Cyclic sieving and m-surjective functions

By

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SENIOR THESIS

Submitted in partial satisfaction of the requirements for the degree of

BACHELOR OF SCIENCE

in

APPLIED MATHEMATICS

in the

COLLEGE OF LETTERS AND SCIENCE

of

UNIVERSITY OF CALIFORNIA

DAVIS

Approved:

Dr. Sean Griffin

2024

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Abstract

In the 90's, Stembridge discovered that plugging $q = -1$ into rank generating functions for plane partitions gives formulas for plane partitions fixed under an involution. In 2004, Reiner, Stanton, and White generalized the discovery to the setting of cyclic group actions on finite sets. They introduced the cyclic sieving phenomenon when a cyclic group acts on a set such that the number of fixed points is equal to the result of plugging a root of unity into a generating function. In the past 12 years, many connections between the cyclic sieving phenomena and tableaux combinatorics have been discovered. In this thesis, we explore these results and prove explicit formulas for minimal degree CSPs.

Acknowledgments

I would like to express my sincere gratitude to Professor Sean Griffin for his guidance, patience, support, and invaluable feedback throughout the process of complete this thesis. Also, I appreciate Professor Brendon Rhoades for his useful suggestions on the project.

I also want to thank to my parents and all my friends for their support. I am truly grateful to their willing to listen to my research and express their understanding and interests in the project, even though they are probably not.

Chapter 1

Introduction

A plane partition π is a finite collection of left-justified cells filled with nonnegative integers such that each column weakly decreases from top to bottom and each row weakly decreases from left to right, i.e. $\pi = [\pi_{i,j}]_{i \leq a, j \leq b}$, $\pi_{i,j} \geq \pi_{i+1,j}$ and $\pi_{i,j} \geq \pi_{i,j+1}$ for all i, j. Alternatively, we can visualize a plane partition in 3-dimensions by stacking $\pi_{i,j}$ many unit cubes on cell (i, j). We denote the set of all plane partitions that fit inside the box with dimensions $[a] \times [b] \times [c]$ by $PP(a, b, c)$. In this case, $\pi_{i,j} \leq c$ for all i, j.

Let $\langle \sigma \rangle$ be a cyclic group with order 2. When σ acts on $\pi \in PP(a, b, c)$, it replaces each nonzero number k with $c - k$ and rotates the plane partition by 180°. We say π is symmetric if $\sigma(\pi) = \pi$, and denote the set of symmetric plane partitions by $PP(a, b, c)^\sigma$.

In the 1990s, Stembridge discovered the remarkable "$q = -1$" phenomenon [4] [9], which is a direct relationship between the generating function for plane partitions and the number of symmetric plane partitions. Let

$$N(q) = \sum_{\pi \in PP(a,b,c)} q^{|\pi|},$$

where $|\pi| = \sum_{\substack{i \leq a \\ j \leq b}} \pi_{i,j}$. When plugging $q = -1$ into the formula, we find that $|PP(a, b, c)^\sigma| = N(-1)$.

Stembridge's discovery was just the opening act. Many combinatorial finite sets were then proved to exhibit similar phenomena with corresponding generating functions and cyclic symmetries. In fact, the "$q = -1$" phenomenon is the first instance of cyclic sieving phenomenon which will be discussed in the following, and $\omega = -1$ is the second root of unity.

In [6], Reiner, Stanton, and White introduced the **cyclic sieving phenomenon**, a generalization of

"$q=-1$" phenomenon, which is defined below.

Definition 1.0.1 (Cyclic Sieving Phenomenon)**.** Let X be a finite set and G be a finite cyclic group which acts on X. For a group element $g \in G$, we denote the fixed point set of X by

$$X^g = \{x \in X : gx = x\}.$$

Let ω be a primitive root of unity which has the same order as g. Let $f(q)$ be a polynomial in q with coefficients in $\mathbb{Z}_{\geq 0}$. The triple $(X, G, f(q))$ is said to exhibit the cyclic sieving phenomenon, or CSP, if, for all $g \in G$, we have

$$|X^g| = f(\omega_g).$$

See [8] for more details.

When CSP is extended to the product of two cyclic groups acting on a set X, the *bicyclic sieving phenomenon* is introduced, which was first defined in [1].

Definition 1.0.2 (Bicyclic Sieving Phenomenon)**.** Let X be a finite set, and let $G = \langle g \rangle$, $G' = \langle g' \rangle$ be finite cyclic groups with $G \times G'$ acting on X. For group elements $g^r \in G$ and $(g')^s \in G'$, we denote the fixed point subset of X by

$$X^{(g^r,(g')^s)} = \{x \in X : (g^r, (g')^s)x = x\}.$$

Let $\omega = e^{2\pi i/|G|}$, $\omega' = e^{2\pi i/|G'|}$, and let $f(q,t)$ be a polynomial. The triple $(X, G \times G', f(q,t))$ is said to exhibit the bicyclic sieving phenomenon, or biCSP, if, for all $g^r \in G$ and $(g')^s \in G'$, we have

$$|X^{(g^r,(g')^s)}| = f(\omega^r, (\omega')^s).$$

Rhoades shows the following connection between cyclic sieving phenomenon and tableau combinatorics as follows.

Theorem 1.1. *Let λ be a partition such that $\lambda = (n^m)$ for some positive integers n, m, then the triple*

$$\left(SYT(\lambda), \langle \partial \rangle, f^\lambda(q)\right)$$

exhibits the cyclic sieving phenomenon.

We will define the standard Young tableau SYT, the group $\langle \partial \rangle$ with the operator of promotion of SYT, and the polynomial $f^{\lambda}(q)$ in Section 2.2.

