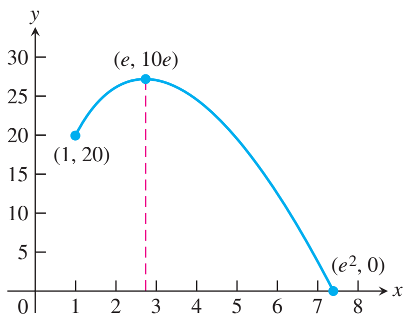



**FIGURE 4.8** The extreme values of $f(x) = 10x(2 - \ln x)$ on $[1, e^2]$ occur at $x = e$ and $x = e^2$ (Example 3).

**Solution** Figure 4.8 suggests that $f$ has its absolute maximum value near $x = 3$ and its absolute minimum value of 0 at $x = e^2$. Let's verify this observation.

We evaluate the function at the critical points and endpoints and take the largest and smallest of the resulting values.

The first derivative is

$$f'(x) = 10(2 - \ln x) - 10x\left(\frac{1}{x}\right) = 10(1 - \ln x).$$

The only critical point in the domain $[1, e^2]$ is the point $x = e$, where $\ln x = 1$. The values of $f$ at this one critical point and at the endpoints are

Critical point value: $f(e) = 10e$

Endpoint values: $f(1) = 10(2 - \ln 1) = 20$

$f(e^2) = 10e^2(2 - 2\ln e) = 0.$

We can see from this list that the function's absolute maximum value is $10e \approx 27.2$; it occurs at the critical interior point $x = e$. The absolute minimum value is 0 and occurs at the right endpoint $x = e^2$. ■

**EXAMPLE 4** Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

**Solution** We evaluate the function at the critical points and endpoints and take the largest and smallest of the resulting values.

The first derivative

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

has no zeros but is undefined at the interior point $x = 0$. The values of $f$ at this one critical point and at the endpoints are

Critical point value: $f(0) = 0$

Endpoint values: $f(-2) = (-2)^{2/3} = \sqrt[3]{4}$

$f(3) = (3)^{2/3} = \sqrt[3]{9}.$

We can see from this list that the function's absolute maximum value is $\sqrt[3]{9} \approx 2.08$, and it occurs at the right endpoint $x = 3$. The absolute minimum value is 0, and it occurs at the interior point $x = 0$ where the graph has a cusp (Figure 4.9). ■

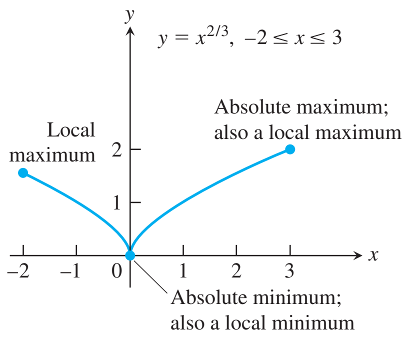


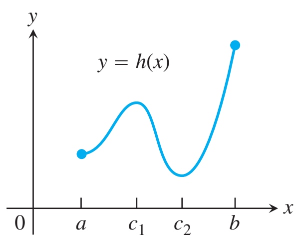

**FIGURE 4.9** The extreme values of $f(x) = x^{2/3}$ on $[-2, 3]$ occur at $x = 0$ and $x = 3$ (Example 4).

## Exercises 4.1

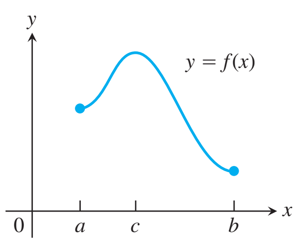
### Finding Extrema from Graphs

In Exercises 1–6, determine from the graph whether the function has any absolute extreme values on $[a, b]$. Then explain how your answer is consistent with Theorem 1.

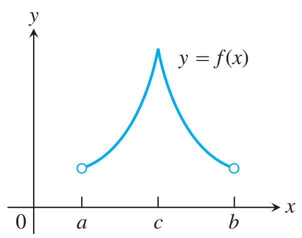
1.



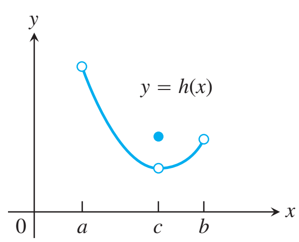

2.



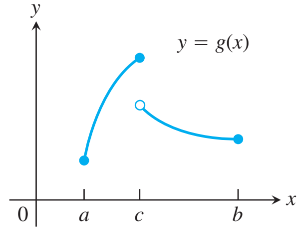

3.



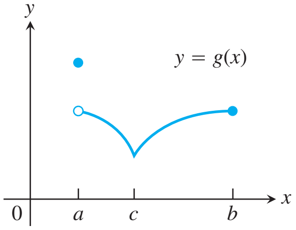

4.



**5.**

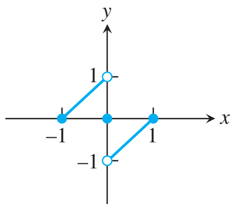



**6.**

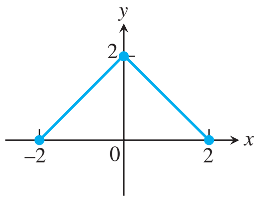



In Exercises 7–10, find the absolute extreme values and where they occur.

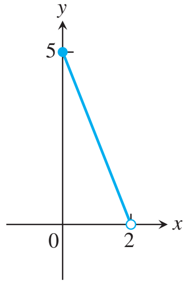
**7.**



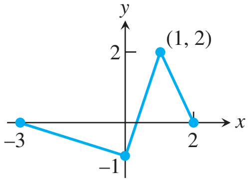

**8.**




**9.**




**10.**




In Exercises 11–14, match the table with a graph.

**11.**

| $x$ | $f'(x)$ |
|---|---|
| $a$ | 0 |
| $b$ | 0 |
| $c$ | 5 |

**12.**

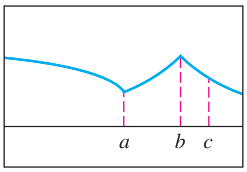
| $x$ | $f'(x)$ |
|---|---|
| $a$ | 0 |
| $b$ | 0 |
| $c$ | $-5$ |

**13.**

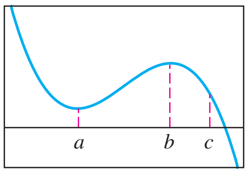
| $x$ | $f'(x)$ |
|---|---|
| $a$ | does not exist |
| $b$ | 0 |
| $c$ | $-2$ |

**14.**

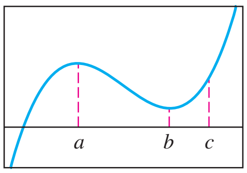
| $x$ | $f'(x)$ |
|---|---|
| $a$ | does not exist |
| $b$ | does not exist |
| $c$ | $-1.7$ |



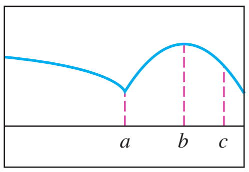

(a)




(b)




(c)




(d)

In Exercises 15–20, sketch the graph of each function and determine whether the function has any absolute extreme values on its domain. Explain how your answer is consistent with Theorem 1.

**15.** $f(x) = |x|, \quad -1 < x < 2$

**16.** $y = \dfrac{6}{x^2 + 2}, \quad -1 < x < 1$

**17.** $g(x) = \begin{cases} -x, & 0 \le x < 1 \\ x - 1, & 1 \le x \le 2 \end{cases}$

**18.** $h(x) = \begin{cases} \dfrac{1}{x}, & -1 \le x < 0 \\ \sqrt{x}, & 0 \le x \le 4 \end{cases}$

**19.** $y = 3 \sin x, \quad 0 < x < 2\pi$

**20.** $f(x) = \begin{cases} x + 1, & -1 \le x < 0 \\ \cos x, & 0 \le x \le \dfrac{\pi}{2} \end{cases}$

## Absolute Extrema on Finite Closed Intervals

In Exercises 21–40, find the absolute maximum and minimum values of each function on the given interval. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

**21.** $f(x) = \dfrac{2}{3}x - 5, \quad -2 \le x \le 3$

**22.** $f(x) = -x - 4, \quad -4 \le x \le 1$

**23.** $f(x) = x^2 - 1, \quad -1 \le x \le 2$

**24.** $f(x) = 4 - x^2, \quad -3 \le x \le 1$

**25.** $F(x) = -\dfrac{1}{x^2}, \quad 0.5 \le x \le 2$

**26.** $F(x) = -\dfrac{1}{x}, \quad -2 \le x \le -1$

**27.** $h(x) = \sqrt[3]{x}, \quad -1 \le x \le 8$

**28.** $h(x) = -3x^{2/3}, \quad -1 \le x \le 1$

**29.** $g(x) = \sqrt{4 - x^2}, \quad -2 \le x \le 1$

**30.** $g(x) = -\sqrt{5 - x^2}, \quad -\sqrt{5} \le x \le 0$

**31.** $f(\theta) = \sin \theta, \quad -\dfrac{\pi}{2} \le \theta \le \dfrac{5\pi}{6}$

**32.** $f(\theta) = \tan \theta, \quad -\dfrac{\pi}{3} \le \theta \le \dfrac{\pi}{4}$

**33.** $g(x) = \csc x, \quad \dfrac{\pi}{3} \le x \le \dfrac{2\pi}{3}$

**34.** $g(x) = \sec x, \quad -\dfrac{\pi}{3} \le x \le \dfrac{\pi}{6}$

**35.** $f(t) = 2 - |t|, \quad -1 \le t \le 3$

**36.** $f(t) = |t - 5|, \quad 4 \le t \le 7$

**37.** $g(x) = xe^{-x}, \quad -1 \le x \le 1$

**38.** $h(x) = \ln(x + 1), \quad 0 \le x \le 3$

**39.** $f(x) = \dfrac{1}{x} + \ln x, \quad 0.5 \le x \le 4$

**40.** $g(x) = e^{-x^2}, \quad -2 \le x \le 1$

In Exercises 41–44, find the function's absolute maximum and minimum values and say where they are assumed.

41. $f(x) = x^{4/3}, \quad -1 \le x \le 8$

42. $f(x) = x^{5/3}, \quad -1 \le x \le 8$

43. $g(\theta) = \theta^{3/5}, \quad -32 \le \theta \le 1$

44. $h(\theta) = 3\theta^{2/3}, \quad -27 \le \theta \le 8$

## Finding Critical Points

In Exercises 45–52, determine all critical points for each function.

45. $y = x^2 - 6x + 7$

46. $f(x) = 6x^2 - x^3$

47. $f(x) = x(4 - x)^3$

48. $g(x) = (x - 1)^2(x - 3)^2$

49. $y = x^2 + \dfrac{2}{x}$

50. $f(x) = \dfrac{x^2}{x - 2}$

51. $y = x^2 - 32\sqrt{x}$

52. $g(x) = \sqrt{2x - x^2}$

## Finding Extreme Values

In Exercises 53–68, find the extreme values (absolute and local) of the function and where they occur.

53. $y = 2x^2 - 8x + 9$

54. $y = x^3 - 2x + 4$

55. $y = x^3 + x^2 - 8x + 5$

56. $y = x^3(x - 5)^2$

57. $y = \sqrt{x^2 - 1}$

58. $y = x - 4\sqrt{x}$

59. $y = \dfrac{1}{\sqrt[3]{1 - x^2}}$

60. $y = \sqrt{3 + 2x - x^2}$

61. $y = \dfrac{x}{x^2 + 1}$

62. $y = \dfrac{x + 1}{x^2 + 2x + 2}$

63. $y = e^x + e^{-x}$

64. $y = e^x - e^{-x}$

65. $y = x \ln x$

66. $y = x^2 \ln x$

67. $y = \cos^{-1}(x^2)$

68. $y = \sin^{-1}(e^x)$

## Local Extrema and Critical Points

In Exercises 69–76, find the critical points, domain endpoints, and extreme values (absolute and local) for each function.

69. $y = x^{2/3}(x + 2)$

70. $y = x^{2/3}(x^2 - 4)$

71. $y = x\sqrt{4 - x^2}$

72. $y = x^2\sqrt{3 - x}$

73. $y = \begin{cases} 4 - 2x, & x \le 1 \\ x + 1, & x > 1 \end{cases}$

74. $y = \begin{cases} 3 - x, & x < 0 \\ 3 + 2x - x^2, & x \ge 0 \end{cases}$

75. $y = \begin{cases} -x^2 - 2x + 4, & x \le 1 \\ -x^2 + 6x - 4, & x > 1 \end{cases}$

76. $y = \begin{cases} -\frac{1}{4}x^2 - \frac{1}{2}x + \frac{15}{4}, & x \le 1 \\ x^3 - 6x^2 + 8x, & x > 1 \end{cases}$

In Exercises 77 and 78, give reasons for your answers.

77. Let $f(x) = (x - 2)^{2/3}$.
   **a.** Does $f'(2)$ exist?
   **b.** Show that the only local extreme value of $f$ occurs at $x = 2$.
   **c.** Does the result in part (b) contradict the Extreme Value Theorem?
   **d.** Repeat parts (a) and (b) for $f(x) = (x - a)^{2/3}$, replacing 2 by $a$.

78. Let $f(x) = |x^3 - 9x|$.
   **a.** Does $f'(0)$ exist?
   **b.** Does $f'(3)$ exist?
   **c.** Does $f'(-3)$ exist?
   **d.** Determine all extrema of $f$.

## Theory and Examples

79. **A minimum with no derivative** The function $f(x) = |x|$ has an absolute minimum value at $x = 0$ even though $f$ is not differentiable at $x = 0$. Is this consistent with Theorem 2? Give reasons for your answer.

80. **Even functions** If an even function $f(x)$ has a local maximum value at $x = c$, can anything be said about the value of $f$ at $x = -c$? Give reasons for your answer.

81. **Odd functions** If an odd function $g(x)$ has a local minimum value at $x = c$, can anything be said about the value of $g$ at $x = -c$? Give reasons for your answer.

82. We know how to find the extreme values of a continuous function $f(x)$ by investigating its values at critical points and endpoints. But what if there *are* no critical points or endpoints? What happens then? Do such functions really exist? Give reasons for your answers.

83. The function

$$V(x) = x(10 - 2x)(16 - 2x), \qquad 0 < x < 5,$$

models the volume of a box.
   **a.** Find the extreme values of $V$.
   **b.** Interpret any values found in part (a) in terms of the volume of the box.

84. **Cubic functions** Consider the cubic function

$$f(x) = ax^3 + bx^2 + cx + d.$$

   **a.** Show that $f$ can have 0, 1, or 2 critical points. Give examples and graphs to support your argument.
   **b.** How many local extreme values can $f$ have?

85. **Maximum height of a vertically moving body** The height of a body moving vertically is given by

$$s = -\frac{1}{2}gt^2 + v_0t + s_0, \qquad g > 0,$$

with $s$ in meters and $t$ in seconds. Find the body's maximum height.

86. **Peak alternating current** Suppose that at any given time $t$ (in seconds) the current $i$ (in amperes) in an alternating current circuit is $i = 2\cos t + 2\sin t$. What is the peak current for this circuit (largest magnitude)?

**T** Graph the functions in Exercises 87–90. Then find the extreme values of the function on the interval and say where they occur.

87. $f(x) = |x - 2| + |x + 3|, \quad -5 \le x \le 5$

88. $g(x) = |x - 1| - |x - 5|, \quad -2 \le x \le 7$

89. $h(x) = |x + 2| - |x - 3|, \quad -\infty < x < \infty$

90. $k(x) = |x + 1| + |x - 3|, \quad -\infty < x < \infty$

## COMPUTER EXPLORATIONS

In Exercises 91–98, you will use a CAS to help find the absolute extrema of the given function over the specified closed interval. Perform the following steps.

**a.** Plot the function over the interval to see its general behavior there.

**b.** Find the interior points where $f' = 0$. (In some exercises, you may have to use the numerical equation solver to approximate a solution.) You may want to plot $f'$ as well.

