Homework 15 Solutions

14.1) (b) Consider the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$. Then, we have

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!}\frac{n!}{2^n} = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1.$$

Hence, the series converges absolutely by the Ratio Test.

(c) Consider the series $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$. Then, we have

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{n^2}{3^n}} = \lim_{n\to\infty} \frac{(\sqrt[n]{n})^2}{3} = \frac{1^2}{3} = \frac{1}{3} < 1.$$

Hence, the series converges absolutely by the Root Test.

(e) Consider the series $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2}$. Notice we have

$$-1 \leq \cos n \leq 1 \ \forall n \in \mathbb{N} \Rightarrow 0 \leq \cos^2 n \leq 1 \ \forall n \in \mathbb{N} \Rightarrow 0 \leq \frac{\cos^2 n}{n^2} \leq \frac{1}{n^2} \ \forall n \in \mathbb{N}.$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the P-series Test ($p = 2 > 1$), the original series converges by the Comparison Test.

(e) Consider the series $\sum_{n=2}^{\infty} \frac{1}{n}$. Notice we have

$$\ln n \leq n \ \forall n \in \mathbb{N} \Rightarrow \frac{1}{n} \leq \frac{1}{\ln n} \text{ for } n \geq 2.$$

Since the series $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges by the P-series Test ($p = 1 \leq 1$), the original series diverges by the Comparison Test.

14.5) Suppose $\sum a_n = A$ and $\sum b_n = B$.

(a) Let $\{s_n\}$ and $\{t_n\}$ be the sequences of partial sums for $\sum a_n = A$ and $\sum b_n = B$, respectively. Since the series both converge, we must have $\lim_{n\to\infty} s_n = A$ and $\lim_{n\to\infty} t_n = B$, so $\lim_{n\to\infty}(s_n + t_n) = A + B$. Since $\{s_n + t_n\}$ is the sequence of partial sums for $\sum(a_n + b_n)$, we conclude $\sum(a_n + b_n) = A + B$.

(b) Suppose $k \in \mathbb{R}$. Using the same notation as in part (a), we have $\lim_{n\to\infty} s_n = A$, so $\lim_{n\to\infty}(ks_n) = kA$. Since $\{ks_n\}$ is the sequence of partial sums for $\sum(ka_n)$, we conclude $\sum(ka_n) = kA$.

14.8) Suppose that $\sum a_n$ and $\sum b_n$ are convergent series of nonnegative numbers. From Worksheet 2 #7, we have

$$\sqrt{a_n b_n} \leq \frac{a_n + b_n}{2} \leq a_n + b_n \quad \forall n \in \mathbb{N}.$$

Since both $\sum a_n$ and $\sum b_n$ converges, we have $\sum(a_n + b_n)$ converges from problem 5a. Therefore, we conclude the series $\sum \sqrt{a_n b_n}$ converges by the Comparison Test.

Worksheet 7 Solutions

1) Let $c \neq 0$. Suppose $\sum_{n=1}^{\infty} ca_n$ does not diverge, so it must converge. Then,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{c}(ca_n).$$

converges by Homework 14.5b (above). Therefore, we conclude if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} ca_n$ diverges for $c \neq 0$, since $c \neq 0$ was arbitrary.

2) Suppose $\sum_{n=1}^{\infty} a_n$ diverges and $\sum_{n=1}^{\infty} b_n$ converges, but $\sum_{n=1}^{\infty}(a_n - b_n)$ converges. Then, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(a_n - b_n) + b_n$ must converge by Homework 14.5a (above). Contradiction!

Therefore, we conclude if $\sum_{n=1}^{\infty} a_n$ diverges and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty}(a_n - b_n)$ diverges.

3) Let $\sum_{n=1}^{\infty} a_n$ be a series and $\sum_{k=1}^{\infty} b_k$ be a series obtained from grouping terms in the series $\sum_{n=1}^{\infty} a_n$.

Suppose $\sum_{n=1}^{\infty} a_n$ converges. Let the sequence $\{n_k\}$ represent the indexes where each grouping ends. In particular, the sequences of terms $\{b_k\}$ will be defined as

$$b_1 = a_1 + a_2 + ... + a_{n_1} \text{ (i.e when } k = 1) \text{ and } b_k = a_{n_k+1} + a_{n_k+2} + ... + a_{n_{k+1}} \text{ for } k > 1.$$

So $\{n_k\}$ is a subsequence of the sequence of natural numbers $\{n\}$. Let $\{s_n\}$ be the sequence of partial sums for $\sum_{n=1}^{\infty} a_n$, then the subsequence $\{s_{n_k}\}$ is the sequence of partial sums for $\sum_{k=1}^{\infty} b_k$ (i.e. $b_k = a_{n_k}$ $\forall k \in \mathbb{N}$). Since $\sum_{n=1}^{\infty} a_n$ converges, let $\sum_{n=1}^{\infty} a_n = s$. Then, $\lim_{n\to\infty} s_n = s$, and we have $\lim_{k\to\infty} s_{n_k} = s$ by Theorem 11.2 since $\{s_{n_k}\}$ is a subsequence of $\{s_n\}$. Thus, $\sum_{k=1}^{\infty} b_k = s$.

Therefore, we conclude if $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{k=1}^{\infty} b_k$ converges to the same sum.

Note: The converse of this statement is FALSE. It is an optional homework to come up with a counterexample.

4) Let $m \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a_n$ be a series. Let $\{s_n\}$ and $\{t_n\}$ be the sequence of partial sums for $\sum_{n=1}^{\infty} a_n$ and

$\sum_{n=m+1}^{\infty} a_n$, respectively. Then, we have

$$s_{m+n} = (a_1 + a_2 + ... + a_m) + (a_{m+1} + a_{m+2} + ... + a_{m+n}) = s_m + t_n \quad \forall n \in \mathbb{N}. \tag{1}$$

($\Rightarrow$) Suppose $\sum_{n=1}^{\infty} a_n$ converges, and let $\sum_{n=1}^{\infty} a_n = s$. So we have $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} s_{n+m} = s$ as well. Hence, by (1) we obtain $\lim_{n\to\infty} t_n = \lim_{n\to\infty}(s_{n+m} - s_m) = s - s_m$, which is finite, and we conclude $\sum_{n=m+1}^{\infty} a_n$ converges.

($\Leftarrow$) Suppose $\sum_{n=m+1}^{\infty} a_n$ converges, and let $\sum_{n=m+1}^{\infty} a_n = t$. So we have $\lim_{n\to\infty} t_n = t$. Then by (1), we get $\lim_{n\to\infty} s_{n+m} = \lim_{n\to\infty}(s_m + t_n) = s_m + t$. Thus, $\lim_{n\to\infty} s_n = s_m + t$ since $\{s_{m+n}\}$ is the tail of $\{s_n\}$, and we conclude $\sum_{n=1}^{\infty} a_n$ converges. Moreover, we also conclude $\sum_{n=1}^{\infty} a_n = s_m + t = s_m + \sum_{n=m+1}^{\infty} a_n$.