k,k+1]} \subsetneq (\mathbb{C}^2)_0^{[k,k+1]}$.*

Proof. When z_j is on the smooth part of the curve component C_j, locally near the point z_j we can reparametrize and find local coordinates such that C_{z_j} is isomorphic to the curve Y. Therefore, the properties of $C_{z_j}^{[k,k+1]}$ follow immediately from Lemma 10.1. □

Corollary 10.3. *When $k > \beta - 1$, the irreducible components of $C_{z_j}^{[k,k+1]}$ have dimension $k - 1$ or less.*

Proof. Because $(\mathbb{C}^2)_0^{[k,k+1]}$ is irreducible of dimension k, as in Theorem 3.4 [9], the nested punctual Hilbert scheme $C_0^{[k,k+1]}$ as a proper closed subset of $(\mathbb{C}^2)_0^{[k,k+1]}$ has dimension less than k. □

Theorem 10.4. *The special nested Hilbert scheme $C^{[n,n+1]}[z]$ has dimension n, can be decomposed as $C^{[n,n+1]}[z] \cong \bigcup_{k=0}^{n} (C \setminus \{z\})^{[n-k]} \times C_z^{[k,k+1]}$. The **top dimensional** components of the special nested punctual Hilbert scheme of points on the curve $C_z^{[k,k+1]}$ are given as follows.*

Pick a positive integer k where $k \leq \beta - 1$, and a tuple of positive integers $m_1^1, ..., m_1^{a_1} | m_2^1, ..., m_2^{a_2} | ... | m_j^1, ..., m_j^{a_j}$ separated by bars, such that $m_j^i \leq \beta$, the sum of all m_j^i equals $n - k$, and that the numbers between each bar are unordered though the bars are ordered.

Each n dimensional component of the special nested Hilbert scheme $C^{[n,n+1]}[z]$ is isomorphic to the closure of $\Sigma_{m_1^1,\ldots,m_1^{a_1}|m_2^1,\ldots,m_2^{a_2}|\ldots|m_j^1,\ldots,m_j^{a_j}} \times (\mathbb{C}^2)^{[k,k+1]}$, where $\Sigma_{m_1^1,\ldots,m_1^{a_1}|m_2^1,\ldots,m_2^{a_2}|\ldots|m_j^1,\ldots,m_j^{a_j}}$ are the components of $(C \setminus \{z\})^{[n-k]}$ as classified in Theorem 5.2 and 5.3.

Proof. Stratify the special nested Hilbert scheme $C^{[n,n+i]}[z] \cong \bigcup_{k=0}^{n} (C \setminus \{z\})^{[n-k]} \times C_z^{[k,k+i]}$ as in Lemma 9.1. The irreducible components of $(C \setminus \{z\})^{[n-k]}$ are the $(n-k)$-dimensional $\Sigma_{m_1^1,\ldots,m_1^{a_1}|m_2^1,\ldots,m_2^{a_2}|\ldots|m_j^1,\ldots,m_j^{a_j}}$ where each $m_j^i \leq \beta$, as classified in Theorem 5.2 and 5.3.[10]

Apply Lemma 10.2. When $k > \beta - 1$, $C_{z_j}^{[k,k+1]}$ has dimension less than k, so its product with $\Sigma_{m_1^1,\ldots,m_1^{a_1}|m_2^1,\ldots,m_2^{a_2}|\ldots|m_j^1,\ldots,m_j^{a_j}}$ has total dimension less than n.

When $k \leq \beta - 1$, $C_z^{[k,k+1]}$ is irreducible of dimension k, and isomorphic to $(\mathbb{C}^2)_0^{[k,k+1]}$, so its product with $\Sigma_{m_1^1,\ldots,m_1^{a_1}|m_2^1,\ldots,m_2^{a_2}|\ldots|m_j^1,\ldots,m_j^{a_j}}$ are components of dimension n. ∎

11. Components of the nested Hilbert scheme of points on a curve

Now we consider projection maps from the nested Hilbert scheme $C^{[n,n+1]}$ to the curve C defined as follows.

Definition 11.1. Define a projection map $p : C^{[n,n+1]} \to C$ as sending $(I, J) \in C^{[n,n+1]}$ to $\pi(I) - \pi(J)$, where π is the Hilbert-Chow morphism.

Remark 11.2. For a singular point $\tilde{z}$, denote the fiber under the projetion map p as $F_{\tilde{z}}$. For a point z_j on the smooth part of the curve component C_j, denote the fiber under the projetion map p as F_{z_j}.