**c.** Find the interior points where $f'$ does not exist.

**d.** Evaluate the function at all points found in parts (b) and (c) and at the endpoints of the interval.

**e.** Find the function's absolute extreme values on the interval and identify where they occur.

**91.** $f(x) = x^4 - 8x^2 + 4x + 2, \quad [-20/25, 64/25]$

**92.** $f(x) = -x^4 + 4x^3 - 4x + 1, \quad [-3/4, 3]$

**93.** $f(x) = x^{2/3}(3 - x), \quad [-2, 2]$

**94.** $f(x) = 2 + 2x - 3x^{2/3}, \quad [-1, 10/3]$

**95.** $f(x) = \sqrt{x} + \cos x, \quad [0, 2\pi]$

**96.** $f(x) = x^{3/4} - \sin x + \frac{1}{2}, \quad [0, 2\pi]$

**97.** $f(x) = \pi x^2 e^{-3x/2}, \quad [0, 5]$

**98.** $f(x) = \ln(2x + x \sin x), \quad [1, 15]$

# 4.2 The Mean Value Theorem

We know that constant functions have zero derivatives, but could there be a more complicated function whose derivative is always zero? If two functions have identical derivatives over an interval, how are the functions related? We answer these and other questions in this chapter by applying the Mean Value Theorem. First we introduce a special case, known as Rolle's Theorem, which is used to prove the Mean Value Theorem.

## Rolle's Theorem

As suggested by its graph, if a differentiable function crosses a horizontal line at two different points, there is at least one point between them where the tangent to the graph is horizontal and the derivative is zero (Figure 4.10). We now state and prove this result.

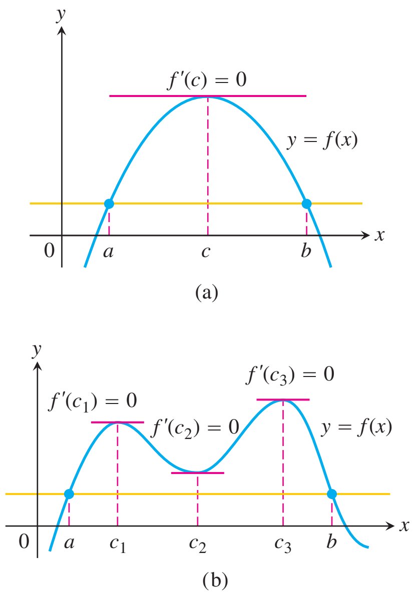



**FIGURE 4.10** Rolle's Theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line. It may have just one (a), or it may have more (b).

**THEOREM 3—Rolle's Theorem** Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior $(a, b)$. If $f(a) = f(b)$, then there is at least one number $c$ in $(a, b)$ at which $f'(c) = 0$.

**Proof** Being continuous, $f$ assumes absolute maximum and minimum values on $[a, b]$ by Theorem 1. These can occur only

1. at interior points where $f'$ is zero,
2. at interior points where $f'$ does not exist,
3. at the endpoints of the function's domain, in this case $a$ and $b$.

By hypothesis, $f$ has a derivative at every interior point. That rules out possibility (2), leaving us with interior points where $f' = 0$ and with the two endpoints $a$ and $b$.

If either the maximum or the minimum occurs at a point $c$ between $a$ and $b$, then $f'(c) = 0$ by Theorem 2 in Section 4.1, and we have found a point for Rolle's Theorem.

If both the absolute maximum and the absolute minimum occur at the endpoints, then because $f(a) = f(b)$ it must be the case that $f$ is a constant function with $f(x) = f(a) = f(b)$ for every $x \in [a, b]$. Therefore $f'(x) = 0$ and the point $c$ can be taken anywhere in the interior $(a, b)$. ■

HISTORICAL BIOGRAPHY

Michel Rolle
(1652–1719)

The hypotheses of Theorem 3 are essential. If they fail at even one point, the graph may not have a horizontal tangent (Figure 4.11).

Rolle's Theorem may be combined with the Intermediate Value Theorem to show when there is only one real solution of an equation $f(x) = 0$, as we illustrate in the next example.

**EXAMPLE 1** Show that the equation

$$x^3 + 3x + 1 = 0$$

has exactly one real solution.

# Exercises 4.2

## Checking the Mean Value Theorem

Find the value or values of $c$ that satisfy the equation

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

in the conclusion of the Mean Value Theorem for the functions and intervals in Exercises 1–8.

1. $f(x) = x^2 + 2x - 1, \quad [0, 1]$
2. $f(x) = x^{2/3}, \quad [0, 1]$
3. $f(x) = x + \frac{1}{x}, \quad \left[\frac{1}{2}, 2\right]$
4. $f(x) = \sqrt{x - 1}, \quad [1, 3]$
5. $f(x) = \sin^{-1} x, \quad [-1, 1]$
6. $f(x) = \ln(x - 1), \quad [2, 4]$
7. $f(x) = x^3 - x^2, \quad [-1, 2]$
8. $g(x) = \begin{cases} x^3, & -2 \le x \le 0 \\ x^2, & 0 < x \le 2 \end{cases}$

Which of the functions in Exercises 9–14 satisfy the hypotheses of the Mean Value Theorem on the given interval, and which do not? Give reasons for your answers.

9. $f(x) = x^{2/3}, \quad [-1, 8]$
10. $f(x) = x^{4/5}, \quad [0, 1]$
11. $f(x) = \sqrt{x(1 - x)}, \quad [0, 1]$
12. $f(x) = \begin{cases} \dfrac{\sin x}{x}, & -\pi \le x < 0 \\ 0, & x = 0 \end{cases}$
13. $f(x) = \begin{cases} x^2 - x, & -2 \le x \le -1 \\ 2x^2 - 3x - 3, & -1 < x \le 0 \end{cases}$
14. $f(x) = \begin{cases} 2x - 3, & 0 \le x \le 2 \\ 6x - x^2 - 7, & 2 < x \le 3 \end{cases}$
15. The function

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

is zero at $x = 0$ and $x = 1$ and differentiable on $(0, 1)$, but its derivative on $(0, 1)$ is never zero. How can this be? Doesn't Rolle's Theorem say the derivative has to be zero somewhere in $(0, 1)$? Give reasons for your answer.

16. For what values of $a$, $m$, and $b$ does the function

$$f(x) = \begin{cases} 3, & x = 0 \\ -x^2 + 3x + a, & 0 < x < 1 \\ mx + b, & 1 \le x \le 2 \end{cases}$$

satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 2]$?

## Roots (Zeros)

17. **a.** Plot the zeros of each polynomial on a line together with the zeros of its first derivative.
    **i)** $y = x^2 - 4$
    **ii)** $y = x^2 + 8x + 15$
    **iii)** $y = x^3 - 3x^2 + 4 = (x + 1)(x - 2)^2$
    **iv)** $y = x^3 - 33x^2 + 216x = x(x - 9)(x - 24)$

    **b.** Use Rolle's Theorem to prove that between every two zeros of $x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ there lies a zero of

$$nx^{n-1} + (n - 1)a_{n-1}x^{n-2} + \cdots + a_1.$$

18. Suppose that $f''$ is continuous on $[a, b]$ and that $f$ has three zeros in the interval. Show that $f''$ has at least one zero in $(a, b)$. Generalize this result.
19. Show that if $f'' > 0$ throughout an interval $[a, b]$, then $f'$ has at most one zero in $[a, b]$. What if $f'' < 0$ throughout $[a, b]$ instead?
20. Show that a cubic polynomial can have at most three real zeros.

Show that the functions in Exercises 21–28 have exactly one zero in the given interval.

21. $f(x) = x^4 + 3x + 1, \quad [-2, -1]$
22. $f(x) = x^3 + \frac{4}{x^2} + 7, \quad (-\infty, 0)$
23. $g(t) = \sqrt{t} + \sqrt{1 + t} - 4, \quad (0, \infty)$
24. $g(t) = \frac{1}{1 - t} + \sqrt{1 + t} - 3.1, \quad (-1, 1)$
25. $r(\theta) = \theta + \sin^2\left(\frac{\theta}{3}\right) - 8, \quad (-\infty, \infty)$
26. $r(\theta) = 2\theta - \cos^2\theta + \sqrt{2}, \quad (-\infty, \infty)$
27. $r(\theta) = \sec\theta - \frac{1}{\theta^3} + 5, \quad (0, \pi/2)$
28. $r(\theta) = \tan\theta - \cot\theta - \theta, \quad (0, \pi/2)$

## Finding Functions from Derivatives

29. Suppose that $f(-1) = 3$ and that $f'(x) = 0$ for all $x$. Must $f(x) = 3$ for all $x$? Give reasons for your answer.
30. Suppose that $f(0) = 5$ and that $f'(x) = 2$ for all $x$. Must $f(x) = 2x + 5$ for all $x$? Give reasons for your answer.
31. Suppose that $f'(x) = 2x$ for all $x$. Find $f(2)$ if
    **a.** $f(0) = 0$ **b.** $f(1) = 0$ **c.** $f(-2) = 3$.
32. What can be said about functions whose derivatives are constant? Give reasons for your answer.

In Exercises 33–38, find all possible functions with the given derivative.

33. **a.** $y' = x$ **b.** $y' = x^2$ **c.** $y' = x^3$
34. **a.** $y' = 2x$ **b.** $y' = 2x - 1$ **c.** $y' = 3x^2 + 2x - 1$
35. **a.** $y' = -\frac{1}{x^2}$ **b.** $y' = 1 - \frac{1}{x^2}$ **c.** $y' = 5 + \frac{1}{x^2}$

**36.** **a.** $y' = \frac{1}{2\sqrt{x}}$ **b.** $y' = \frac{1}{\sqrt{x}}$ **c.** $y' = 4x - \frac{1}{\sqrt{x}}$

**37.** **a.** $y' = \sin 2t$ **b.** $y' = \cos\frac{t}{2}$ **c.** $y' = \sin 2t + \cos\frac{t}{2}$

**38.** **a.** $y' = \sec^2\theta$ **b.** $y' = \sqrt{\theta}$ **c.** $y' = \sqrt{\theta} - \sec^2\theta$

In Exercises 39–42, find the function with the given derivative whose graph passes through the point $P$.

**39.** $f'(x) = 2x - 1, \quad P(0, 0)$

**40.** $g'(x) = \frac{1}{x^2} + 2x, \quad P(-1, 1)$

**41.** $f'(x) = e^{2x}, \quad P\left(0, \frac{3}{2}\right)$

**42.** $r'(t) = \sec t \tan t - 1, \quad P(0, 0)$

## Finding Position from Velocity or Acceleration

Exercises 43–46 give the velocity $v = ds/dt$ and initial position of a body moving along a coordinate line. Find the body's position at time $t$.

**43.** $v = 9.8t + 5, \quad s(0) = 10$

**44.** $v = 32t - 2, \quad s(0.5) = 4$

**45.** $v = \sin \pi t, \quad s(0) = 0$

**46.** $v = \frac{2}{\pi}\cos\frac{2t}{\pi}, \quad s(\pi^2) = 1$

Exercises 47–50 give the acceleration $a = d^2s/dt^2$, initial velocity, and initial position of a body moving on a coordinate line. Find the body's position at time $t$.

**47.** $a = e^t, \quad v(0) = 20, \quad s(0) = 5$

**48.** $a = 9.8, \quad v(0) = -3, \quad s(0) = 0$

**49.** $a = -4\sin 2t, \quad v(0) = 2, \quad s(0) = -3$

**50.** $a = \frac{9}{\pi^2}\cos\frac{3t}{\pi}, \quad v(0) = 0, \quad s(0) = -1$

## Applications

**51.** **Temperature change** It took 14 sec for a mercury thermometer to rise from −19°C to 100°C when it was taken from a freezer and placed in boiling water. Show that somewhere along the way the mercury was rising at the rate of 8.5°C/sec.

**52.** A trucker handed in a ticket at a toll booth showing that in 2 hours she had covered 159 mi on a toll road with speed limit 65 mph. The trucker was cited for speeding. Why?

**53.** Classical accounts tell us that a 170-oar trireme (ancient Greek or Roman warship) once covered 184 sea miles in 24 hours. Explain why at some point during this feat the trireme's speed exceeded 7.5 knots (sea miles per hour).

**54.** A marathoner ran the 26.2-mi New York City Marathon in 2.2 hours. Show that at least twice the marathoner was running at exactly 11 mph, assuming the initial and final speeds are zero.

**55.** Show that at some instant during a 2-hour automobile trip the car's speedometer reading will equal the average speed for the trip.

**56.** **Free fall on the moon** On our moon, the acceleration of gravity is 1.6 m/sec$^2$. If a rock is dropped into a crevasse, how fast will it be going just before it hits bottom 30 sec later?

## Theory and Examples

**57.** **The geometric mean of *a* and *b*** The *geometric mean* of two positive numbers $a$ and $b$ is the number $\sqrt{ab}$. Show that the value of $c$ in the conclusion of the Mean Value Theorem for $f(x) = 1/x$ on an interval of positive numbers $[a, b]$ is $c = \sqrt{ab}$.

**58.** **The arithmetic mean of *a* and *b*** The *arithmetic mean* of two numbers $a$ and $b$ is the number $(a + b)/2$. Show that the value of $c$ in the conclusion of the Mean Value Theorem for $f(x) = x^2$ on any interval $[a, b]$ is $c = (a + b)/2$.

T **59.** Graph the function

$$f(x) = \sin x \sin(x + 2) - \sin^2(x + 1).$$

What does the graph do? Why does the function behave this way? Give reasons for your answers.

**60.** **Rolle's Theorem**

**a.** Construct a polynomial $f(x)$ that has zeros at $x = -2, -1, 0, 1$, and $2$.

**b.** Graph $f$ and its derivative $f'$ together. How is what you see related to Rolle's Theorem?

**c.** Do $g(x) = \sin x$ and its derivative $g'$ illustrate the same phenomenon as $f$ and $f'$?

**61.** **Unique solution** Assume that $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Also assume that $f(a)$ and $f(b)$ have opposite signs and that $f' \neq 0$ between $a$ and $b$. Show that $f(x) = 0$ exactly once between $a$ and $b$.

**62.** **Parallel tangents** Assume that $f$ and $g$ are differentiable on $[a, b]$ and that $f(a) = g(a)$ and $f(b) = g(b)$. Show that there is at least one point between $a$ and $b$ where the tangents to the graphs of $f$ and $g$ are parallel or the same line. Illustrate with a sketch.

**63.** Suppose that $f'(x) \leq 1$ for $1 \leq x \leq 4$. Show that $f(4) - f(1) \leq 3$.

**64.** Suppose that $0 < f'(x) < 1/2$ for all $x$-values. Show that $f(-1) < f(1) < 2 + f(-1)$.

**65.** Show that $|\cos x - 1| \leq |x|$ for all $x$-values. (*Hint:* Consider $f(t) = \cos t$ on $[0, x]$.)

**66.** Show that for any numbers $a$ and $b$, the sine inequality $|\sin b - \sin a| \leq |b - a|$ is true.

**67.** If the graphs of two differentiable functions $f(x)$ and $g(x)$ start at the same point in the plane and the functions have the same rate of change at every point, do the graphs have to be identical? Give reasons for your answer.

**68.** If $|f(w) - f(x)| \leq |w - x|$ for all values $w$ and $x$ and $f$ is a differentiable function, show that $-1 \leq f'(x) \leq 1$ for all $x$-values.

**69.** Assume that $f$ is differentiable on $a \leq x \leq b$ and that $f(b) < f(a)$. Show that $f'$ is negative at some point between $a$ and $b$.

**70.** Let $f$ be a function defined on an interval $[a, b]$. What conditions could you place on $f$ to guarantee that

$$\min f' \leq \frac{f(b) - f(a)}{b - a} \leq \max f',$$

where $\min f'$ and $\max f'$ refer to the minimum and maximum values of $f'$ on $[a, b]$? Give reasons for your answers.

T **71.** Use the inequalities in Exercise 70 to estimate $f(0.1)$ if $f'(x) = 1/(1 + x^4 \cos x)$ for $0 \le x \le 0.1$ and $f(0) = 1$.