Theorem 1.2 (Oh and Rhoades [5])**.** *Let n and k be positive integers, then the triple $(Z_{n,k}, \mathbb{Z}_n \times \mathbb{Z}_k, Z_{n,k}(q,t))$ exhibits the bicyclic sieving phenomenon, where $Z_{n,k} = \{(a_1, a_2, \ldots, a_n) | \{a_1, a_2, \ldots, a_n\} = \{\omega, \omega^2, \ldots, \omega^k\}\}$, and*

$$Z_{n,k}(q,t) = \sum_{T \in SYT(n)} q^{\mathrm{maj}(T)} \cdot \begin{bmatrix} n - \mathrm{des}(T) - 1 \\ n-k \end{bmatrix}_q \cdot f^{\mathrm{shape}(T)}(t).$$

The thesis is organized as follows. In chapter 2, some necessary definitions and examples of G-modules and tableaux combinatorics are provided. We also introduce examples of CSPs and biCSPs which are connected to tableaux combinatorics. In chapter 3, we discuss the minimum degree of the polynomials for the set $Z_{z,k}^{(m)}$ which still exhibit CSPs and present proofs.

Chapter 2

Background

2.1 G-module

In this section, we give definitions and examples of G-modules.

Definition 2.1.1 (group)**.** A **group** is a nonempty set G with a binary operation $f : G * G \to G$ such that the following conditions are satisfied:

1. (associative) For any $a, b, c \in G$, $a * (b * c) = (a * b) * c$.

2. (closed) For any $a, b \in G$, $a * b \in G$ as well.

3. (identity) There exists an element $e \in G$ so that $a * e = e * a = a$.

4. (inverse) For every $a \in G$, there exists $a^{-1} \in G$ so that $a * a^{-1} = a^{-1} * a = e$.

Example 1. The symmetric group is an example of a group. Consider the finite set $X = \{1, 2, \ldots, n\}$ and the symmetric group of permutations of X, denoted as S_n. By using cycle notation for permutations, for $(2, 1, 3), (2, 3) \in S_3$, $(2, 1, 3)(2, 3) = (1, 3) \in S_3$. Here, we use the convention that the permutations are performed from right to left.

Definition 2.1.2 (group action)**.** An action of a group G on a set X is a function $f : G \times X \to X$ with $(g, x) \mapsto g \cdot x$ such that

- for the identity e in G, $e \cdot x = x$ for all $x \in X$.

- for all $g_1, g_2 \in G$, $x \in X$, $(g_1 \cdot g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$.

Example 2. The symmetric group $G = S_n$ acts on the set $X = \{1, \ldots, n\}$ as follows. For every element $\sigma \in S_n$ and $x \in X$, $\sigma \cdot x = \sigma(x)$. For $\sigma = (2, 1, 3) \in S_3$, the group action on every element in X is

$$(2, 1, 3) \cdot 1 = 3$$
$$(2, 1, 3) \cdot 2 = 1$$
$$(2, 1, 3) \cdot 3 = 2.$$

Definition 2.1.3 (permutation matrix)**.** Given an ordering $X = \{x_1, \ldots, x_n\}$ and $g \in G$ for G be a group, $[g]_x$ is the matrix whose (i, j) entry is 1 if $g(x_j) = x_i$ and 0 otherwise.

Lemma 2.1. *Given $g, h \in G$ for G a group and an ordering $X = \{x_1, \ldots, x_n\}$, $[g]_x \cdot [h]_x = [g \cdot h]_x$.*

Proof. Let i, j be the indices of the columns and rows for the matrix $[g]_X$, respectively, and let j, ℓ be the indices of the columns and rows for the matrix $[h]_X$, respectively.

For every (i, j) entry in the matrix $[g]_X$ and every (j, ℓ) entry in the matrix $[h]_X$,

$$([g]_X)_{ij} = \begin{cases} 1 & \text{if } g(x_j) = x_i \\ 0 & \text{otherwise} \end{cases}, \; ([h]_X)_{j\ell} = \begin{cases} 1 & \text{if } h(x_\ell) = x_j \\ 0 & \text{otherwise} \end{cases}$$

This follows that for every (i, ℓ) entry in the matrix $([g]_X \cdot [h]_X)_{i\ell}$,

$$\begin{aligned} ([g]_X \cdot [h]_X)_{i\ell} &= \sum_j ([g]_X)_{ij} \cdot ([h]_X)_{j\ell} \\ &= \#\{j \mid g(x_j) = x_i,\ h(x_\ell) = x_j\} \\ &= \begin{cases} 1 & \text{if } gh(x_\ell) = x_i \\ 0 & \text{otherwise} \end{cases} \\ &= ([g \cdot h]_X)_{i\ell} \end{aligned}$$

Thus, $[g]_X \cdot [h]_X = [g \cdot h]_X$. $\square$

Example 3. Let G be the symmetric group S_3, and $g = (1, 3, 2)$, $h = (1, 2)$, where $g, h \in S_3$. According to the definition of permutation matrix,

$$[(1,3,2)]_X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, [(1,2)]_X = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus,

$$[(1,3,2)]_X \cdot [(1,2)]_X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Alternatively, compute $(1,3,2) \cdot (1,2) = (2,3)$, and

$$[(2,3)]_X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Therefore, $[(1,3,2)]_X \cdot [(1,2)]_X = [(1,3,2) \cdot (1,2)]_X$, which verifies that $[g]_X \cdot [h]_X = [g \cdot h]_X$.