We give a lower bound of the dimension of the components of the nested Hilbert scheme of points on a curve, $C^{[n,n+1]}$.

Lemma 11.3. *The irreducible components of the nested Hilbert scheme of points on a curve $C^{[n,n+1]}$ have dimension at least $n+1$.*

Proof. By work of Cheah [11], the nested Hilbert scheme of points on the affine plane, $(\mathbb{C}^2)^{[n,n+1]}$, is smooth of dimension $2n+2$.

The nested Hilbert scheme of points on a curve, $C^{[n,n+1]}$, is a subscheme of $(\mathbb{C}^2)^{[n,n+1]}$ obtained by imposing one extra condition that the polynomial $f(x,y)$ defining the curve C to be contained as an element in the pair of ideals (I,J). It's sufficient to require only $f(x,y) \in I$ because $I \subset J$.

Fix one such pair of ideals $(I,J) \subset (\mathbb{C}^2)^{[n,n+1]}$. Consider the affine chart $U_\mu \times V_\nu$ of $(\mathbb{C}^2)^{[n+1]} \times (\mathbb{C}^2)^{[n]}$, where U_μ is the chart on $(\mathbb{C}^2)^{[n+1]}$ containing the ideal I, and V_ν is the chart on $(\mathbb{C}^2)^{[n]}$ containing the ideal J.

When imposing the condition $f(x,y) \in I \in U_\mu$, we write $f(x,y)$ as a linear combination of the monomial basis B_μ, and the coefficients of those basis elements should be set to 0 because we want $f(x,y) \in I$. There are $n+1$ basis elements, thus the corresponding coefficients being set to 0 gives $n+1$ equations on $(\mathbb{C}^2)^{[n+1]} \times (\mathbb{C}^2)^{[n]}$, further cutting up the subscheme $(\mathbb{C}^2)^{[n,n+1]}$ inside $(\mathbb{C}^2)^{[n+1]} \times (\mathbb{C}^2)^{[n]}$.

We have $n+1$ equations cutting an $2n+2$ dimensional nested Hilbert scheme of points on the plane $(\mathbb{C}^2)^{[n,n+1]}$, so the irreducible components of the nested Hilbert scheme of points on the curve $C^{[n,n+1]}$ have dimension at least $n+1$.

$\square$

Keeping this in mind, we study both the fibers $F_{\tilde{z}}$ of the map p at the singular points, and the fibers F_{z_j} at the smooth points of the curve C.

Lemma 11.4. *The fibers $F_{\tilde{z}}$ and F_{z_j} both have dimension at most n.*

Proof. Fix a point $\tilde{z}$ on the curve C that is singular, the fiber $F_{\tilde{z}}$ is the pair of ideals $(I, J) \subset C^{[n,n+1]}$ such that $\pi(I) - \pi(J) = \tilde{z}$. So the fiber $F_{\tilde{z}}$ is exactly isomorphic to the special nested Hilbert scheme of points on the curve $C^{[n,n+1]}[\tilde{z}]$.

Use Lemma 9.1 and decompose the special nested Hilbert scheme of points on the curve $C^{[n,n+i]}[\tilde{z}] \cong \bigcup_{k=0}^{n}(C \setminus \{\tilde{z}\})^{[n-k]} \times C_{\tilde{z}}^{[k,k+1]}$.

The Hilbert scheme of $n-k$ points on the curve C has dimension $n-k$, as proved in Theorem 5.4, and the nested punctual Hilbert scheme of points $C_{\tilde{z}}^{[k,k+1]}$ has dimension at most k, as it is a subset of the nested punctual Hilbert scheme of points on the plane $(\mathbb{C}^2)^{[k,k+1]}$. So the fiber $F_{\tilde{z}}$ has dimension at most n in total.