T **72.** Use the inequalities in Exercise 70 to estimate $f(0.1)$ if $f'(x) = 1/(1 - x^4)$ for $0 \le x \le 0.1$ and $f(0) = 2$.

**73.** Let $f$ be differentiable at every value of $x$ and suppose that $f(1) = 1$, that $f' < 0$ on $(-\infty, 1)$, and that $f' > 0$ on $(1, \infty)$.

**a.** Show that $f(x) \ge 1$ for all $x$.

**b.** Must $f'(1) = 0$? Explain.

**74.** Let $f(x) = px^2 + qx + r$ be a quadratic function defined on a closed interval $[a, b]$. Show that there is exactly one point $c$ in $(a, b)$ at which $f$ satisfies the conclusion of the Mean Value Theorem.

**75.** Use the same-derivative argument, as was done to prove the Product and Power Rules for logarithms, to prove the Quotient Rule property.

**76.** Use the same-derivative argument to prove the identities

**a.** $\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}$ **b.** $\sec^{-1} x + \csc^{-1} x = \dfrac{\pi}{2}$

**77.** Starting with the equation $e^{x_1}e^{x_2} = e^{x_1 + x_2}$, derived in the text, show that $e^{-x} = 1/e^x$ for any real number $x$. Then show that $e^{x_1}/e^{x_2} = e^{x_1 - x_2}$ for any numbers $x_1$ and $x_2$.

**78.** Show that $(e^{x_1})^{x_2} = e^{x_1 x_2} = (e^{x_2})^{x_1}$ for any numbers $x_1$ and $x_2$.

# 4.3 Monotonic Functions and the First Derivative Test

In sketching the graph of a differentiable function it is useful to know where it increases (rises from left to right) and where it decreases (falls from left to right) over an interval. This section gives a test to determine where it increases and where it decreases. We also show how to test the critical points of a function to identify whether local extreme values are present.

## Increasing Functions and Decreasing Functions

As another corollary to the Mean Value Theorem, we show that functions with positive derivatives are increasing functions and functions with negative derivatives are decreasing functions. A function that is increasing or decreasing on an interval is said to be **monotonic** on the interval.

**COROLLARY 3** Suppose that $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$.

If $f'(x) > 0$ at each point $x \in (a, b)$, then $f$ is increasing on $[a, b]$.
If $f'(x) < 0$ at each point $x \in (a, b)$, then $f$ is decreasing on $[a, b]$.

**Proof** Let $x_1$ and $x_2$ be any two points in $[a, b]$ with $x_1 < x_2$. The Mean Value Theorem applied to $f$ on $[x_1, x_2]$ says that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

for some $c$ between $x_1$ and $x_2$. The sign of the right-hand side of this equation is the same as the sign of $f'(c)$ because $x_2 - x_1$ is positive. Therefore, $f(x_2) > f(x_1)$ if $f'$ is positive on $(a, b)$ and $f(x_2) < f(x_1)$ if $f'$ is negative on $(a, b)$. ■

Corollary 3 is valid for infinite as well as finite intervals. To find the intervals where a function $f$ is increasing or decreasing, we first find all of the critical points of $f$. If $a < b$ are two critical points for $f$, and if the derivative $f'$ is continuous but never zero on the interval $(a, b)$, then by the Intermediate Value Theorem applied to $f'$, the derivative must be everywhere positive on $(a, b)$, or everywhere negative there. One way we can determine the sign of $f'$ on $(a, b)$ is simply by evaluating the derivative at a single point $c$ in $(a, b)$. If $f'(c) > 0$, then $f'(x) > 0$ for all $x$ in $(a, b)$ so $f$ is increasing on $[a, b]$ by Corollary 3; if $f'(c) < 0$, then $f$ is decreasing on $[a, b]$. The next example illustrates how we use this procedure.

**EXAMPLE 3** Find the critical points of

$$f(x) = (x^2 - 3)e^x.$$

Identify the intervals on which $f$ is increasing and decreasing. Find the function's local and absolute extreme values.

**Solution** The function $f$ is continuous and differentiable for all real numbers, so the critical points occur only at the zeros of $f'$.

Using the Derivative Product Rule, we find the derivative

$$\begin{aligned} f'(x) &= (x^2 - 3) \cdot \frac{d}{dx} e^x + \frac{d}{dx}(x^2 - 3) \cdot e^x \\ &= (x^2 - 3) \cdot e^x + (2x) \cdot e^x \\ &= (x^2 + 2x - 3)e^x. \end{aligned}$$

Since $e^x$ is never zero, the first derivative is zero if and only if

$$x^2 + 2x - 3 = 0$$

$$(x + 3)(x - 1) = 0.$$

The zeros $x = -3$ and $x = 1$ partition the $x$-axis into intervals as follows.

| **Interval** | $x < -3$ | $-3 < x < 1$ | $1 < x$ |
|---|---|---|---|
| **Sign of $f'$** | $+$ | $-$ | $+$ |
| **Behavior of $f$** | increasing | decreasing | increasing |

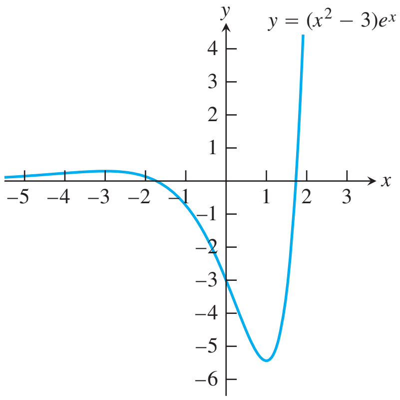



**FIGURE 4.23** The graph of $f(x) = (x^2 - 3)e^x$ (Example 3).

We can see from the table that there is a local maximum (about 0.299) at $x = -3$ and a local minimum (about $-5.437$) at $x = 1$. The local minimum value is also an absolute minimum because $f(x) > 0$ for $|x| > \sqrt{3}$. There is no absolute maximum. The function increases on $(-\infty, -3)$ and $(1, \infty)$ and decreases on $(-3, 1)$. Figure 4.23 shows the graph. ■

# Exercises 4.3

### Analyzing Functions from Derivatives

Answer the following questions about the functions whose derivatives are given in Exercises 1–14:

**a.** What are the critical points of $f$?

**b.** On what intervals is $f$ increasing or decreasing?

**c.** At what points, if any, does $f$ assume local maximum and minimum values?

**1.** $f'(x) = x(x - 1)$ **2.** $f'(x) = (x - 1)(x + 2)$

**3.** $f'(x) = (x - 1)^2(x + 2)$ **4.** $f'(x) = (x - 1)^2(x + 2)^2$

**5.** $f'(x) = (x - 1)e^{-x}$

**6.** $f'(x) = (x - 7)(x + 1)(x + 5)$

**7.** $f'(x) = \dfrac{x^2(x - 1)}{x + 2}, \quad x \neq -2$

**8.** $f'(x) = \dfrac{(x - 2)(x + 4)}{(x + 1)(x - 3)}, \quad x \neq -1, 3$

**9.** $f'(x) = 1 - \dfrac{4}{x^2}, \quad x \neq 0$ **10.** $f'(x) = 3 - \dfrac{6}{\sqrt{x}}, \quad x \neq 0$

**11.** $f'(x) = x^{-1/3}(x + 2)$ **12.** $f'(x) = x^{-1/2}(x - 3)$

**13.** $f'(x) = (\sin x - 1)(2\cos x + 1), 0 \le x \le 2\pi$

**14.** $f'(x) = (\sin x + \cos x)(\sin x - \cos x), 0 \le x \le 2\pi$

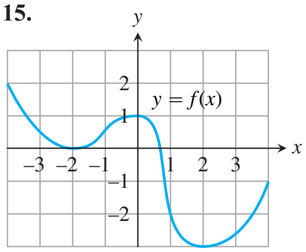
### Identifying Extrema

In Exercises 15–44:

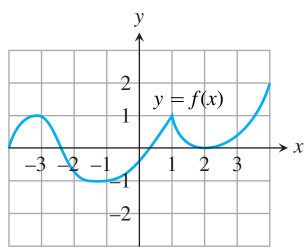
**a.** Find the open intervals on which the function is increasing and decreasing.

**b.** Identify the function's local and absolute extreme values, if any, saying where they occur.

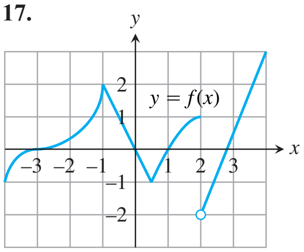
**15.**



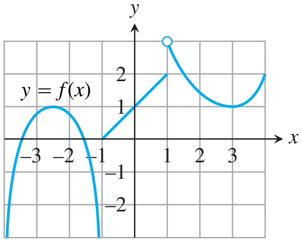

**16.**



**17.**




**18.**




**19.** $g(t) = -t^2 - 3t + 3$

**20.** $g(t) = -3t^2 + 9t + 5$

**21.** $h(x) = -x^3 + 2x^2$

**22.** $h(x) = 2x^3 - 18x$

**23.** $f(\theta) = 3\theta^2 - 4\theta^3$

**24.** $f(\theta) = 6\theta - \theta^3$

**25.** $f(r) = 3r^3 + 16r$

**26.** $h(r) = (r + 7)^3$

**27.** $f(x) = x^4 - 8x^2 + 16$

**28.** $g(x) = x^4 - 4x^3 + 4x^2$

**29.** $H(t) = \frac{3}{2}t^4 - t^6$

**30.** $K(t) = 15t^3 - t^5$

**31.** $f(x) = x - 6\sqrt{x - 1}$

**32.** $g(x) = 4\sqrt{x} - x^2 + 3$

**33.** $g(x) = x\sqrt{8 - x^2}$

**34.** $g(x) = x^2\sqrt{5 - x}$

**35.** $f(x) = \dfrac{x^2 - 3}{x - 2}, \quad x \neq 2$

**36.** $f(x) = \dfrac{x^3}{3x^2 + 1}$

**37.** $f(x) = x^{1/3}(x + 8)$

**38.** $g(x) = x^{2/3}(x + 5)$

**39.** $h(x) = x^{1/3}(x^2 - 4)$

**40.** $k(x) = x^{2/3}(x^2 - 4)$

**41.** $f(x) = e^{2x} + e^{-x}$

**42.** $f(x) = e^{\sqrt{x}}$

**43.** $f(x) = x \ln x$

**44.** $f(x) = x^2 \ln x$

In Exercises 45–56:

**a.** Identify the function's local extreme values in the given domain, and say where they occur.

**b.** Which of the extreme values, if any, are absolute?

T **c.** Support your findings with a graphing calculator or computer grapher.

**45.** $f(x) = 2x - x^2, \quad -\infty < x \leq 2$

**46.** $f(x) = (x + 1)^2, \quad -\infty < x \leq 0$

**47.** $g(x) = x^2 - 4x + 4, \quad 1 \leq x < \infty$

**48.** $g(x) = -x^2 - 6x - 9, \quad -4 \leq x < \infty$

**49.** $f(t) = 12t - t^3, \quad -3 \leq t < \infty$

**50.** $f(t) = t^3 - 3t^2, \quad -\infty < t \leq 3$

**51.** $h(x) = \dfrac{x^3}{3} - 2x^2 + 4x, \quad 0 \leq x < \infty$

**52.** $k(x) = x^3 + 3x^2 + 3x + 1, \quad -\infty < x \leq 0$

**53.** $f(x) = \sqrt{25 - x^2}, \quad -5 \leq x \leq 5$

**54.** $f(x) = \sqrt{x^2 - 2x - 3}, \quad 3 \leq x < \infty$

**55.** $g(x) = \dfrac{x - 2}{x^2 - 1}, \quad 0 \leq x < 1$

**56.** $g(x) = \dfrac{x^2}{4 - x^2}, \quad -2 < x \leq 1$

In Exercises 57–64:

**a.** Find the local extrema of each function on the given interval, and say where they occur.

T **b.** Graph the function and its derivative together. Comment on the behavior of $f$ in relation to the signs and values of $f'$.

**57.** $f(x) = \sin 2x, \quad 0 \leq x \leq \pi$

**58.** $f(x) = \sin x - \cos x, \quad 0 \leq x \leq 2\pi$

**59.** $f(x) = \sqrt{3}\cos x + \sin x, \quad 0 \leq x \leq 2\pi$

**60.** $f(x) = -2x + \tan x, \quad \dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$

**61.** $f(x) = \dfrac{x}{2} - 2\sin\dfrac{x}{2}, \quad 0 \leq x \leq 2\pi$

**62.** $f(x) = -2\cos x - \cos^2 x, \quad -\pi \leq x \leq \pi$

**63.** $f(x) = \csc^2 x - 2\cot x, \quad 0 < x < \pi$

**64.** $f(x) = \sec^2 x - 2\tan x, \quad \dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$

## Theory and Examples

Show that the functions in Exercises 65 and 66 have local extreme values at the given values of $\theta$, and say which kind of local extreme the function has.

**65.** $h(\theta) = 3\cos\dfrac{\theta}{2}, \quad 0 \leq \theta \leq 2\pi, \quad$ at $\theta = 0$ and $\theta = 2\pi$

**66.** $h(\theta) = 5\sin\dfrac{\theta}{2}, \quad 0 \leq \theta \leq \pi, \quad$ at $\theta = 0$ and $\theta = \pi$

**67.** Sketch the graph of a differentiable function $y = f(x)$ through the point $(1, 1)$ if $f'(1) = 0$ and

**a.** $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$;

**b.** $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$;

**c.** $f'(x) > 0$ for $x \neq 1$;

**d.** $f'(x) < 0$ for $x \neq 1$.

**68.** Sketch the graph of a differentiable function $y = f(x)$ that has

**a.** a local minimum at $(1, 1)$ and a local maximum at $(3, 3)$;

**b.** a local maximum at $(1, 1)$ and a local minimum at $(3, 3)$;

**c.** local maxima at $(1, 1)$ and $(3, 3)$;

**d.** local minima at $(1, 1)$ and $(3, 3)$.

**69.** Sketch the graph of a continuous function $y = g(x)$ such that

**a.** $g(2) = 2$, $0 < g' < 1$ for $x < 2$, $g'(x) \to 1^-$ as $x \to 2^-$, $-1 < g' < 0$ for $x > 2$, and $g'(x) \to -1^+$ as $x \to 2^+$;

**b.** $g(2) = 2$, $g' < 0$ for $x < 2$, $g'(x) \to -\infty$ as $x \to 2^-$, $g' > 0$ for $x > 2$, and $g'(x) \to \infty$ as $x \to 2^+$.

**70.** Sketch the graph of a continuous function $y = h(x)$ such that

**a.** $h(0) = 0$, $-2 \leq h(x) \leq 2$ for all $x$, $h'(x) \to \infty$ as $x \to 0^-$, and $h'(x) \to \infty$ as $x \to 0^+$;

**b.** $h(0) = 0$, $-2 \leq h(x) \leq 0$ for all $x$, $h'(x) \to \infty$ as $x \to 0^-$, and $h'(x) \to -\infty$ as $x \to 0^+$.

**71.** Discuss the extreme-value behavior of the function $f(x) = x\sin(1/x)$, $x \neq 0$. How many critical points does this function have? Where are they located on the $x$-axis? Does $f$ have an absolute minimum? An absolute maximum? (See Exercise 49 in Section 2.3.)

**72.** Find the intervals on which the function $f(x) = ax^2 + bx + c$, $a \neq 0$, is increasing and decreasing. Describe the reasoning behind your answer.

**73.** Determine the values of constants $a$ and $b$ so that $f(x) = ax^2 + bx$ has an absolute maximum at the point $(1, 2)$.

**74.** Determine the values of constants $a$, $b$, $c$, and $d$ so that $f(x) = ax^3 + bx^2 + cx + d$ has a local maximum at the point $(0, 0)$ and a local minimum at the point $(1, -1)$.