Definition 2.1.4 (G-module)**.** Let X be a set, and take its linear combinations to form a complex vector space $V = \mathbb{C}X$. In other words, if $X = \{a_1, a_2, \dots, a_n\}$, then $\mathbb{C}X$ is the set of formal linear combinations of elements in X,

$$\mathbb{C}X = \{c_1 \cdot \underline{a_1} + c_2 \cdot \underline{a_2} + \cdots + c_n \cdot \underline{a_n} : c_i \in \mathbb{C}\ \forall i\}.$$

The elements of X are underlined because they are regarded as vectors. If a given group G acts on X, then it also acts on $\mathbb{C}X$ by extending the action linearly.

Example 4. If the set $X = \binom{[3]}{2} = \{\{1,2\},\{1,3\},\{2,3\}\}$, then the vector space

$$V = \mathbb{C}X = \{c_1 \cdot \underline{1} + c_2 \cdot \underline{2} + c_3 \cdot \underline{3} \mid c_1, c_2, c_3 \in \mathbb{C}\},$$

where $\underline{1} = \{1,2\}$, $\underline{2} = \{1,3\}$, $\underline{3} = \{2,3\}$.

Let us choose the ordered basis $\{\underline{1}, \underline{2}, \underline{3}\}$. Then for the group element $\sigma = (2,1,3) \in G$, the group action on every element in X is

$$(2,1,3) \cdot \underline{1} = \underline{2},$$

$$(2,1,3) \cdot \underline{2} = \underline{3},$$

$$(2,1,3) \cdot \underline{3} = \underline{1}.$$

In the matrix form, the permutation $(2,1,3)$ acts on V by

$$[(2,1,3)]_X = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

2.2 Tableaux and related combinatorics

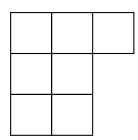
In this section, we recall the definitions of partition and tableaux and introduce tableaux operations on tableaux. We also explore some combinatorial statistics, which are essential in the CSPs and biCSPs that will be discussed in the following sections.

Definition 2.2.1 (partition)**.** A **partition** of $n \in \mathbb{N}$, denoted as $\lambda \vdash n$, is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, in which $\sum_{n=1}^{k} \lambda_i = n$ and $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k$. Let $(n_1^{a_1}, \ldots, n_k^{a_k})$ denote the partition with a_i many parts of size n_i for all i.

Example 5. All partitions of 3 are $(1, 1, 1)$, $(2, 1)$, and (3). The partition $(1, 1, 1)$ can be denoted by (1^3), and $(3, 3, 2, 2)$ can be denoted by $(3^2, 2^2)$.

Definition 2.2.2. A **Young diagram** of λ, given by $\lambda \vdash n$, is composed of k rows of left-justified cells with λ_i many cells in row i for $1 \leqslant i \leqslant k$.

Example 6. The Young diagram of the partition $(3, 2, 2) \vdash 7$ is shown in the following.



Definition 2.2.3 (standard Young tableau)**.** A **standard Young tableau** T of shape λ with $\lambda \vdash n$ is obtained by filling the cells in the Young diagram with each of the positive integers in $[n]$ used exactly once such that each column increases from top to bottom and each row increases from left to right in the diagram. Denote the set of the standard Young tableaux with shape λ as $\mathrm{SYT}(\lambda)$.

Example 7. Given the shape $(2, 1) \vdash 3$,

$$SYT(2,1) = \left\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \right\}.$$

Definition 2.2.4 (semistandard Young tableau)**.** A **semistandard Young tableau** T of shape λ with $\lambda \vdash n$ is obtained by filling the cells in the Young diagram with positive integers such that each row weakly increases from left to right and each column strictly increases from top to bottom. The **content** μ of T, denoted by $\mu \vDash n$, is a sequence of positive integers $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$, in which μ_i is equal to the number of i's in T. Denote the set of semistandard Young tableaux with shape λ and content μ as $\mathrm{SSYT}(\lambda, \mu)$.

Example 8. Given the shape $\lambda = (3,2) \vdash 5$ and content $\mu = (2,2,1)$,

$$SSYT(\lambda,\mu) = \left\{ \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}, \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \right\}.$$

A **combinatorial statistic** is a function from a set of combinatorial objects to $\mathbb{Z}^+$. *Charge*, *cocharge*, *descent*, and *major index* are combinatorial statistics on SYT and SSYT that have important connections to CSPs.

Definition 2.2.5 (reading word)**.** The **reading word** of a semistandard Young tableau T is the sequence of numbers obtained by connecting the rows of T from the bottom row to top.

Example 9. Let T be the following semistandard Young tableau.

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

The reading word of T is

$$rw\left(\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right) = 323112.$$

Definition 2.2.6 (charge)**.** Suppose T is a semistandard Young tableau whose content is a partition. We acquire the **charge** on T by the following process:

1. Read the word $rw(T)$ from right to left.

2. Find the first 1 in the word we meet and label it with 0.

3. Every time after labeling the number k, continue to read the word until a $k+1$ is found. Label it with i, where i is the number of times we've finished reading the whole word up to this point.

4. Repeat step 3 until the largest number in the word is labeled. Delete all labeled numbers and obtain a new word.

5. Repeat step 1-4 until the new word we obtain in step 4 is empty. The *charge* on T is the sum of the labels of the numbers in the original reading word of T.

The *charge* on T is denoted as $\mathrm{ch}(T)$.

Example 10. Let T be the semistandard Young tableau in Example 9 and following the process above, we obtain the charge of T by the following steps:

1. After performing the first three steps, we obtain $3_0 2_0 3 1 1_0 2$.

2. Deleting the labeled numbers and repeating the process, we get $3_1 1_0 2_1$.

3. Combining the first two steps, we get the labeled reading word of T as $3_0 2_0 3_1 1_0 1_0 2_1$.

Then

$$\mathrm{ch}(T) = \mathrm{ch}(323112) = 0 + 0 + 1 + 0 + 0 + 1 = 2.$$

We also give the definitions of two statistics, descent and major index, which will be applied in a CSP in the following discussion.