Fix a point z_j on the smooth part of some curve component C_j. The argument is exactly the same as above and we have the isomorphism $F_{z_j} \cong C^{[n,n+1]}[z_j]$. As proved in Theorem

10.4, the special nested Hilbert scheme of points $C^{[n,n+1]}[z_j]$ at a smooth point z_j on the curve C has dimension n, so the fiber F_{z_j} also has dimension n. □

Theorem 11.5. *The nested Hilbert scheme of points $C^{[n,n+1]}$ has dimension $n+1$. The components of the nested Hilbert scheme of points $C^{[n,n+1]}$ are all locally trivial fibrations over the base C, and the fibers are*

$$\Sigma_{m_1^1,\ldots,m_1^{a_1}|m_2^1,\ldots,m_2^{a_2}|\ldots|m_t^1,\ldots,m_t^{a_t}} \times (\mathbb{C}_0^2)^{[k,k+1]}$$

where $\Sigma_{m_1^1,\ldots,m_1^{a_1}|m_2^1,\ldots,m_2^{a_2}|\ldots|m_t^1,\ldots,m_t^{a_t}}$ are the components of $C^{[n-k]}$ as classified in Theorems 5.2 and 5.3, $(\mathbb{C}_0^2)^{[k,k+1]}$ is the nested punctual Hilbert scheme of points on the plane $\mathbb{C}^2$.

Proof. The fiber $F_{\tilde{z}}$ at the singular points has dimension n or less as proved in Lemma 11.4, and its product with the 0 dimensional singularity points $\tilde{z}$ has dimension n or less. We know from Lemma 11.3 that the dimension of the irreducible components of $C^{[n,n+1]}$ is at least $n+1$, so the product does not have dimension big enough to be a component.

The fiber F_{z_j} has dimension n as in the proof of Lemma 11.4. And the product $F_{z_j} \times C_j$ with the curve component has dimension $n+1$.

All of the n dimensional components of the fiber F_{z_j} are classified in Theorem 10.4, and they are given by $\Sigma_{m_1^1,\ldots,m_1^{a_1}|,m_2^1,\ldots,m_2^{a_2}|,\ldots,m_j^1,\ldots,m_j^{a_j}} \times (\mathbb{C}^2)^{[k,k+1]}$. So these n dimensional components of F_{z_j} taking product with its corresponding curve component C_j is the $n+1$ dimensional components of the nested Hilbert scheme $C^{[n,n+1]}$.

Because all the components of $C^{[n,n+1]}$ are at least $n+1$ dimensional by Lemma 11.3, there are no other components of $C^{[n,n+1]}$. Thus, the classification is complete. The nested Hilbert scheme of points $C^{[n,n+1]}$ has dimension $n+1$. □

Corollary 11.6. *The nested Hilbert scheme of points $C^{[n,n+1]}$ is a local complete intersection in $(\mathbb{C}^2)^{[k,k+1]}$.*

Proof. The $n+1$ nested Hilbert scheme of points $C^{[n,n+1]}$ is cut out by $n+1$ equations in the $2n+2$ dimensional $(\mathbb{C}^2)^{[k,k+1]}$, so $C^{[n,n+1]}$ is a local complete intersection in $(\mathbb{C}^2)^{[k,k+1]}$.

□

From the nested Hilbert scheme of points on the curve C, $C^{[n,n+1]}$, we can project it to the Hilbert scheme of n and $n+1$ points on the curve, $C^{[n]}$ and $C^{[n+1]}$.

Definition 11.7. Construct two projection maps p_+ and p_- as follows: $p_- : C^{[n,n+1]} \to C^{[n+1]}$, and $p_+ : C^{[n,n+1]} \to C^{[n]} \times C$.

Define $p_+((I,J)) = (I,z)$, where $z = \pi(I) - \pi(j)$ and π is the Hilbert-Chow map.

Define $p_-((I,J)) = I$.

Lemma 11.8. *The map p_- is proper.*

Proof. Use Lemma 9.1 and decompose the special nested Hilbert scheme of points on the curve $C^{[n,n+i]}[z] \cong \bigcup_{k=0}^{n}(C \setminus \{z\})^{[n-k]} \times C_0^{[k,k+i]}$.

The projection map $p_-((I,J)) = I$ under the decomposition becomes $p_-((I_1,I_2,J_2)) = (I_1,J_2)$, where I_1,I_2,J_2 are the same ideals that appeared in the proof of Lemma 9.1. Recall from the proof of Lemma 9.1 that $I = I_1 \cap I_2$, $J = J_1 \cap J_2$, $I_1 = J_1$, $I_1 \in C^{[n-k]}$, and $(I_2,J_2) \in C_0^{[k,k+i]}$. So p_- can be decomposed as $p_- = \mathrm{id} \times p_-'$ where id is the identity map on $C^{[n-k]}$, and p_-' is the projection map on $C_0^{[k,k+i]}$ that sends $(I_2,J_2) \in C_0^{[k,k+i]}$ to I_2.

The map p_-' is projective and therefore proper, and its product with the identity map is also proper.

$\square$

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