**75.** Locate and identify the absolute extreme values of

**a.** $\ln(\cos x)$ on $[-\pi/4, \pi/3]$,

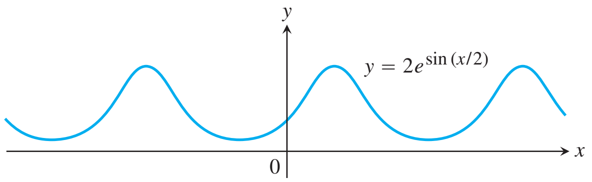
**b.** $\cos(\ln x)$ on $[1/2, 2]$.

**76.** **a.** Prove that $f(x) = x - \ln x$ is increasing for $x > 1$.

**b.** Using part (a), show that $\ln x < x$ if $x > 1$.

**77.** Find the absolute maximum and minimum values of $f(x) = e^x - 2x$ on $[0, 1]$.

**78.** Where does the periodic function $f(x) = 2e^{\sin(x/2)}$ take on its extreme values and what are these values?




**79.** Find the absolute maximum value of $f(x) = x^2 \ln(1/x)$ and say where it is assumed.

**80.** **a.** Prove that $e^x \ge 1 + x$ if $x \ge 0$.

**b.** Use the result in part (a) to show that

$$e^x \ge 1 + x + \frac{1}{2}x^2.$$

**81.** Show that increasing functions and decreasing functions are one-to-one. That is, show that for any $x_1$ and $x_2$ in $I$, $x_2 \ne x_1$ implies $f(x_2) \ne f(x_1)$.

Use the results of Exercise 81 to show that the functions in Exercises 82–86 have inverses over their domains. Find a formula for $df^{-1}/dx$ using Theorem 3, Section 3.8.

**82.** $f(x) = (1/3)x + (5/6)$

**83.** $f(x) = 27x^3$

**84.** $f(x) = 1 - 8x^3$

**85.** $f(x) = (1 - x)^3$

**86.** $f(x) = x^{5/3}$

# 4.4 Concavity and Curve Sketching

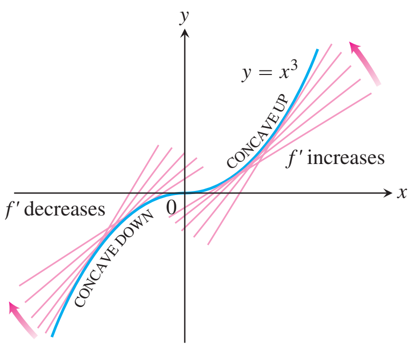



**FIGURE 4.24** The graph of $f(x) = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$ (Example 1a).

We have seen how the first derivative tells us where a function is increasing, where it is decreasing, and whether a local maximum or local minimum occurs at a critical point. In this section we see that the second derivative gives us information about how the graph of a differentiable function bends or turns. With this knowledge about the first and second derivatives, coupled with our previous understanding of asymptotic behavior and symmetry studied in Sections 2.6 and 1.1, we can now draw an accurate graph of a function. By organizing all of these ideas into a coherent procedure, we give a method for sketching graphs and revealing visually the key features of functions. Identifying and knowing the locations of these features is of major importance in mathematics and its applications to science and engineering, especially in the graphical analysis and interpretation of data.

## Concavity

As you can see in Figure 4.24, the curve $y = x^3$ rises as $x$ increases, but the portions defined on the intervals $(-\infty, 0)$ and $(0, \infty)$ turn in different ways. As we approach the origin from the left along the curve, the curve turns to our right and falls below its tangents. The slopes of the tangents are decreasing on the interval $(-\infty, 0)$. As we move away from the origin along the curve to the right, the curve turns to our left and rises above its tangents. The slopes of the tangents are increasing on the interval $(0, \infty)$. This turning or bending behavior defines the *concavity* of the curve.

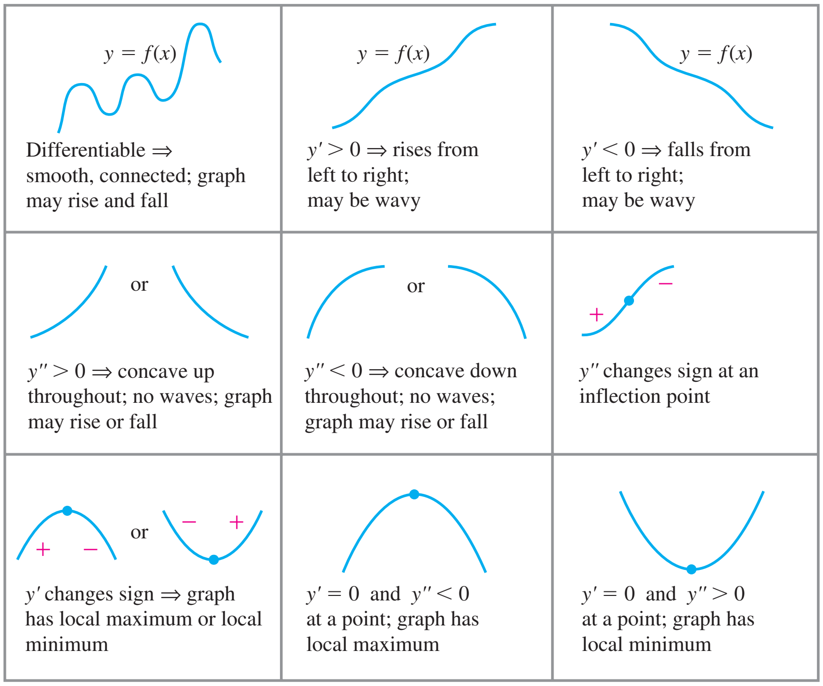
**DEFINITION** The graph of a differentiable function $y = f(x)$ is

**(a)** **concave up** on an open interval $I$ if $f'$ is increasing on $I$;

**(b)** **concave down** on an open interval $I$ if $f'$ is decreasing on $I$.

If $y = f(x)$ has a second derivative, we can apply Corollary 3 of the Mean Value Theorem to the first derivative function. We conclude that $f'$ increases if $f'' > 0$ on $I$, and decreases if $f'' < 0$.

figure summarizes how the derivative and second derivative affect the shape of a graph.

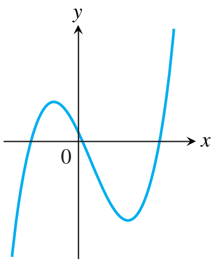



## Exercises 4.4

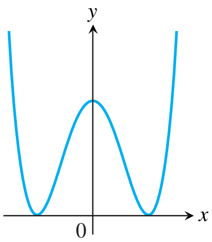
### Analyzing Functions from Graphs

Identify the inflection points and local maxima and minima of the functions graphed in Exercises 1–8. Identify the intervals on which the functions are concave up and concave down.

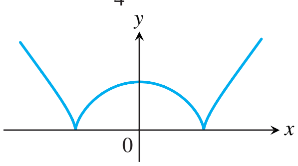
**1.** $y = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{3}$



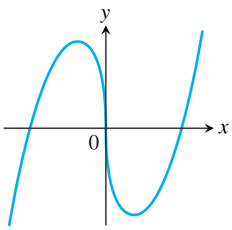

**2.** $y = \frac{x^4}{4} - 2x^2 + 4$



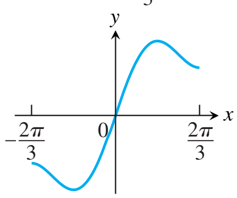

**3.** $y = \frac{3}{4}(x^2 - 1)^{2/3}$



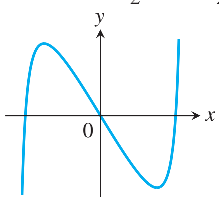

**4.** $y = \frac{9}{14}x^{1/3}(x^2 - 7)$



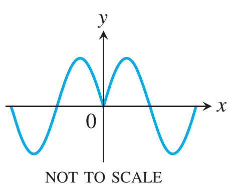

**5.** $y = x + \sin 2x, -\frac{2\pi}{3} \le x \le \frac{2\pi}{3}$



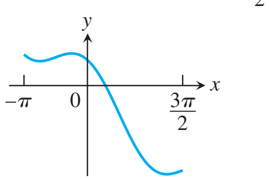

**6.** $y = \tan x - 4x, -\frac{\pi}{2} < x < \frac{\pi}{2}$




**7.** $y = \sin|x|, -2\pi \le x \le 2\pi$




**8.** $y = 2\cos x - \sqrt{2}x, -\pi \le x \le \frac{3\pi}{2}$




### Graphing Equations

Use the steps of the graphing procedure on page 248 to graph the equations in Exercises 9–58. Include the coordinates of any local and absolute extreme points and inflection points.

**9.** $y = x^2 - 4x + 3$

**10.** $y = 6 - 2x - x^2$

**11.** $y = x^3 - 3x + 3$

**12.** $y = x(6 - 2x)^2$

**13.** $y = -2x^3 + 6x^2 - 3$

**14.** $y = 1 - 9x - 6x^2 - x^3$

**15.** $y = (x - 2)^3 + 1$

**16.** $y = 1 - (x + 1)^3$

**17.** $y = x^4 - 2x^2 = x^2(x^2 - 2)$

**18.** $y = -x^4 + 6x^2 - 4 = x^2(6 - x^2) - 4$

**19.** $y = 4x^3 - x^4 = x^3(4 - x)$

**20.** $y = x^4 + 2x^3 = x^3(x + 2)$

**21.** $y = x^5 - 5x^4 = x^4(x - 5)$

**22.** $y = x\left(\frac{x}{2} - 5\right)^4$

**23.** $y = x + \sin x, \quad 0 \le x \le 2\pi$

**24.** $y = x - \sin x, \quad 0 \le x \le 2\pi$

**25.** $y = \sqrt{3}x - 2\cos x, \quad 0 \le x \le 2\pi$

**26.** $y = \frac{4}{3}x - \tan x, \quad \frac{-\pi}{2} < x < \frac{\pi}{2}$

**27.** $y = \sin x \cos x, \quad 0 \le x \le \pi$

**28.** $y = \cos x + \sqrt{3}\sin x, \quad 0 \le x \le 2\pi$

**29.** $y = x^{1/5}$

**30.** $y = x^{2/5}$

**31.** $y = \dfrac{x}{\sqrt{x^2 + 1}}$

**32.** $y = \dfrac{\sqrt{1 - x^2}}{2x + 1}$

**33.** $y = 2x - 3x^{2/3}$

**34.** $y = 5x^{2/5} - 2x$

**35.** $y = x^{2/3}\left(\frac{5}{2} - x\right)$

**36.** $y = x^{2/3}(x - 5)$

**37.** $y = x\sqrt{8 - x^2}$

**38.** $y = (2 - x^2)^{3/2}$

**39.** $y = \sqrt{16 - x^2}$

**40.** $y = x^2 + \dfrac{2}{x}$

**41.** $y = \dfrac{x^2 - 3}{x - 2}$

**42.** $y = \sqrt[3]{x^3 + 1}$

**43.** $y = \dfrac{8x}{x^2 + 4}$

**44.** $y = \dfrac{5}{x^4 + 5}$

**45.** $y = |x^2 - 1|$

**46.** $y = |x^2 - 2x|$

**47.** $y = \sqrt{|x|} = \begin{cases} \sqrt{-x}, & x < 0 \\ \sqrt{x}, & x \ge 0 \end{cases}$

**48.** $y = \sqrt{|x - 4|}$

**49.** $y = xe^{1/x}$

**50.** $y = \dfrac{e^x}{x}$

**51.** $y = \ln(3 - x^2)$

**52.** $y = x(\ln x)^2$

**53.** $y = e^x - 2e^{-x} - 3x$

**54.** $y = xe^{-x}$

**55.** $y = \ln(\cos x)$

**56.** $y = \dfrac{\ln x}{\sqrt{x}}$

**57.** $y = \dfrac{1}{1 + e^{-x}}$

**58.** $y = \dfrac{e^x}{1 + e^x}$

## Sketching the General Shape, Knowing $y'$

Each of Exercises 59–80 gives the first derivative of a continuous function $y = f(x)$. Find $y''$ and then use steps 2–4 of the graphing procedure on page 248 to sketch the general shape of the graph of $f$.

**59.** $y' = 2 + x - x^2$

**60.** $y' = x^2 - x - 6$

**61.** $y' = x(x - 3)^2$

**62.** $y' = x^2(2 - x)$

**63.** $y' = x(x^2 - 12)$

**64.** $y' = (x - 1)^2(2x + 3)$

**65.** $y' = (8x - 5x^2)(4 - x)^2$

**66.** $y' = (x^2 - 2x)(x - 5)^2$

**67.** $y' = \sec^2 x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

**68.** $y' = \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

**69.** $y' = \cot\frac{\theta}{2}, \quad 0 < \theta < 2\pi$

**70.** $y' = \csc^2\frac{\theta}{2}, \quad 0 < \theta < 2\pi$

**71.** $y' = \tan^2\theta - 1, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

**72.** $y' = 1 - \cot^2\theta, \quad 0 < \theta < \pi$

**73.** $y' = \cos t, \quad 0 \le t \le 2\pi$

**74.** $y' = \sin t, \quad 0 \le t \le 2\pi$

**75.** $y' = (x + 1)^{-2/3}$

**76.** $y' = (x - 2)^{-1/3}$

**77.** $y' = x^{-2/3}(x - 1)$

**78.** $y' = x^{-4/5}(x + 1)$

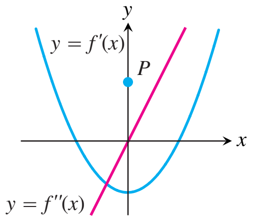
**79.** $y' = 2|x| = \begin{cases} -2x, & x \le 0 \\ 2x, & x > 0 \end{cases}$

**80.** $y' = \begin{cases} -x^2, & x \le 0 \\ x^2, & x > 0 \end{cases}$

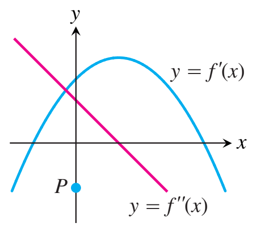
## Sketching $y$ from Graphs of $y'$ and $y''$

Each of Exercises 81–84 shows the graphs of the first and second derivatives of a function $y = f(x)$. Copy the picture and add to it a sketch of the approximate graph of $f$, given that the graph passes through the point $P$.

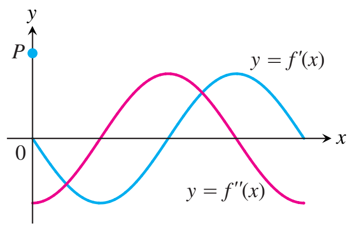
**81.**



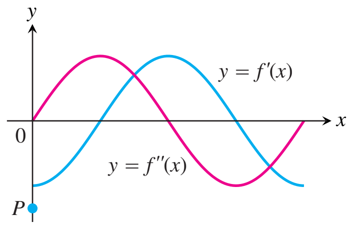

**82.**




**83.**




**84.**




## Graphing Rational Functions

Graph the rational functions in Exercises 85–102.

**85.** $y = \dfrac{2x^2 + x - 1}{x^2 - 1}$

**86.** $y = \dfrac{x^2 - 49}{x^2 + 5x - 14}$

**87.** $y = \dfrac{x^4 + 1}{x^2}$

**88.** $y = \dfrac{x^2 - 4}{2x}$

**89.** $y = \dfrac{1}{x^2 - 1}$

**90.** $y = \dfrac{x^2}{x^2 - 1}$

**91.** $y = -\dfrac{x^2 - 2}{x^2 - 1}$ **92.** $y = \dfrac{x^2 - 4}{x^2 - 2}$

**93.** $y = \dfrac{x^2}{x + 1}$ **94.** $y = -\dfrac{x^2 - 4}{x + 1}$

**95.** $y = \dfrac{x^2 - x + 1}{x - 1}$ **96.** $y = -\dfrac{x^2 - x + 1}{x - 1}$

**97.** $y = \dfrac{x^3 - 3x^2 + 3x - 1}{x^2 + x - 2}$ **98.** $y = \dfrac{x^3 + x - 2}{x - x^2}$

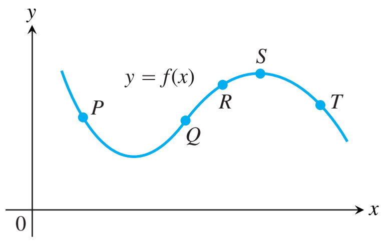
**99.** $y = \dfrac{x}{x^2 - 1}$ **100.** $y = \dfrac{x - 1}{x^2(x - 2)}$

**101.** $y = \dfrac{8}{x^2 + 4}$ (Agnesi's witch)

**102.** $y = \dfrac{4x}{x^2 + 4}$ (Newton's serpentine)

## Theory and Examples

**103.** The accompanying figure shows a portion of the graph of a twice-differentiable function $y = f(x)$. At each of the five labeled points, classify $y'$ and $y''$ as positive, negative, or zero.