Definition 2.2.7. Given a permutation $\sigma = \sigma_1 \sigma_2 \dots \sigma_n$, the descent set, denoted as $\mathrm{Des}(\sigma)$, is given by

$$\mathrm{Des}(\sigma) = \{i : \sigma_i > \sigma_{i+1}\},$$

and the major index of the permutation is

$$\mathrm{maj}(\sigma) = \sum_{i \in \mathrm{Des}(\sigma)} i.$$

In addition, given a standard Young tableau T, the descent set $\mathrm{Des}(T)$ is given by

$$\mathrm{Des}(T) = \{i : i + 1 \text{ appears in a strictly lower row than } i\},$$

and $\mathrm{des}(T)$ denotes the number of elements in the set $\mathrm{Des}(T)$. The major index, $\mathrm{maj}(T)$, is the sum of the elements in $\mathrm{Des}(T)$.

Example 11. For a permutation $\sigma = 31524$, the descent set $\mathrm{Des}(\sigma) = \{1, 3\}$ since $\sigma_1 = 3 > \sigma_2 = 1$ and $\sigma_3 = 5 > \sigma_4 = 2$. Thus, $\mathrm{maj}(\sigma) = 1 + 3 = 4$.

For the standard Young tableau T below, the descent set $\mathrm{Des}(T) = \{2, 3, 5\}$, so the descent number des$(T) = 3$. The major index of T is $\mathrm{maj}(T) = 2 + 3 + 5 = 10$.

$$T \;=\; \begin{array}{|c|c|c|}\hline 1 & 2 & 7 \\ \hline 3 & 5 \\ \cline{1-2} 4 & 6 \\ \cline{1-2}\end{array}$$

Definition 2.2.8 (hook)**.** Given any Young diagram, the hook of the cell c in (i, j), i.e. in the i^{th} row and the j^{th} column, is the set of cells to the right of c in the same row or below c in the same column. The hooklength h_c is the number of cells in the hook of c.

Example 12. Given the Young diagram of the partition $(3, 2, 2) \vdash 7$, the hook of the cell c in $(1, 2)$ is shown by dots below. In this case, the hooklength $h_c = 4$.

$$\begin{array}{|c|c|c|}\hline & \cdot & \cdot \\ \hline & \cdot \\ \cline{1-2} & \cdot \\ \cline{1-2}\end{array}$$

Definition 2.2.9 (promotion)**.** Given a standard Young tableau T, we acquire its promotion, denoted ∂T, by the following process:

1. Replace the $(1, 1)$ cell of T by a dot.
2. When the dot is in the (i, j) cell, find $x = \min\{T_{(i+1,j)}, T_{(i,j+1)}\}$, where $T_{i,j}$ is the (i, j) entry of T, and exchange the entries of the dot and x. If only one of the $(i + 1, j), (i, j + 1)$ entries exists, then replace the dot with that entry directly. Repeat this step until (i, j) is a corner, i.e. T does not contain the cells of $(i + 1, j)$ and $(i, j + 1)$.
3. We obtain the promotion ∂T by subtracting all elements of T after step 2 by 1 and replacing the dot by the largest number in T.

Example 13. Let $T \;=\; \begin{array}{|c|c|c|}\hline 1 & 2 & 7 \\ \hline 3 & 5 \\ \cline{1-2} 4 & 6 \\ \cline{1-2}\end{array}$, the promotion ∂T is acquired in the following:

$$\begin{array}{|c|c|c|}\hline \cdot & 2 & 7 \\ \hline 3 & 5 \\ \cline{1-2} 4 & 6 \\ \cline{1-2}\end{array} \to \begin{array}{|c|c|c|}\hline 2 & \cdot & 7 \\ \hline 3 & 5 \\ \cline{1-2} 4 & 6 \\ \cline{1-2}\end{array} \to \begin{array}{|c|c|c|}\hline 2 & 5 & 7 \\ \hline 3 & \cdot \\ \cline{1-2} 4 & 6 \\ \cline{1-2}\end{array} \to \begin{array}{|c|c|c|}\hline 2 & 5 & 7 \\ \hline 3 & 6 \\ \cline{1-2} 4 & \cdot \\ \cline{1-2}\end{array} \to \begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 3 & 7 \\ \cline{1-2}\end{array}$$

2.3 Cyclic sieving phenomenon

Recall that the cyclic sieving phenomenon needs three elements in the tuple: a finite set X, a finite cyclic group G which acts on X, and a polynomial $f(q)$ with coefficients in $\mathbb{Z}_{\geq 0}$. The triple $(X, G, f(q))$ exhibits the cyclic sieving phenomenon (CSP) if for all $g \in G$, we have

$$|X^g| = f(\omega_g),$$

where X^g is the fixed point set of X and ω is a primitive root of unity with the same order as G.

For the rest of this section, we state some main results of CSPs and biCSPs involving standard Young tableaux.

Definition 2.3.1. A **q-analogue of** n is defined as the polyomial

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1}.$$

Observe that when substituting $q = 1$, we get the integer n.

Here, we give the definition of the polynomial $f^\lambda(q)$. Recall that h_c is the hook of a certain cell in a given Young diagram, which is introduced in the previous section.

Definition 2.3.2 (Rhoades [7])**.** Given the shape $\lambda \vdash n$, the polynomial $f^\lambda(q)$ is defined as

$$f^\lambda(q) = \frac{[n]_q!}{\prod_{c \in \lambda} [h_c]_q}.$$

By a result of Haiman [3], the order of ∂ as an operator on $SYT(n^m)$ divides $m \cdot n$.