**104.** Sketch a smooth connected curve $y = f(x)$ with

$f(-2) = 8,$ $f'(2) = f'(-2) = 0,$

$f(0) = 4,$ $f'(x) < 0$ for $|x| < 2,$

$f(2) = 0,$ $f''(x) < 0$ for $x < 0,$

$f'(x) > 0$ for $|x| > 2,$ $f''(x) > 0$ for $x > 0.$

**105.** Sketch the graph of a twice-differentiable function $y = f(x)$ with the following properties. Label coordinates where possible.

| $x$ | $y$ | Derivatives |
|---|---|---|
| $x < 2$ | | $y' < 0,\ y'' > 0$ |
| 2 | 1 | $y' = 0,\ y'' > 0$ |
| $2 < x < 4$ | | $y' > 0,\ y'' > 0$ |
| 4 | 4 | $y' > 0,\ y'' = 0$ |
| $4 < x < 6$ | | $y' > 0,\ y'' < 0$ |
| 6 | 7 | $y' = 0,\ y'' < 0$ |
| $x > 6$ | | $y' < 0,\ y'' < 0$ |

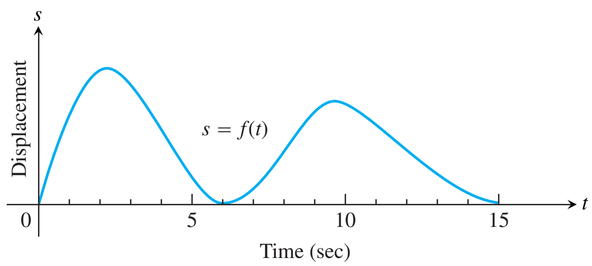
**106.** Sketch the graph of a twice-differentiable function $y = f(x)$ that passes through the points $(-2, 2)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$, and $(2, 2)$ and whose first two derivatives have the following sign patterns.

$y'$: + (before $-2$), − (between $-2$ and $0$), + (between $0$ and $2$), − (after $2$)

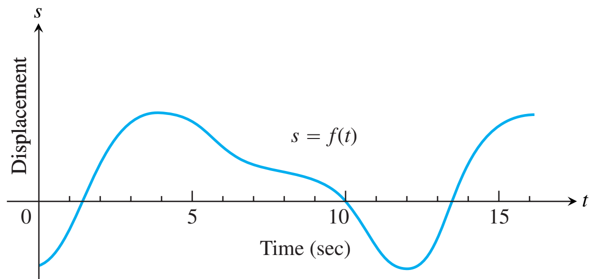
$y''$: − (before $-1$), + (between $-1$ and $1$), − (after $1$)

**Motion Along a Line** The graphs in Exercises 107 and 108 show the position $s = f(t)$ of an object moving up and down on a coordinate line. **(a)** When is the object moving away from the origin? toward the origin? At approximately what times is the **(b)** velocity equal to zero? **(c)** acceleration equal to zero? **(d)** When is the acceleration positive? negative?

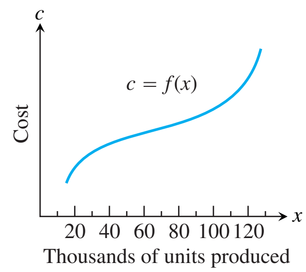
**107.**



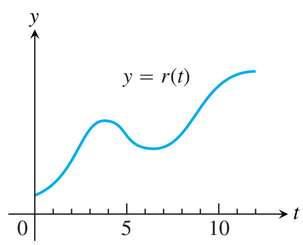

**108.**




**109. Marginal cost** The accompanying graph shows the hypothetical cost $c = f(x)$ of manufacturing $x$ items. At approximately what production level does the marginal cost change from decreasing to increasing?




**110.** The accompanying graph shows the monthly revenue of the Widget Corporation for the last 12 years. During approximately what time intervals was the marginal revenue increasing? Decreasing?




**111.** Suppose the derivative of the function $y = f(x)$ is

$$y' = (x - 1)^2(x - 2).$$

At what points, if any, does the graph of $f$ have a local minimum, local maximum, or point of inflection? (*Hint:* Draw the sign pattern for $y'$.)

**112.** Suppose the derivative of the function $y = f(x)$ is

$$y' = (x-1)^2(x-2)(x-4).$$

At what points, if any, does the graph of $f$ have a local minimum, local maximum, or point of inflection?

**113.** For $x > 0$, sketch a curve $y = f(x)$ that has $f(1) = 0$ and $f'(x) = 1/x$. Can anything be said about the concavity of such a curve? Give reasons for your answer.

**114.** Can anything be said about the graph of a function $y = f(x)$ that has a continuous second derivative that is never zero? Give reasons for your answer.

**115.** If $b$, $c$, and $d$ are constants, for what value of $b$ will the curve $y = x^3 + bx^2 + cx + d$ have a point of inflection at $x = 1$? Give reasons for your answer.

**116. Parabolas**

**a.** Find the coordinates of the vertex of the parabola $y = ax^2 + bx + c, a \neq 0$.

**b.** When is the parabola concave up? Concave down? Give reasons for your answers.

**117. Quadratic curves** What can you say about the inflection points of a quadratic curve $y = ax^2 + bx + c, a \neq 0$? Give reasons for your answer.

**118. Cubic curves** What can you say about the inflection points of a cubic curve $y = ax^3 + bx^2 + cx + d, a \neq 0$? Give reasons for your answer.

**119.** Suppose that the second derivative of the function $y = f(x)$ is

$$y'' = (x+1)(x-2).$$

For what $x$-values does the graph of $f$ have an inflection point?

**120.** Suppose that the second derivative of the function $y = f(x)$ is

$$y'' = x^2(x-2)^3(x+3).$$

For what $x$-values does the graph of $f$ have an inflection point?

**121.** Find the values of constants $a$, $b$, and $c$ so that the graph of $y = ax^3 + bx^2 + cx$ has a local maximum at $x = 3$, local minimum at $x = -1$, and inflection point at $(1, 11)$.

**122.** Find the values of constants $a$, $b$, and $c$ so that the graph of $y = (x^2 + a)/(bx + c)$ has a local minimum at $x = 3$ and a local maximum at $(-1, -2)$.

**COMPUTER EXPLORATIONS**

In Exercises 123–126, find the inflection points (if any) on the graph of the function and the coordinates of the points on the graph where the function has a local maximum or local minimum value. Then graph the function in a region large enough to show all these points simultaneously. Add to your picture the graphs of the function's first and second derivatives. How are the values at which these graphs intersect the $x$-axis related to the graph of the function? In what other ways are the graphs of the derivatives related to the graph of the function?

**123.** $y = x^5 - 5x^4 - 240$ **124.** $y = x^3 - 12x^2$

**125.** $y = \frac{4}{5}x^5 + 16x^2 - 25$

**126.** $y = \frac{x^4}{4} - \frac{x^3}{3} - 4x^2 + 12x + 20$

**127.** Graph $f(x) = 2x^4 - 4x^2 + 1$ and its first two derivatives together. Comment on the behavior of $f$ in relation to the signs and values of $f'$ and $f''$.

**128.** Graph $f(x) = x\cos x$ and its second derivative together for $0 \leq x \leq 2\pi$. Comment on the behavior of the graph of $f$ in relation to the signs and values of $f''$.

## 4.5 Indeterminate Forms and L'Hôpital's Rule

HISTORICAL BIOGRAPHY

Guillaume François Antoine de l'Hôpital (1661–1704)
Johann Bernoulli (1667–1748)

John (Johann) Bernoulli discovered a rule using derivatives to calculate limits of fractions whose numerators and denominators both approach zero or $+\infty$. The rule is known today as **l'Hôpital's Rule**, after Guillaume de l'Hôpital. He was a French nobleman who wrote the first introductory differential calculus text, where the rule first appeared in print. Limits involving transcendental functions often require some use of the rule for their calculation.

### Indeterminate Form 0/0

If we want to know how the function

$$F(x) = \frac{x - \sin x}{x^3}$$

behaves *near* $x = 0$ (where it is undefined), we can examine the limit of $F(x)$ as $x \to 0$. We cannot apply the Quotient Rule for limits (Theorem 1 of Chapter 2) because the limit of the denominator is 0. Moreover, in this case, *both* the numerator and denominator approach 0, and $0/0$ is undefined. Such limits may or may not exist in general, but the limit does exist for the function $F(x)$ under discussion by applying l'Hôpital's Rule, as we will see in Example 1d.

# Exercises 4.5

### Finding Limits in Two Ways

In Exercises 1–6, use l'Hôpital's Rule to evaluate the limit. Then evaluate the limit using a method studied in Chapter 2.

1. $\lim_{x\to -2} \dfrac{x+2}{x^2-4}$

2. $\lim_{x\to 0} \dfrac{\sin 5x}{x}$

3. $\lim_{x\to\infty} \dfrac{5x^2-3x}{7x^2+1}$

4. $\lim_{x\to 1} \dfrac{x^3-1}{4x^3-x-3}$

5. $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$

6. $\lim_{x\to\infty} \dfrac{2x^2+3x}{x^3+x+1}$

### Applying l'Hôpital's Rule

Use l'Hôpital's rule to find the limits in Exercises 7–50.

7. $\lim_{x\to 2} \dfrac{x-2}{x^2-4}$

8. $\lim_{x\to -5} \dfrac{x^2-25}{x+5}$

9. $\lim_{t\to -3} \dfrac{t^3-4t+15}{t^2-t-12}$

10. $\lim_{t\to 1} \dfrac{3t^3-3}{4t^3-t-3}$

11. $\lim_{x\to\infty} \dfrac{5x^3-2x}{7x^3+3}$

12. $\lim_{x\to\infty} \dfrac{x-8x^2}{12x^2+5x}$

13. $\lim_{t\to 0} \dfrac{\sin t^2}{t}$

14. $\lim_{t\to 0} \dfrac{\sin 5t}{2t}$

15. $\lim_{x\to 0} \dfrac{8x^2}{\cos x-1}$

16. $\lim_{x\to 0} \dfrac{\sin x - x}{x^3}$

17. $\lim_{\theta\to\pi/2} \dfrac{2\theta-\pi}{\cos(2\pi-\theta)}$

18. $\lim_{\theta\to -\pi/3} \dfrac{3\theta+\pi}{\sin(\theta+(\pi/3))}$

19. $\lim_{\theta\to\pi/2} \dfrac{1-\sin\theta}{1+\cos 2\theta}$

20. $\lim_{x\to 1} \dfrac{x-1}{\ln x - \sin\pi x}$

21. $\lim_{x\to 0} \dfrac{x^2}{\ln(\sec x)}$

22. $\lim_{x\to\pi/2} \dfrac{\ln(\csc x)}{(x-(\pi/2))^2}$

23. $\lim_{t\to 0} \dfrac{t(1-\cos t)}{t-\sin t}$

24. $\lim_{t\to 0} \dfrac{t\sin t}{1-\cos t}$

25. $\lim_{x\to(\pi/2)^-} \left(x-\dfrac{\pi}{2}\right)\sec x$

26. $\lim_{x\to(\pi/2)^-} \left(\dfrac{\pi}{2}-x\right)\tan x$

27. $\lim_{\theta\to 0} \dfrac{3^{\sin\theta}-1}{\theta}$

28. $\lim_{\theta\to 0} \dfrac{(1/2)^{\theta}-1}{\theta}$

29. $\lim_{x\to 0} \dfrac{x2^x}{2^x-1}$

30. $\lim_{x\to 0} \dfrac{3^x-1}{2^x-1}$

31. $\lim_{x\to\infty} \dfrac{\ln(x+1)}{\log_2 x}$

32. $\lim_{x\to\infty} \dfrac{\log_2 x}{\log_3(x+3)}$

33. $\lim_{x\to 0^+} \dfrac{\ln(x^2+2x)}{\ln x}$

34. $\lim_{x\to 0^+} \dfrac{\ln(e^x-1)}{\ln x}$

35. $\lim_{y\to 0} \dfrac{\sqrt{5y+25}-5}{y}$

36. $\lim_{y\to 0} \dfrac{\sqrt{ay+a^2}-a}{y}, \quad a>0$

37. $\lim_{x\to\infty} (\ln 2x - \ln(x+1))$

38. $\lim_{x\to 0^+} (\ln x - \ln\sin x)$

39. $\lim_{x\to 0^+} \dfrac{(\ln x)^2}{\ln(\sin x)}$

40. $\lim_{x\to 0^+} \left(\dfrac{3x+1}{x}-\dfrac{1}{\sin x}\right)$

41. $\lim_{x\to 1^+} \left(\dfrac{1}{x-1}-\dfrac{1}{\ln x}\right)$

42. $\lim_{x\to 0^+} (\csc x - \cot x + \cos x)$

43. $\lim_{\theta\to 0} \dfrac{\cos\theta-1}{e^\theta-\theta-1}$

44. $\lim_{h\to 0} \dfrac{e^h-(1+h)}{h^2}$

45. $\lim_{t\to\infty} \dfrac{e^t+t^2}{e^t-t}$

46. $\lim_{x\to\infty} x^2e^{-x}$

47. $\lim_{x\to 0} \dfrac{x-\sin x}{x\tan x}$

48. $\lim_{x\to 0} \dfrac{(e^x-1)^2}{x\sin x}$

49. $\lim_{\theta\to 0} \dfrac{\theta-\sin\theta\cos\theta}{\tan\theta-\theta}$

50. $\lim_{x\to 0} \dfrac{\sin 3x - 3x + x^2}{\sin x \sin 2x}$

### Indeterminate Powers and Products

Find the limits in Exercise 51–66.

51. $\lim_{x\to 1^+} x^{1/(1-x)}$

52. $\lim_{x\to 1^+} x^{1/(x-1)}$

53. $\lim_{x\to\infty} (\ln x)^{1/x}$

54. $\lim_{x\to e^+} (\ln x)^{1/(x-e)}$

55. $\lim_{x\to 0^+} x^{-1/\ln x}$

56. $\lim_{x\to\infty} x^{1/\ln x}$

57. $\lim_{x\to\infty} (1+2x)^{1/(2\ln x)}$

58. $\lim_{x\to 0} (e^x+x)^{1/x}$

59. $\lim_{x\to 0^+} x^x$

60. $\lim_{x\to 0^+} \left(1+\dfrac{1}{x}\right)^x$

61. $\lim_{x\to\infty} \left(\dfrac{x+2}{x-1}\right)^x$

62. $\lim_{x\to\infty} \left(\dfrac{x^2+1}{x+2}\right)^{1/x}$

63. $\lim_{x\to 0^+} x^2\ln x$

64. $\lim_{x\to 0^+} x(\ln x)^2$

65. $\lim_{x\to 0^+} x\tan\left(\dfrac{\pi}{2}-x\right)$

66. $\lim_{x\to 0^+} \sin x \cdot \ln x$

### Theory and Applications

L'Hôpital's Rule does not help with the limits in Exercises 67–74. Try it—you just keep on cycling. Find the limits some other way.

67. $\lim_{x\to\infty} \dfrac{\sqrt{9x+1}}{\sqrt{x+1}}$

68. $\lim_{x\to 0^+} \dfrac{\sqrt{x}}{\sqrt{\sin x}}$

69. $\lim_{x\to(\pi/2)^-} \dfrac{\sec x}{\tan x}$

70. $\lim_{x\to 0^+} \dfrac{\cot x}{\csc x}$

71. $\lim_{x\to\infty} \dfrac{2^x-3^x}{3^x+4^x}$

72. $\lim_{x\to -\infty} \dfrac{2^x+4^x}{5^x-2^x}$

73. $\lim_{x\to\infty} \dfrac{e^{x^2}}{xe^x}$

74. $\lim_{x\to 0^+} \dfrac{x}{e^{-1/x}}$

75. Which one is correct, and which one is wrong? Give reasons for your answers.

 a. $\lim_{x\to 3} \dfrac{x-3}{x^2-3} = \lim_{x\to 3} \dfrac{1}{2x} = \dfrac{1}{6}$ b. $\lim_{x\to 3} \dfrac{x-3}{x^2-3} = \dfrac{0}{6} = 0$

76. Which one is correct, and which one is wrong? Give reasons for your answers.

 a. $$\lim_{x\to 0} \frac{x^2-2x}{x^2-\sin x} = \lim_{x\to 0} \frac{2x-2}{2x-\cos x} = \lim_{x\to 0} \frac{2}{2+\sin x} = \frac{2}{2+0} = 1$$

 b. $$\lim_{x\to 0} \frac{x^2-2x}{x^2-\sin x} = \lim_{x\to 0} \frac{2x-2}{2x-\cos x} = \frac{-2}{0-1} = 2$$

**77.** Only one of these calculations is correct. Which one? Why are the others wrong? Give reasons for your answers.

**a.** $\lim_{x\to 0^+} x\ln x = 0\cdot(-\infty) = 0$

**b.** $\lim_{x\to 0^+} x\ln x = 0\cdot(-\infty) = -\infty$

**c.** $\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \dfrac{\ln x}{(1/x)} = \dfrac{-\infty}{\infty} = -1$

**d.** $\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \dfrac{\ln x}{(1/x)}$

$$= \lim_{x\to 0^+} \frac{(1/x)}{(-1/x^2)} = \lim_{x\to 0^+} (-x) = 0$$

**78.** Find all values of $c$ that satisfy the conclusion of Cauchy's Mean Value Theorem for the given functions and interval.

**a.** $f(x) = x,\quad g(x) = x^2,\quad (a, b) = (-2, 0)$

**b.** $f(x) = x,\quad g(x) = x^2,\quad (a, b)$ arbitrary

**c.** $f(x) = x^3/3 - 4x,\quad g(x) = x^2,\quad (a, b) = (0, 3)$

**79. Continuous extension** Find a value of $c$ that makes the function

$$f(x) = \begin{cases} \dfrac{9x - 3\sin 3x}{5x^3}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

continuous at $x = 0$. Explain why your value of $c$ works.

**80.** For what values of $a$ and $b$ is

$$\lim_{x\to 0}\left(\frac{\tan 2x}{x^3} + \frac{a}{x^2} + \frac{\sin bx}{x}\right) = 0?$$

T **81. $\infty - \infty$ Form**

**a.** Estimate the value of

$$\lim_{x\to\infty}\left(x - \sqrt{x^2 + x}\right)$$

by graphing $f(x) = x - \sqrt{x^2 + x}$ over a suitably large interval of $x$-values.

**b.** Now confirm your estimate by finding the limit with l'Hôpital's Rule. As the first step, multiply $f(x)$ by the fraction $(x + \sqrt{x^2 + x})/(x + \sqrt{x^2 + x})$ and simplify the new numerator.

**82.** Find $\lim_{x\to\infty}\left(\sqrt{x^2 + 1} - \sqrt{x}\right)$.

T **83. 0/0 Form** Estimate the value of

$$\lim_{x\to 1}\frac{2x^2 - (3x + 1)\sqrt{x} + 2}{x - 1}$$

by graphing. Then confirm your estimate with l'Hôpital's Rule.

**84.** This exercise explores the difference between the limit

$$\lim_{x\to\infty}\left(1 + \frac{1}{x^2}\right)^x$$

and the limit

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e.$$

**a.** Use l'Hôpital's Rule to show that

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e.$$

T **b.** Graph

$$f(x) = \left(1 + \frac{1}{x^2}\right)^x \quad\text{and}\quad g(x) = \left(1 + \frac{1}{x}\right)^x$$

together for $x \geq 0$. How does the behavior of $f$ compare with that of $g$? Estimate the value of $\lim_{x\to\infty} f(x)$.

**c.** Confirm your estimate of $\lim_{x\to\infty} f(x)$ by calculating it with l'Hôpital's Rule.

**85.** Show that

$$\lim_{k\to\infty}\left(1 + \frac{r}{k}\right)^k = e^r.$$

**86.** Given that $x > 0$, find the maximum value, if any, of

**a.** $x^{1/x}$

**b.** $x^{1/x^2}$

**c.** $x^{1/x^n}$ ($n$ a positive integer)

**d.** Show that $\lim_{x\to\infty} x^{1/x^n} = 1$ for every positive integer $n$.

**87.** Use limits to find horizontal asymptotes for each function.

**a.** $y = x\tan\left(\dfrac{1}{x}\right)$ **b.** $y = \dfrac{3x + e^{2x}}{2x + e^{3x}}$

**88.** Find $f'(0)$ for $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$

T **89. The continuous extension of $(\sin x)^x$ to $[0, \pi]$**

**a.** Graph $f(x) = (\sin x)^x$ on the interval $0 \leq x \leq \pi$. What value would you assign to $f$ to make it continuous at $x = 0$?

**b.** Verify your conclusion in part (a) by finding $\lim_{x\to 0^+} f(x)$ with l'Hôpital's Rule.

**c.** Returning to the graph, estimate the maximum value of $f$ on $[0, \pi]$. About where is max $f$ taken on?

**d.** Sharpen your estimate in part (c) by graphing $f'$ in the same window to see where its graph crosses the $x$-axis. To simplify your work, you might want to delete the exponential factor from the expression for $f'$ and graph just the factor that has a zero.

T **90. The function $(\sin x)^{\tan x}$** (*Continuation of Exercise 89.*)

**a.** Graph $f(x) = (\sin x)^{\tan x}$ on the interval $-7 \leq x \leq 7$. How do you account for the gaps in the graph? How wide are the gaps?

**b.** Now graph $f$ on the interval $0 \leq x \leq \pi$. The function is not defined at $x = \pi/2$, but the graph has no break at this point. What is going on? What value does the graph appear to give for $f$ at $x = \pi/2$? (*Hint:* Use l'Hôpital's Rule to find lim $f$ as $x \to (\pi/2)^-$ and $x \to (\pi/2)^+$.)

**c.** Continuing with the graphs in part (b), find max $f$ and min $f$ as accurately as you can and estimate the values of $x$ at which they are taken on.

**EXAMPLE 5** Suppose that $r(x) = 9x$ and $c(x) = x^3 - 6x^2 + 15x$, where $x$ represents millions of MP3 players produced. Is there a production level that maximizes profit? If so, what is it?

**Solution** Notice that $r'(x) = 9$ and $c'(x) = 3x^2 - 12x + 15$.

$$3x^2 - 12x + 15 = 9 \qquad \text{Set } c'(x) = r'(x).$$
$$3x^2 - 12x + 6 = 0$$

The two solutions of the quadratic equation are

$$x_1 = \frac{12 - \sqrt{72}}{6} = 2 - \sqrt{2} \approx 0.586 \qquad \text{and}$$

$$x_2 = \frac{12 + \sqrt{72}}{6} = 2 + \sqrt{2} \approx 3.414.$$

The possible production levels for maximum profit are $x \approx 0.586$ million MP3 players or $x \approx 3.414$ million. The second derivative of $p(x) = r(x) - c(x)$ is $p''(x) = -c''(x)$ since $r''(x)$ is everywhere zero. Thus, $p''(x) = 6(2 - x)$, which is negative at $x = 2 + \sqrt{2}$ and positive at $x = 2 - \sqrt{2}$. By the Second Derivative Test, a maximum profit occurs at about $x = 3.414$ (where revenue exceeds costs) and maximum loss occurs at about $x = 0.586$. The graphs of $r(x)$ and $c(x)$ are shown in Figure 4.43. ■

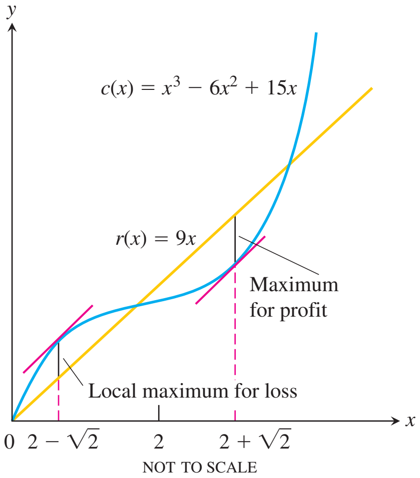


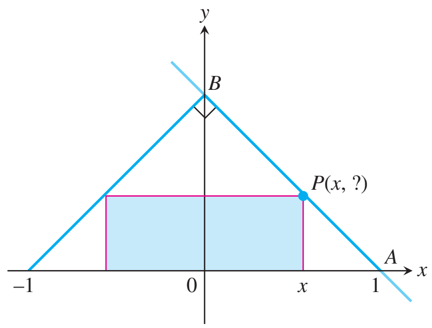

**FIGURE 4.43** The cost and revenue curves for Example 5.

# Exercises 4.6

### Mathematical Applications

Whenever you are maximizing or minimizing a function of a single variable, we urge you to graph it over the domain that is appropriate to the problem you are solving. The graph will provide insight before you calculate and will furnish a visual context for understanding your answer.

1. **Minimizing perimeter** What is the smallest perimeter possible for a rectangle whose area is 16 in$^2$, and what are its dimensions?
2. Show that among all rectangles with an 8-m perimeter, the one with largest area is a square.
3. The figure shows a rectangle inscribed in an isosceles right triangle whose hypotenuse is 2 units long.
   a. Express the $y$-coordinate of $P$ in terms of $x$. (*Hint:* Write an equation for the line $AB$.)
   b. Express the area of the rectangle in terms of $x$.
   c. What is the largest area the rectangle can have, and what are its dimensions?




4. A rectangle has its base on the $x$-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have, and what are its dimensions?
5. You are planning to make an open rectangular box from an 8-in.-by-15-in. piece of cardboard by cutting congruent squares from the corners and folding up the sides. What are the dimensions of the box of largest volume you can make this way, and what is its volume?
6. You are planning to close off a corner of the first quadrant with a line segment 20 units long running from $(a, 0)$ to $(0, b)$. Show that the area of the triangle enclosed by the segment is largest when $a = b$.
7. **The best fencing plan** A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 m of wire at your disposal, what is the largest area you can enclose, and what are its dimensions?
8. **The shortest fence** A 216 m$^2$ rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. What dimensions for the outer rectangle will require the smallest total length of fence? How much fence will be needed?
9. **Designing a tank** Your iron works has contracted to design and build a 500 ft$^3$, square-based, open-top, rectangular steel holding tank for a paper company. The tank is to be made by welding thin stainless steel plates together along their edges. As the production engineer, your job is to find dimensions for the base and height that will make the tank weigh as little as possible.

**a.** What dimensions do you tell the shop to use?

**b.** Briefly describe how you took weight into account.

**10. Catching rainwater** A 1125 $\text{ft}^3$ open-top rectangular tank with a square base $x$ ft on a side and $y$ ft deep is to be built with its top flush with the ground to catch runoff water. The costs associated with the tank involve not only the material from which the tank is made but also an excavation charge proportional to the product $xy$.

**a.** If the total cost is

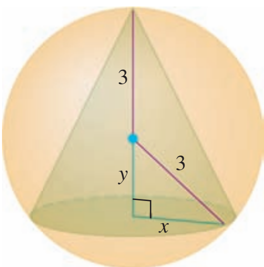
$$c = 5(x^2 + 4xy) + 10xy,$$

what values of $x$ and $y$ will minimize it?

**b.** Give a possible scenario for the cost function in part (a).

**11. Designing a poster** You are designing a rectangular poster to contain 50 $\text{in}^2$ of printing with a 4-in. margin at the top and bottom and a 2-in. margin at each side. What overall dimensions will minimize the amount of paper used?

**12.** Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.




**13.** Two sides of a triangle have lengths $a$ and $b$, and the angle between them is $\theta$. What value of $\theta$ will maximize the triangle's area? (*Hint:* $A = (1/2)ab \sin\theta$.)

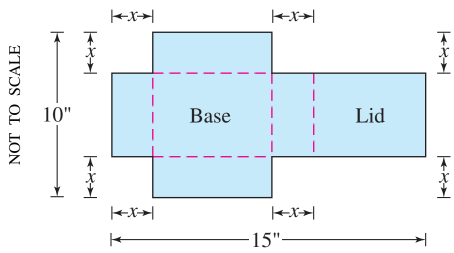
**14. Designing a can** What are the dimensions of the lightest open-top right circular cylindrical can that will hold a volume of 1000 $\text{cm}^3$? Compare the result here with the result in Example 2.

**15. Designing a can** You are designing a 1000 $\text{cm}^3$ right circular cylindrical can whose manufacture will take waste into account. There is no waste in cutting the aluminum for the side, but the top and bottom of radius $r$ will be cut from squares that measure $2r$ units on a side. The total amount of aluminum used up by the can will therefore be

$$A = 8r^2 + 2\pi rh$$

rather than the $A = 2\pi r^2 + 2\pi rh$ in Example 2. In Example 2, the ratio of $h$ to $r$ for the most economical can was 2 to 1. What is the ratio now?

T **16. Designing a box with a lid** A piece of cardboard measures 10 in. by 15 in. Two equal squares are removed from the corners of a 10-in. side as shown in the figure. Two equal rectangles are removed from the other corners so that the tabs can be folded to form a rectangular box with lid.




**a.** Write a formula $V(x)$ for the volume of the box.

**b.** Find the domain of $V$ for the problem situation and graph $V$ over this domain.

**c.** Use a graphical method to find the maximum volume and the value of $x$ that gives it.

**d.** Confirm your result in part (c) analytically.

T **17. Designing a suitcase** A 24-in.-by-36-in. sheet of cardboard is folded in half to form a 24-in.-by-18-in. rectangle as shown in the accompanying figure. Then four congruent squares of side length $x$ are cut from the corners of the folded rectangle. The sheet is unfolded, and the six tabs are folded up to form a box with sides and a lid.

**a.** Write a formula $V(x)$ for the volume of the box.

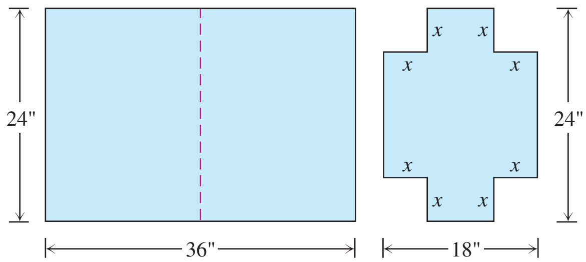
**b.** Find the domain of $V$ for the problem situation and graph $V$ over this domain.

**c.** Use a graphical method to find the maximum volume and the value of $x$ that gives it.

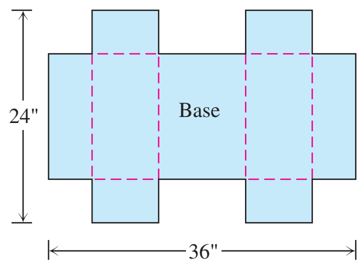
**d.** Confirm your result in part (c) analytically.

**e.** Find a value of $x$ that yields a volume of 1120 $\text{in}^3$.

**f.** Write a paragraph describing the issues that arise in part (b).



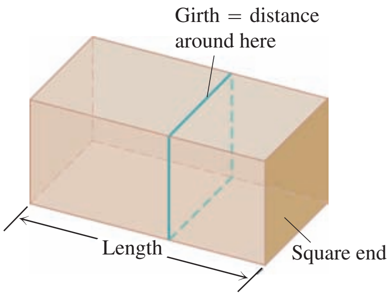

The sheet is then unfolded.




**18.** A rectangle is to be inscribed under the arch of the curve $y = 4\cos(0.5x)$ from $x = -\pi$ to $x = \pi$. What are the dimensions of the rectangle with largest area, and what is the largest area?