In 2010, Rhoades [7] discovered a cyclic sieving phenomenon connecting standard Young tableaux, its promotion, and the polynomial $f^\lambda(q)$.

Theorem 2.2. *Let the partition $\lambda = (n^m)$ for some positive integers n, m, then the triple*

$$\big(SYT(\lambda), \mathbb{Z}_{mn}, f^\lambda(q)\big)$$

exhibits the cyclic sieving phenomenon, where $i \in \mathbb{Z}_{mn}$ acts on T by $i \cdot T = \partial^i(T)$.

Example 14. Consider the partition $\lambda = (3^2)$. Then the set

$$SYT(\lambda) = \left\{ \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array} \right\}.$$

The promotion of each of the elements in $SYT(\lambda)$ is shown below.

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array} \to \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array} \to \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array} \to \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array} \to \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array} \to \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array}$$

Then ∂ acts as the following permutation of the set $SYT(\lambda)$

$$\partial = \left(\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array} \right) \left(\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array} \right).$$

Observe that when ∂^2 acts on $SYT(\lambda)$, we have 2 fixed points:

$$\partial^2 \left(\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array} \right) = \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array} \text{ and } \partial^2 \left(\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array} \right) = \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array}.$$

Similarly, when ∂^3 acts on $SYT(\lambda)$, we get 3 fixed points. Indeed, when computing the polynomial $f^\lambda(q)$ with $\lambda = (3^2)$, so

$$f^\lambda(q) = \frac{[6]_q!}{[4]_q \cdot [3]_q \cdot [2]_q \cdot [3]_q \cdot [2]_q \cdot [1]_q} = \frac{[6]_q \cdot [5]_q}{[3]_q \cdot [2]_q} = 1 + q^2 + q^3 + q^4 + q^6.$$

Let $\omega = e^{2\pi i/6}$. When plugging ω^2 into the polynomial, $f^\lambda(\omega^2) = 1+\omega^4+\omega^6+\omega^8+\omega^{12} = 1+\omega^4+1+\omega^2+1 = 2$, which matches the number of fixed points we obtain when ∂^2 acting on $SYT(\lambda)$.
In the same way, $f^\lambda(\omega^3) = 1 + \omega^6 + \omega^9 + \omega^{12} + \omega^{18} = 1 + 1 - 1 + 1 + 1 = 3$, which matches the number of fixed points we obtain when ∂^3 acting on $SYT(\lambda)$.
Therefore, Theorem 2.3.2 holds true in our example.

Theorem 2.3 (Oh and Rhoades [5])**.** *Let n and k be positive integers and let $\omega = e^{2\pi i/k}$, $\omega' = e^{2\pi i/n}$, then the triple $(Z_{n,k}, \mathbb{Z}_n \times \mathbb{Z}_k, Z_{n,k}(q,t))$ exhibits the bicyclic sieving phenomenon, where*

$$Z_{n,k} = \left\{ (a_1, a_2, \ldots, a_n) | \{a_1, a_2, \ldots, a_n\} = \{\omega, \omega^2, \ldots, \omega^k\} \right\},$$

and

$$Z_{n,k}(q,t) = \sum_{T \in SYT(n)} q^{\mathrm{maj}(T)} \cdot \begin{bmatrix} n - \mathrm{des}(T) - 1 \\ n-k \end{bmatrix}_q \cdot f^{\mathrm{shape}(T)}(t) \cdot t^{n(\lambda)},$$

where $n(\lambda) = \sum_i (i-1) \cdot \lambda_i$.

The cyclic group $\mathbb{Z}_n$ *acts on* $Z_{n,k}$ *by the rotation* $(a_1, \ldots, a_n) \mapsto (a_2, \ldots, a_n, a_1)$*, and the cyclic group* $\mathbb{Z}_k$ *acts on* $Z_{n,k}$ *by* $(a_1, \ldots, a_n) \mapsto (a_1 \cdot \omega, \ldots, a_n \cdot \omega)$*.*

Example 15. Given $n = 3$ and $k = 2$. Then $\omega = e^{2\pi i/2}$, $\omega' = e^{2\pi i/3}$, and

$$Z_{3,2} = \{(\omega, \omega, \omega^2), (\omega, \omega^2, \omega), (\omega^2, \omega, \omega), (\omega, \omega^2, \omega^2), (\omega^2, \omega, \omega^2), (\omega^2, \omega^2, \omega)\}.$$

Observe that the fixed points exist only when $(3,2) \in \mathbb{Z}_3 \times \mathbb{Z}_2$ acts on $Z_{3,2}$, and $\left| Z_{3,2}^{(3,2)} \right| = |Z_{3,2}| = 6$.

We also need to compute $Z_{n,k}(q,t)$. For $n = 3$, all possible SYTs are

$$T_1 = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array} \quad T_2 = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array} \quad T_3 = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \cline{1-1} \end{array} \quad T_4 = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}.$$

According to the definitions of descent number and major index,

$$\mathrm{des}(T_1) = 0,\ \mathrm{maj}(T_1) = 0;$$

$$\mathrm{des}(T_2) = 1,\ \mathrm{maj}(T_2) = 2;$$

$$\mathrm{des}(T_3) = 1,\ \mathrm{maj}(T_3) = 1;$$

$$\mathrm{des}(T_4) = 2,\ \mathrm{maj}(T_4) = 3.$$

In addition,

$$f^{\mathrm{shape}(T_1)}(t) = \frac{[3]_t!}{[3]_t!} = 1,$$
$$f^{\mathrm{shape}(T_2)}(t) = f^{\mathrm{shape}(T_3)}(t) = \frac{[3]_t!}{[3]_t \cdot [1]_t \cdot [1]_t} = [2]_t = 1 + t,$$
$$f^{\mathrm{shape}(T_4)}(t) = \frac{[3]_t!}{[3]_t!} = 1.$$