**19.** Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius 10 cm. What is the maximum volume?

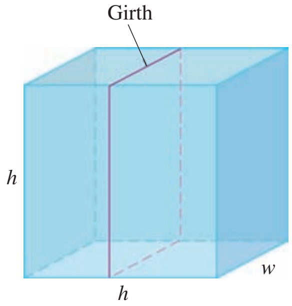
**20. a.** The U.S. Postal Service will accept a box for domestic shipment only if the sum of its length and girth (distance around) does not exceed 108 in. What dimensions will give a box with a square end the largest possible volume?




**T b.** Graph the volume of a 108-in. box (length plus girth equals 108 in.) as a function of its length and compare what you see with your answer in part (a).

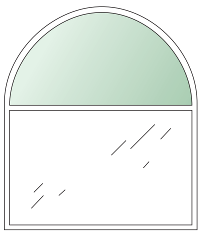
**21.** (*Continuation of Exercise 20.*)

**a.** Suppose that instead of having a box with square ends you have a box with square sides so that its dimensions are $h$ by $h$ by $w$ and the girth is $2h + 2w$. What dimensions will give the box its largest volume now?



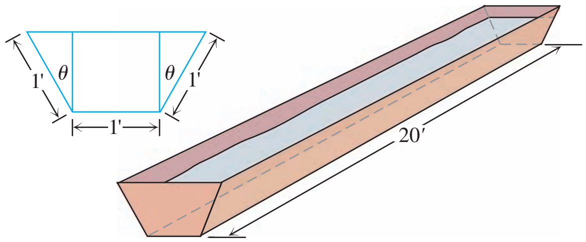

**T b.** Graph the volume as a function of $h$ and compare what you see with your answer in part (a).

**22.** A window is in the form of a rectangle surmounted by a semicircle. The rectangle is of clear glass, whereas the semicircle is of tinted glass that transmits only half as much light per unit area as clear glass does. The total perimeter is fixed. Find the proportions of the window that will admit the most light. Neglect the thickness of the frame.



**23.** A silo (base not included) is to be constructed in the form of a cylinder surmounted by a hemisphere. The cost of construction per square unit of surface area is twice as great for the hemisphere as it is for the cylindrical sidewall. Determine the dimensions to be used if the volume is fixed and the cost of construction is to be kept to a minimum. Neglect the thickness of the silo and waste in construction.

**24.** The trough in the figure is to be made to the dimensions shown. Only the angle $\theta$ can be varied. What value of $\theta$ will maximize the trough's volume?

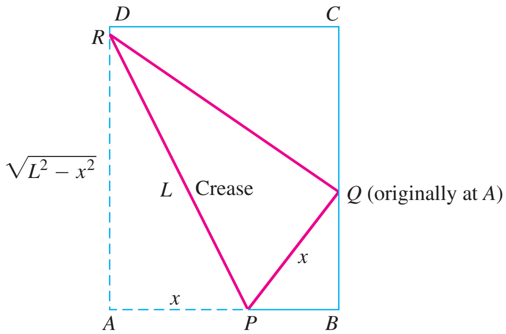



**25. Paper folding** A rectangular sheet of 8.5-in.-by-11-in. paper is placed on a flat surface. One of the corners is placed on the opposite longer edge, as shown in the figure, and held there as the paper is smoothed flat. The problem is to make the length of the crease as small as possible. Call the length $L$. Try it with paper.

**a.** Show that $L^2 = 2x^3/(2x - 8.5)$.

**b.** What value of $x$ minimizes $L^2$?

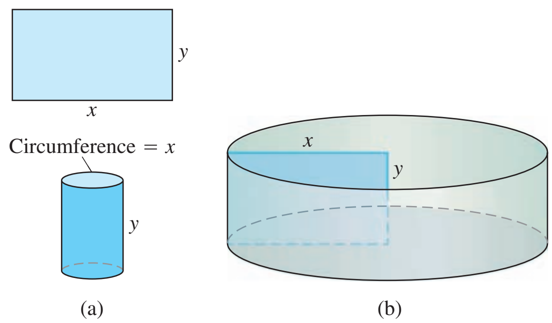
**c.** What is the minimum value of $L$?



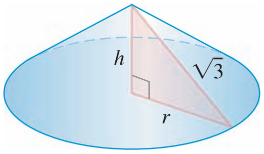

**26. Constructing cylinders** Compare the answers to the following two construction problems.

**a.** A rectangular sheet of perimeter 36 cm and dimensions $x$ cm by $y$ cm is to be rolled into a cylinder as shown in part (a) of the figure. What values of $x$ and $y$ give the largest volume?

**b.** The same sheet is to be revolved about one of the sides of length $y$ to sweep out the cylinder as shown in part (b) of the figure. What values of $x$ and $y$ give the largest volume?



**27. Constructing cones** A right triangle whose hypotenuse is $\sqrt{3}$ m long is revolved about one of its legs to generate a right circular cone. Find the radius, height, and volume of the cone of greatest volume that can be made this way.




**28.** Find the point on the line $\frac{x}{a} + \frac{y}{b} = 1$ that is closest to the origin.

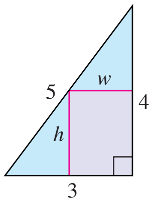
**29.** Find a positive number for which the sum of it and its reciprocal is the smallest (least) possible.

**30.** Find a postitive number for which the sum of its reciprocal and four times its square is the smallest possible.

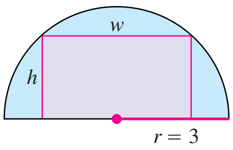
**31.** A wire $b$ m long is cut into two pieces. One piece is bent into an equilateral triangle and the other is bent into a circle. If the sum of the areas enclosed by each part is a minimum, what is the length of each part?

**32.** Answer Exercise 31 if one piece is bent into a square and the other into a circle.

**33.** Determine the dimensions of the rectangle of largest area that can be inscribed in the right triangle shown in the accompanying figure.




**34.** Determine the dimensions of the rectangle of largest area that can be inscribed in a semicircle of radius 3. (See accompanying figure.)




**35.** What value of $a$ makes $f(x) = x^2 + (a/x)$ have

**a.** a local minimum at $x = 2$?

**b.** a point of inflection at $x = 1$?

**36.** What values of $a$ and $b$ make $f(x) = x^3 + ax^2 + bx$ have

**a.** a local maximum at $x = -1$ and a local minimum at $x = 3$?

**b.** a local minimum at $x = 4$ and a point of inflection at $x = 1$?

## Physical Applications

**37. Vertical motion** The height above ground of an object moving vertically is given by

$$s = -16t^2 + 96t + 112,$$

with $s$ in feet and $t$ in seconds. Find

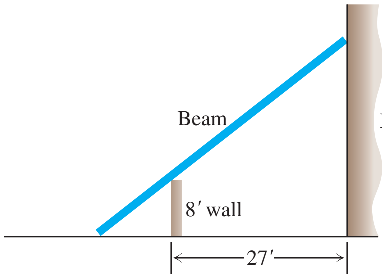
**a.** the object's velocity when $t = 0$;

**b.** its maximum height and when it occurs;

**c.** its velocity when $s = 0$.

**38. Quickest route** Jane is 2 mi offshore in a boat and wishes to reach a coastal village 6 mi down a straight shoreline from the point nearest the boat. She can row 2 mph and can walk 5 mph. Where should she land her boat to reach the village in the least amount of time?

**39. Shortest beam** The 8-ft wall shown here stands 27 ft from the building. Find the length of the shortest straight beam that will reach to the side of the building from the ground outside the wall.




**40. Motion on a line** The positions of two particles on the $s$-axis are $s_1 = \sin t$ and $s_2 = \sin(t + \pi/3)$, with $s_1$ and $s_2$ in meters and $t$ in seconds.

**a.** At what time(s) in the interval $0 \le t \le 2\pi$ do the particles meet?

**b.** What is the farthest apart that the particles ever get?

**c.** When in the interval $0 \le t \le 2\pi$ is the distance between the particles changing the fastest?

**41.** The intensity of illumination at any point from a light source is proportional to the square of the reciprocal of the distance between the point and the light source. Two lights, one having an intensity eight times that of the other, are 6 m apart. How far from the stronger light is the total illumination least?

**42. Projectile motion** The *range* $R$ of a projectile fired from the origin over horizontal ground is the distance from the origin to the point of impact. If the projectile is fired with an initial velocity $v_0$ at an angle $\alpha$ with the horizontal, then in Chapter 13 we find that

$$R = \frac{v_0^2}{g}\sin 2\alpha,$$

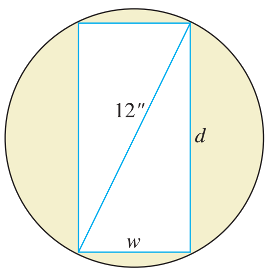
where $g$ is the downward acceleration due to gravity. Find the angle $\alpha$ for which the range $R$ is the largest possible.

T **43. Strength of a beam** The strength $S$ of a rectangular wooden beam is proportional to its width times the square of its depth. (See the accompanying figure.)

**a.** Find the dimensions of the strongest beam that can be cut from a 12-in.-diameter cylindrical log.

**b.** Graph $S$ as a function of the beam's width $w$, assuming the proportionality constant to be $k = 1$. Reconcile what you see with your answer in part (a).

**c.** On the same screen, graph $S$ as a function of the beam's depth $d$, again taking $k = 1$. Compare the graphs with one another and with your answer in part (a). What would be the effect of changing to some other value of $k$? Try it.



T **44. Stiffness of a beam** The stiffness $S$ of a rectangular beam is proportional to its width times the cube of its depth.

**a.** Find the dimensions of the stiffest beam that can be cut from a 12-in.-diameter cylindrical log.

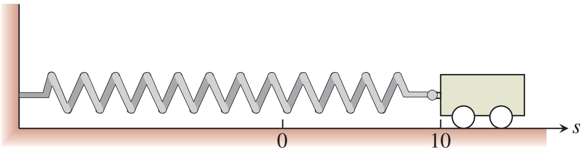
**b.** Graph $S$ as a function of the beam's width $w$, assuming the proportionality constant to be $k = 1$. Reconcile what you see with your answer in part (a).

**c.** On the same screen, graph $S$ as a function of the beam's depth $d$, again taking $k = 1$. Compare the graphs with one another and with your answer in part (a). What would be the effect of changing to some other value of $k$? Try it.

**45. Frictionless cart** A small frictionless cart, attached to the wall by a spring, is pulled 10 cm from its rest position and released at time $t = 0$ to roll back and forth for 4 sec. Its position at time $t$ is $s = 10 \cos \pi t$.

**a.** What is the cart's maximum speed? When is the cart moving that fast? Where is it then? What is the magnitude of the acceleration then?

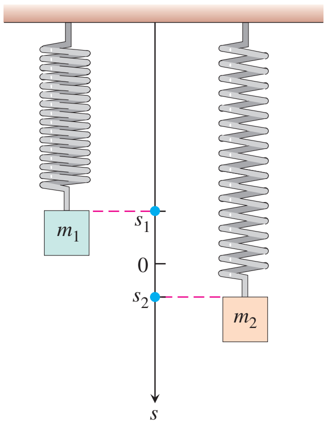
**b.** Where is the cart when the magnitude of the acceleration is greatest? What is the cart's speed then?




**46.** Two masses hanging side by side from springs have positions $s_1 = 2 \sin t$ and $s_2 = \sin 2t$, respectively.

**a.** At what times in the interval $0 < t$ do the masses pass each other? (*Hint:* $\sin 2t = 2 \sin t \cos t$.)

**b.** When in the interval $0 \le t \le 2\pi$ is the vertical distance between the masses the greatest? What is this distance? (*Hint:* $\cos 2t = 2 \cos^2 t - 1$.)




**47. Distance between two ships** At noon, ship $A$ was 12 nautical miles due north of ship $B$. Ship $A$ was sailing south at 12 knots (nautical miles per hour; a nautical mile is 2000 yd) and continued to do so all day. Ship $B$ was sailing east at 8 knots and continued to do so all day.

**a.** Start counting time with $t = 0$ at noon and express the distance $s$ between the ships as a function of $t$.

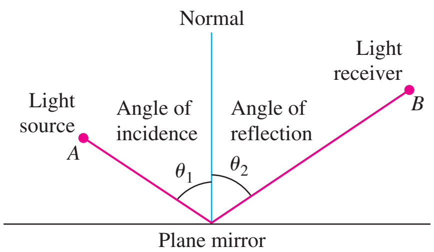
**b.** How rapidly was the distance between the ships changing at noon? One hour later?

**c.** The visibility that day was 5 nautical miles. Did the ships ever sight each other?

T **d.** Graph $s$ and $ds/dt$ together as functions of $t$ for $-1 \le t \le 3$, using different colors if possible. Compare the graphs and reconcile what you see with your answers in parts (b) and (c).

**e.** The graph of $ds/dt$ looks as if it might have a horizontal asymptote in the first quadrant. This in turn suggests that $ds/dt$ approaches a limiting value as $t \to \infty$. What is this value? What is its relation to the ships' individual speeds?

**48. Fermat's principle in optics** Light from a source $A$ is reflected by a plane mirror to a receiver at point $B$, as shown in the accompanying figure. Show that for the light to obey Fermat's principle, the angle of incidence must equal the angle of reflection, both measured from the line normal to the reflecting surface. (This result can also be derived without calculus. There is a purely geometric argument, which you may prefer.)




**49. Tin pest** When metallic tin is kept below 13.2°C, it slowly becomes brittle and crumbles to a gray powder. Tin objects eventually crumble to this gray powder spontaneously if kept in a cold climate for years. The Europeans who saw tin organ pipes in their churches crumble away years ago called the change *tin pest* because it seemed to be contagious, and indeed it was, for the gray powder is a catalyst for its own formation.

A *catalyst* for a chemical reaction is a substance that controls the rate of reaction without undergoing any permanent change in itself. An *autocatalytic reaction* is one whose product is a catalyst for its own formation. Such a reaction may proceed slowly at first if the amount of catalyst present is small and slowly again at the end, when most of the original substance is used up. But in between, when both the substance and its catalyst product are abundant, the reaction proceeds at a faster pace.

In some cases, it is reasonable to assume that the rate $v = dx/dt$ of the reaction is proportional both to the amount of the original substance present and to the amount of product. That is, $v$ may be considered to be a function of $x$ alone, and

$$v = kx(a - x) = kax - kx^2,$$

where

$x =$ the amount of product

$a =$ the amount of substance at the beginning

$k =$ a positive constant.

At what value of $x$ does the rate $v$ have a maximum? What is the maximum value of $v$?

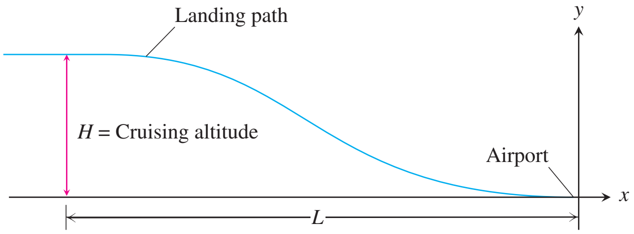
**50. Airplane landing path** An airplane is flying at altitude $H$ when it begins its descent to an airport runway that is at horizontal ground distance $L$ from the airplane, as shown in the figure. Assume that the landing path of the airplane is the graph of a cubic polynomial function $y = ax^3 + bx^2 + cx + d$, where $y(-L) = H$ and $y(0) = 0$.

**a.** What is $dy/dx$ at $x = 0$?

**b.** What is $dy/dx$ at $x = -L$?

**c.** Use the values for $dy/dx$ at $x = 0$ and $x = -L$ together with $y(0) = 0$ and $y(-L) = H$ to show that

$$y(x) = H\left[2\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2\right].$$




## Business and Economics

**51.** It costs you $c$ dollars each to manufacture and distribute backpacks. If the backpacks sell at $x$ dollars each, the number sold is given by

$$n = \frac{a}{x - c} + b(100 - x),$$

where $a$ and $b$ are positive constants. What selling price will bring a maximum profit?