Therefore,

$$
\begin{aligned}
Z_{3,2}(q,t) &= q^0 \cdot \binom{3-0-1}{3-2}_q \cdot 1 \cdot t^0 + q^2 \cdot \binom{3-1-1}{3-2}_q \cdot (1+t) \cdot t + q^1 \cdot \binom{3-1-1}{3-2}_q \cdot (1+t) \cdot t \\
&\quad + q^3 \cdot \binom{3-2-1}{3-2}_q \cdot 1 \cdot t^3 \\
&= \binom{2}{1}_q + q^2 \cdot \binom{1}{1}_q \cdot (t+t^2) + q \cdot \binom{1}{1}_q \cdot (t+t^2) + q^3 \cdot \binom{0}{1}_q \cdot t^3 \\
&= (1+q) + q^2 \cdot (t+t^2) + q \cdot (t+t^2) + 0 \\
&= 1 + q + qt + qt^2 + q^2t + q^2t^2
\end{aligned}
$$

For $q = (\omega')^3 = 1$ and $t = \omega^2 = 1$,

$$Z_{3,2}(1,1) = 1 + 1 + 1 \cdot 1 + 1 \cdot 1^2 + 1^2 \cdot 1 + 1^2 \cdot 1^2 = 6.$$

Chapter 3

Minimum degree CSP for m-surjective functions

In [6], Reiner, Stanton, and White proposed and proved a condition that needs to hold for a triple $(X, f(q), C_n)$ to exhibit cyclic sieving phenomenon, which is shown below.

Proposition 3.1 (Reiner-Stanton-White). *A triple $(X, f(q), C_n)$ exhibits the cyclic sieving phenomenon if and only if*

$$f(q) = \sum_{\ell=0}^{n-1} a_\ell q^\ell \mod q^n - 1,$$

where $a_\ell = \#\ C_n$-orbits of X for which the order of the stabilizer subgroup of an element of that orbit divides ℓ.

In this chapter, we will define m-surjective functions and consider different cases of a specific m-surjective function so that the degree of the polynomial can be minimized and the CSP still holds. We will compute these polynomials for some sets $Z_{n,k}^{(m)}$ by finding the orbit sizes.

Definition 3.0.1. Define a function $f : [n] \to [k]$ for some integers n and k. We say the function f is m-surjective if $\left|f^{-1}(i)\right| \geq m \ \forall\ i \in [k]$.

Definition 3.0.2. The set $Z_{n,k}^{(m)}$ is constructed by the following:

$Z_{n,k}^{(m)} = \left\{(a_1, a_2, \ldots, a_n) | \{a_1, a_2, \ldots, a_n\} = \{\omega, \omega^2, \ldots, \omega^k\} \text{ and each } \omega^i \text{ appears at least } m \text{ times} \ \forall\ 1 \leq i \leq k\right\}.$

According to the definition 3.0.1, $Z_{n,k}^{(m)}$ is in bijection with m-surjective functions.

Example 16. When $n = 5$, $k = 2$, and $m = 2$, then the set $Z_{3,2}^{(2)}$ is

$$\{(\omega,\omega,\omega^2,\omega^2,\omega^2),(\omega,\omega^2,\omega,\omega^2,\omega^2),(\omega,\omega^2,\omega^2,\omega,\omega^2),(\omega,\omega^2,\omega^2,\omega^2,\omega),(\omega^2,\omega,\omega,\omega^2,\omega^2),$$

$$(\omega^2,\omega,\omega^2,\omega,\omega^2),(\omega^2,\omega,\omega^2,\omega^2,\omega),(\omega^2,\omega^2,\omega,\omega,\omega^2),(\omega^2,\omega^2,\omega,\omega^2,\omega),(\omega^2,\omega^2,\omega^2,\omega,\omega),$$

$$(\omega^2,\omega^2,\omega,\omega,\omega),(\omega^2,\omega,\omega^2,\omega,\omega),(\omega^2,\omega,\omega,\omega^2,\omega),(\omega^2,\omega,\omega,\omega,\omega^2),(\omega,\omega^2,\omega^2,\omega,\omega),$$

$$(\omega,\omega^2,\omega,\omega^2,\omega),(\omega,\omega^2,\omega,\omega,\omega^2),(\omega,\omega,\omega^2,\omega^2,\omega),(\omega,\omega,\omega^2,\omega,\omega^2),(\omega,\omega,\omega,\omega^2,\omega^2)\}.$$

Definition 3.0.3. The cyclic group $\mathbb{Z}_n$ acts on the set $Z_{n,k}^{(m)}$ by rotating the positions $(a_1, a_2, \ldots, a_n)$ to $(a_2, a_3, \ldots, a_n, a_1)$.

$Z_{n,k}$ is a special case of $Z_{n,k}^{(m)}$ when $m = 1$. In other words, $Z_{n,k}^{(m)}$ is a generalization of $Z_{n,k}$. These sets are motivated by work of Griffin [2].

Lemma 3.1. *For $\ell \neq 1$, $\#\left(Z_{n,k}^{(m)}\right)^{[\ell]} \neq 0$ if and only if $\lfloor \frac{a}{k} \rfloor \geq \frac{m \cdot a}{n}$, where $a = \gcd(\ell, n)$.*

Proof. Observed that $\#\left(Z_{n,k}^{(m)}\right)^{[\ell]} = \#\left(Z_{n,k}^{(m)}\right)^{[a]}$ for $a = \gcd(\ell, n)$, so it suffices to prove the lemma when $\ell | n$.

For $\ell | n$, let $n = \ell \cdot d$ for some integers d. When $[\ell]$ acts on $Z_{n,k}^{(m)}$, and the fixed points are denoted as

$$\left(Z_{n,k}^{(m)}\right)^{[\ell]} = \left\{(a_1, a_2, \ldots, a_n) \in Z_{n,k}^{(m)} | a_i = a_j \text{ for } i \equiv j \mod \ell\right\}.$$

In the backward direction, suppose $\lfloor \frac{\ell}{k} \rfloor \geq \frac{m \cdot \ell}{n}$, then $\lfloor \frac{\ell}{k} \rfloor \cdot d \geq m$. Let $(a_1, \ldots, a_n)$ be defined by

$$\begin{array}{ccc}
a_1 = \omega, & a_2 = \omega^2, & \ldots, a_\ell = \omega^\ell = \omega^{\ell \mod k}, \\
a_{\ell+1} = \omega, & a_{\ell+2} = \omega^2, & \ldots, a_{2\ell} = \omega^{2\ell} = \omega^{2\ell \mod k}, \\
\vdots & \vdots & \vdots \\
a_{1+(d-1)\ell} = \omega, & \ldots & a_n = a_{d\ell} = \omega^{d\ell \mod k},
\end{array}$$

so $a_{i+j\ell} = \omega^i$ for $0 < i \leq \ell$ and $0 \leq j \leq d - 1$.

Then we have d rows and ℓ columns, and all elements in the same column are equal to the same ω^i for some $1 \leq i \leq k$. For each i, ω^i appears at least $\lfloor \frac{\ell}{k} \rfloor \cdot d$ times, and given that $\lfloor \frac{\ell}{k} \rfloor \cdot d \geq m$, ω^i appears at least

m times for all i. $(a_1, \ldots, a_n) \in \left(Z_{n,k}^{(m)}\right)^{[\ell]}$, so $\# \left(Z_{n,k}^{(m)}\right)^{[\ell]} \neq 0$.

Conversely, suppose that $\# \left(Z_{n,k}^{(m)}\right)^{[\ell]} \neq 0$, there exists some $\vec{a} = (a_1, \ldots, a_n)$ such that $(a_1, \ldots, a_n)^{[\ell]} = (a_{1+\ell}, a_{2+\ell}, \ldots, a_n, a_1, a_2, \ldots, a_\ell) = (a_1, \ldots, a_n)$.

Therefore, $a_1 = a_{1+\ell}$, $a_2 = a_{2+\ell}$, $\ldots$, so put all equally a_i into one set for all i, and we have ℓ many sets and each contains d many elements.

Case 1: $d|m$. Then $m = d \cdot q$ for some integer q. If there exists $\vec{a} \in \left(Z_{n,k}^{(m)}\right)^{[\ell]}$, then each of the $\omega^1, \ldots, \omega^k$ appears at least m times with $m = d \cdot q$.

Since each set contains d many elements, each of the $\omega^1, \ldots, \omega^k$ appears in at least q sets, which implies $\ell \geq k \cdot q$, i.e. $\frac{\ell}{k} \geq q$.

Since q is an integer, then $\lfloor \frac{\ell}{k} \rfloor \geq q = \frac{m}{d}$, i.e. $\lfloor \frac{\ell}{k} \rfloor \cdot d \geq m$.

Case 2: $d \nmid m$. Then $m = d \cdot q + r$ for some integers q, r such that $0 < r < d$.

If there exists $\vec{a} \in \left(Z_{n,k}^{(m)}\right)^{[\ell]}$, then each of the $\omega^1, \ldots, \omega^k$ appears at least m times, i.e. appears at least $(d \cdot q + r)$ many times.

Since each set contains d many elements, each of the $\omega^1, \ldots, \omega^k$ appears in at least $(q+1)$ sets.

Thus, $\ell \geq k \cdot (q+1)$, which implies $\frac{\ell}{k} \geq (q+1)$.

Since $(q+1)$ is an integer, then $\lfloor \frac{\ell}{k} \rfloor \geq (q+1) > \frac{m}{d}$, which implies $\lfloor \frac{\ell}{k} \rfloor \cdot d > m$.

Combining two cases, we have $\lfloor \frac{\ell}{k} \rfloor \cdot d \geq m$. □

Corollary 3.1. *If* $\gcd(\ell, n) = 1$ *and* $\ell > 1$, *or if* $\ell = 1$ *and* $k > 1$, *then* $\# \left(Z_{n,k}^{(m)}\right)^{[\ell]} = 0$.

Proof. Suppose $\gcd(\ell, n) = 1$ and $\ell \neq 1$, then $0 = \lfloor \frac{\ell}{k} \rfloor \geq \frac{m}{n} \geq 0$. By lemma 3.1, $\# \left(Z_{n,k}^{(m)}\right)^{[\ell]} = 0$.

On the other hand, suppose $\ell = 1$ and $k > 1$. Observed that the fixed points exist only when $(\alpha_1, \ldots, \alpha_n) = (\alpha_2, \ldots, \alpha_n, \alpha_1)$, which implies that all α_i are the same. This is impossible since $k > 1$, i.e. we have more than one ω^i and each of which appears at least m times in the tuple. Thus, $\# \left(Z_{n,k}^{(m)}\right)^{[1]} = 0$ in this case. □

Theorem 3.2. *Let* k, m, n *be positive integers with* $n > 1$. *If* k *is greater than all proper divisors of* n, *then* $\left(Z_{n,k}^{(m)},\ \mathbb{Z}_n,\ [n]_q \cdot \frac{\#\left(Z_{n,k}^{(m)}\right)}{n} \right)$ *exhibits the cyclic sieving phenomenon, where* $\mathbb{Z}_n$ *acts on* $Z_{n,k}^{(m)}$ *by rotation* $(a_1, \ldots, a_n) \mapsto (a_2, \ldots, a_n, a_1)$.

Proof. Let $f(q) = [n]_q \cdot \frac{\#\left(Z_{n,k}^{(m)}\right)}{n}$. We need to show that $\#\left(Z_{n,k}^{(m)}\right)^{[\ell]} = f(\zeta^\ell)$ for all ℓ, where $\zeta = exp\left(\frac{2\pi i}{n}\right)$. It suffices to check for $0 \le \ell < n$.

Case 1: $1 < \ell < n$.

Since k is greater than all proper divisors of n, then $k > \gcd(\ell, n)$, which implies that $\left\lfloor \frac{\gcd(\ell,n)}{k} \right\rfloor = 0$.

By lemma 3.1, since $0 = \left\lfloor \frac{\gcd(\ell,n)}{k} \right\rfloor < \frac{m \cdot \gcd(\ell,n)}{n}$, then $\left(Z_{n,k}^{(m)}\right)^{[\ell]} = 0$ for all $1 < \ell < n$.

Also, since $[n]_q = (q-\zeta)\cdot(q-\zeta^2)\cdots\left(q-\zeta^{n-1}\right)$, then $f(\zeta^\ell) = 0$ for all $1 < \ell < n$.

Thus, $\#\left(Z_{n,k}^{(m)}\right)^{[\ell]} = 0 = f(\zeta^\ell)$ for all $1 < \ell < n$.

Case 2: $\ell = 1$.

Since we are assuming k is greater than all proper divisions of n, then $k > 1$. By corollary 3.1, $\#\left(Z_{n,k}^{(m)}\right)^{[1]} = 0$ for $\ell = 1$ and $k > 1$.

Also, $f(\zeta) = 0$ since ζ is a root of $[n]_q$.

Thus, $\#\left(Z_{n,k}^{(m)}\right)^{[1]} = 0 = f(\zeta)$ in this case.

Case 3: $\ell = 0$.

Notice that

$$\#\left(Z_{n,k}^{(m)}\right)^{[0]} = \#Z_{n,k}^{(m)} = [n]_1 \cdot \frac{\#Z_{n,k}^{(m)}}{n} = f(1) = f(\zeta^0),$$

so $\#\left(Z_{n,k}^{(m)}\right)^{[\ell]} = f(\zeta^\ell)$ holds for $\ell = 0$.

We claim that the number $\frac{\#\left(Z_{n,k}^{(m)}\right)}{n}$ is an integer. Indeed, when $\mathbb{Z}_n$ acts on $Z_{n,k}^{(m)}$, let x_i be representatives of distinct orbits for $i = 1, \ldots, m$, then $\#Z_{n,k}^{(m)} = \sum_i \# \operatorname{Orb}(x_i)$. According to the Orbit-Stabilizer theorem, $\#\mathbb{Z}_n = \# \operatorname{Orb}(x_i) \cdot \# \operatorname{Stab}(x_i)$. In **Case 1** and **Case 2**, we conclude that $\left(\#Z_{n,k}^{(m)}\right)^{[\ell]} = 0$ for $\ell = 1, \ldots, n-1$, which implies that there is no stabilizer of all $x \in Z_{n,k}^{(m)}$ for $\ell = 1, \ldots, n-1$. Therefore, for all $x \in Z_{n,k}^{(m)}$, $\operatorname{Stab}(x) = \{[0]\}$, i.e. $\# \operatorname{Stab}(x_i) = 1$. Since $\#\mathbb{Z}_n = n$, then $\# \operatorname{Orb}(x_i) = n$ for all $i = 1, \ldots, m$, which follows that $\#Z_{n,k}^{(m)} = n \cdot \# \operatorname{Orb}(x_i)$. Hence, $[n]_q \cdot \frac{\#\left(Z_{n,k}^{(m)}\right)}{n}$ is a polynomial with integer coefficients.

Thus, the triple exhibits the CSP. $\square$

The Theorem 3.2 can be proved by Proposition 3.1 as well.

Proof. Since k is greater than all proper divisors of n, then all orbits of C_n acting on $Z_{n,k}^{(m)}$ have order n, and all stabilizer subgroups have order 1. Thus,

$$\begin{aligned} a_\ell &= \#\text{ of } C_n\text{-orbits s.t. the stabilizer-order divides } \ell \\ &= \frac{\left|Z_{n,k}^{(m)}\right|}{n} \end{aligned}$$

According to Proposition 3.1,

$$\begin{aligned} f(q) &= \sum_{\ell=0}^{n-1} a_\ell q^\ell \mod q^n - 1 \\ &= \sum_{\ell=0}^{n-1} \frac{\left|Z_{n,k}^{(m)}\right|}{n} \cdot q^\ell \\ &= \frac{\left|Z_{n,k}^{(m)}\right|}{n} \cdot [n]_q \end{aligned}$$

and the triple $\left(Z_{n,k}^{(m)}, f(q), C_n\right)$ exhibits CSP. $\square$

Example 17. For $n = 4$, $k = 2$, and $m = 1$, then the CSP does not hold since k isn't greater than all proper divisors of n.

Indeed, since $\#Z_{4,2}^{(1)} = 14$, then $f(q) = [n]_q \cdot \frac{\#\left(Z_{4,2}^{(1)}\right)}{n} = [4]_q \cdot \frac{14}{4}$, which is not an integer, so the CSP cannot hold.

This example shows that the hypothesis in the Theorem 3.2 is necessary.

The computation of a_ℓ in other cases and the examples of bicyclic sieving phenomenon for $Z_{n,k}^{(m)}$ are left as future work.

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