**52.** You operate a tour service that offers the following rates:

$200 per person if 50 people (the minimum number to book the tour) go on the tour.

For each additional person, up to a maximum of 80 people total, the rate per person is reduced by $2.

It costs $6000 (a fixed cost) plus $32 per person to conduct the tour. How many people does it take to maximize your profit?

**53. Wilson lot size formula** One of the formulas for inventory management says that the average weekly cost of ordering, paying for, and holding merchandise is

$$A(q) = \frac{km}{q} + cm + \frac{hq}{2},$$

where $q$ is the quantity you order when things run low (shoes, radios, brooms, or whatever the item might be), $k$ is the cost of placing an order (the same, no matter how often you order), $c$ is the cost of one item (a constant), $m$ is the number of items sold each week (a constant), and $h$ is the weekly holding cost per item (a constant that takes into account things such as space, utilities, insurance, and security).

**a.** Your job, as the inventory manager for your store, is to find the quantity that will minimize $A(q)$. What is it? (The formula you get for the answer is called the *Wilson lot size formula*.)

**b.** Shipping costs sometimes depend on order size. When they do, it is more realistic to replace $k$ by $k + bq$, the sum of $k$ and a constant multiple of $q$. What is the most economical quantity to order now?

**54. Production level** Prove that the production level (if any) at which average cost is smallest is a level at which the average cost equals marginal cost.

**55.** Show that if $r(x) = 6x$ and $c(x) = x^3 - 6x^2 + 15x$ are your revenue and cost functions, then the best you can do is break even (have revenue equal cost).

**56. Production level** Suppose that $c(x) = x^3 - 20x^2 + 20{,}000x$ is the cost of manufacturing $x$ items. Find a production level that will minimize the average cost of making $x$ items.

**57.** You are to construct an open rectangular box with a square base and a volume of 48 ft$^3$. If material for the bottom costs \$6/ft$^2$ and material for the sides costs \$4/ft$^2$, what dimensions will result in the least expensive box? What is the minimum cost?

**58.** The 800-room Mega Motel chain is filled to capacity when the room charge is $50 per night. For each $10 increase in room charge, 40 fewer rooms are filled each night. What charge per room will result in the maximum revenue per night?

## Biology

**59. Sensitivity to medicine** (*Continuation of Exercise 72, Section 3.3.*) Find the amount of medicine to which the body is most sensitive by finding the value of $M$ that maximizes the derivative $dR/dM$, where

$$R = M^2\left(\frac{C}{2} - \frac{M}{3}\right)$$

and $C$ is a constant.

**60. How we cough**

**a.** When we cough, the trachea (windpipe) contracts to increase the velocity of the air going out. This raises the questions of how much it should contract to maximize the velocity and whether it really contracts that much when we cough.

Under reasonable assumptions about the elasticity of the tracheal wall and about how the air near the wall is slowed by friction, the average flow velocity $v$ can be modeled by the equation

$$v = c(r_0 - r)r^2 \text{ cm/sec}, \qquad \frac{r_0}{2} \le r \le r_0,$$

where $r_0$ is the rest radius of the trachea in centimeters and $c$ is a positive constant whose value depends in part on the length of the trachea.

Show that $v$ is greatest when $r = (2/3)r_0$; that is, when the trachea is about 33% contracted. The remarkable fact is that X-ray photographs confirm that the trachea contracts about this much during a cough.

**T** **b.** Take $r_0$ to be 0.5 and $c$ to be 1 and graph $v$ over the interval $0 \le r \le 0.5$. Compare what you see with the claim that $v$ is at a maximum when $r = (2/3)r_0$.

## Theory and Examples

**61. An inequality for positive integers** Show that if $a$, $b$, $c$, and $d$ are positive integers, then

$$\frac{(a^2 + 1)(b^2 + 1)(c^2 + 1)(d^2 + 1)}{abcd} \ge 16.$$

**62. The derivative $dt/dx$ in Example 4**

**a.** Show that

$$f(x) = \frac{x}{\sqrt{a^2 + x^2}}$$

is an increasing function of $x$.

**b.** Show that

$$g(x) = \frac{d - x}{\sqrt{b^2 + (d - x)^2}}$$

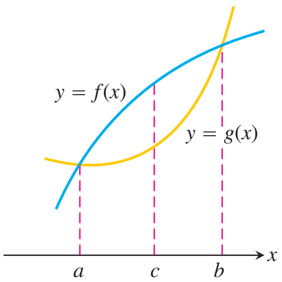
is a decreasing function of $x$.

**c.** Show that

$$\frac{dt}{dx} = \frac{x}{c_1\sqrt{a^2 + x^2}} - \frac{d - x}{c_2\sqrt{b^2 + (d - x)^2}}$$

is an increasing function of $x$.

**63.** Let $f(x)$ and $g(x)$ be the differentiable functions graphed here. Point $c$ is the point where the vertical distance between the curves is the greatest. Is there anything special about the tangents to the two curves at $c$? Give reasons for your answer.




**64.** You have been asked to determine whether the function $f(x) = 3 + 4\cos x + \cos 2x$ is ever negative.

**a.** Explain why you need to consider values of $x$ only in the interval $[0, 2\pi]$.

**b.** Is $f$ ever negative? Explain.

**65. a.** The function $y = \cot x - \sqrt{2}\csc x$ has an absolute maximum value on the interval $0 < x < \pi$. Find it.

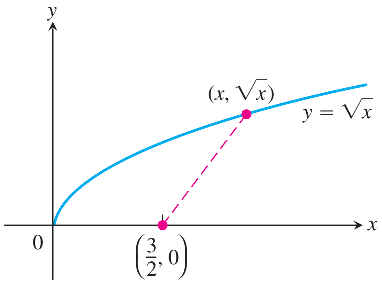
T **b.** Graph the function and compare what you see with your answer in part (a).

**66. a.** The function $y = \tan x + 3\cot x$ has an absolute minimum value on the interval $0 < x < \pi/2$. Find it.

T **b.** Graph the function and compare what you see with your answer in part (a).

**67. a.** How close does the curve $y = \sqrt{x}$ come to the point $(3/2, 0)$? (*Hint:* If you minimize the *square* of the distance, you can avoid square roots.)

T **b.** Graph the distance function $D(x)$ and $y = \sqrt{x}$ together and reconcile what you see with your answer in part (a).




**68. a.** How close does the semicircle $y = \sqrt{16 - x^2}$ come to the point $(1, \sqrt{3})$?

T **b.** Graph the distance function and $y = \sqrt{16 - x^2}$ together and reconcile what you see with your answer in part (a).

# 4.7 Newton's Method

In this section we study a numerical method, called *Newton's method* or the *Newton–Raphson method*, which is a technique to approximate the solution to an equation $f(x) = 0$. Essentially it uses tangent lines in place of the graph of $y = f(x)$ near the points where $f$ is zero. (A value of $x$ where $f$ is zero is a *root* of the function $f$ and a *solution* of the equation $f(x) = 0$.)

## Procedure for Newton's Method

The goal of Newton's method for estimating a solution of an equation $f(x) = 0$ is to produce a sequence of approximations that approach the solution. We pick the first number $x_0$ of the sequence. Then, under favorable circumstances, the method does the rest by moving step by step toward a point where the graph of $f$ crosses the $x$-axis (Figure 4.44). At each step the method approximates a zero of $f$ with a zero of one of its linearizations. Here is how it works.

The initial estimate, $x_0$, may be found by graphing or just plain guessing. The method then uses the tangent to the curve $y = f(x)$ at $(x_0, f(x_0))$ to approximate the curve, calling

## Convergence of the Approximations

In Chapter 10 we define precisely the idea of *convergence* for the approximations $x_n$ in Newton's method. Intuitively, we mean that as the number $n$ of approximations increases without bound, the values $x_n$ get arbitrarily close to the desired root $r$. (This notion is similar to the idea of the limit of a function $g(t)$ as $t$ approaches infinity, as defined in Section 2.6.)

In practice, Newton's method usually gives convergence with impressive speed, but this is not guaranteed. One way to test convergence is to begin by graphing the function to estimate a good starting value for $x_0$. You can test that you are getting closer to a zero of the function by evaluating $|f(x_n)|$, and check that the approximations are converging by evaluating $|x_n - x_{n+1}|$.

*Newton's method does not always converge*. For instance, if

$$f(x) = \begin{cases} -\sqrt{r - x}, & x < r \\ \sqrt{x - r}, & x \geq r, \end{cases}$$

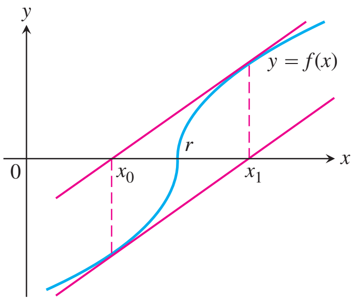



**FIGURE 4.49** Newton's method fails to converge. You go from $x_0$ to $x_1$ and back to $x_0$, never getting any closer to $r$.

the graph will be like the one in Figure 4.49. If we begin with $x_0 = r - h$, we get $x_1 = r + h$, and successive approximations go back and forth between these two values. No amount of iteration brings us closer to the root than our first guess.

*If Newton's method does converge, it converges to a root*. Be careful, however. There are situations in which the method appears to converge but there is no root there. Fortunately, such situations are rare.

*When Newton's method converges to a root, it may not be the root you have in mind*. Figure 4.50 shows two ways this can happen.

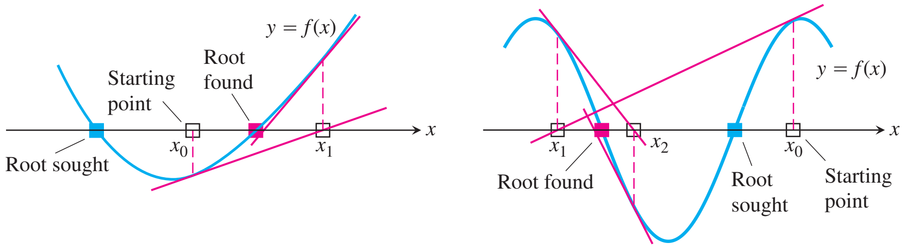



**FIGURE 4.50** If you start too far away, Newton's method may miss the root you want.

## Exercises 4.7

### Root Finding

1. Use Newton's method to estimate the solutions of the equation $x^2 + x - 1 = 0$. Start with $x_0 = -1$ for the left-hand solution and with $x_0 = 1$ for the solution on the right. Then, in each case, find $x_2$.

2. Use Newton's method to estimate the one real solution of $x^3 + 3x + 1 = 0$. Start with $x_0 = 0$ and then find $x_2$.

3. Use Newton's method to estimate the two zeros of the function $f(x) = x^4 + x - 3$. Start with $x_0 = -1$ for the left-hand zero and with $x_0 = 1$ for the zero on the right. Then, in each case, find $x_2$.

4. Use Newton's method to estimate the two zeros of the function $f(x) = 2x - x^2 + 1$. Start with $x_0 = 0$ for the left-hand zero and with $x_0 = 2$ for the zero on the right. Then, in each case, find $x_2$.

5. Use Newton's method to find the positive fourth root of 2 by solving the equation $x^4 - 2 = 0$. Start with $x_0 = 1$ and find $x_2$.

6. Use Newton's method to find the negative fourth root of 2 by solving the equation $x^4 - 2 = 0$. Start with $x_0 = -1$ and find $x_2$.

7. **Guessing a root** Suppose that your first guess is lucky, in the sense that $x_0$ is a root of $f(x) = 0$. Assuming that $f'(x_0)$ is defined and not 0, what happens to $x_1$ and later approximations?

8. **Estimating pi** You plan to estimate $\pi/2$ to five decimal places by using Newton's method to solve the equation $\cos x = 0$. Does it matter what your starting value is? Give reasons for your answer.

### Theory and Examples

9. **Oscillation** Show that if $h > 0$, applying Newton's method to

$$f(x) = \begin{cases} \sqrt{x}, & x \geq 0 \\ \sqrt{-x}, & x < 0 \end{cases}$$

leads to $x_1 = -h$ if $x_0 = h$ and to $x_1 = h$ if $x_0 = -h$. Draw a picture that shows what is going on.

**10. Approximations that get worse and worse** Apply Newton's method to $f(x) = x^{1/3}$ with $x_0 = 1$ and calculate $x_1, x_2, x_3$, and $x_4$. Find a formula for $|x_n|$. What happens to $|x_n|$ as $n \rightarrow \infty$? Draw a picture that shows what is going on.

**11.** Explain why the following four statements ask for the same information:

**i)** Find the roots of $f(x) = x^3 - 3x - 1$.

**ii)** Find the $x$-coordinates of the intersections of the curve $y = x^3$ with the line $y = 3x + 1$.

**iii)** Find the $x$-coordinates of the points where the curve $y = x^3 - 3x$ crosses the horizontal line $y = 1$.

**iv)** Find the values of $x$ where the derivative of $g(x) = (1/4)x^4 - (3/2)x^2 - x + 5$ equals zero.

**12. Locating a planet** To calculate a planet's space coordinates, we have to solve equations like $x = 1 + 0.5\sin x$. Graphing the function $f(x) = x - 1 - 0.5\sin x$ suggests that the function has a root near $x = 1.5$. Use one application of Newton's method to improve this estimate. That is, start with $x_0 = 1.5$ and find $x_1$. (The value of the root is 1.49870 to five decimal places.) Remember to use radians.

T **13. Intersecting curves** The curve $y = \tan x$ crosses the line $y = 2x$ between $x = 0$ and $x = \pi/2$. Use Newton's method to find where.

T **14. Real solutions of a quartic** Use Newton's method to find the two real solutions of the equation $x^4 - 2x^3 - x^2 - 2x + 2 = 0$.

T **15. a.** How many solutions does the equation $\sin 3x = 0.99 - x^2$ have?

**b.** Use Newton's method to find them.

**16. Intersection of curves**

**a.** Does $\cos 3x$ ever equal $x$? Give reasons for your answer.

**b.** Use Newton's method to find where.

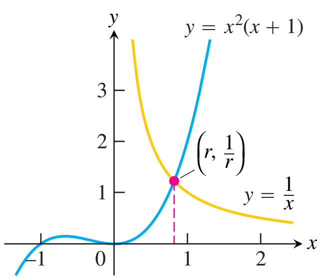
**17.** Find the four real zeros of the function $f(x) = 2x^4 - 4x^2 + 1$.

T **18. Estimating pi** Estimate $\pi$ to as many decimal places as your calculator will display by using Newton's method to solve the equation $\tan x = 0$ with $x_0 = 3$.

**19. Intersection of curves** At what value(s) of $x$ does $\cos x = 2x$?

**20. Intersection of curves** At what value(s) of $x$ does $\cos x = -x$?

**21.** The graphs of $y = x^2(x + 1)$ and $y = 1/x$ $(x > 0)$ intersect at one point $x = r$. Use Newton's method to estimate the value of $r$ to four decimal places.




**22.** The graphs of $y = \sqrt{x}$ and $y = 3 - x^2$ intersect at one point $x = r$. Use Newton's method to estimate the value of $r$ to four decimal places.

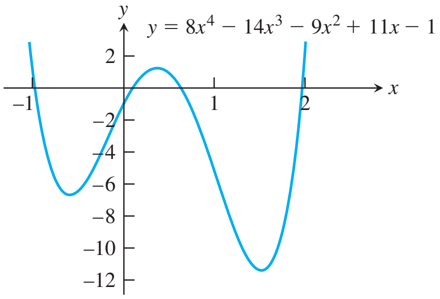
**23. Intersection of curves** At what value(s) of $x$ does $e^{-x^2} = x^2 - x + 1$?

**24. Intersection of curves** At what value(s) of $x$ does $\ln(1 - x^2) = x - 1$?

**25.** Use the Intermediate Value Theorem from Section 2.5 to show that $f(x) = x^3 + 2x - 4$ has a root between $x = 1$ and $x = 2$. Then find the root to five decimal places.

**26. Factoring a quartic** Find the approximate values of $r_1$ through $r_4$ in the factorization

$$8x^4 - 14x^3 - 9x^2 + 11x - 1 = 8(x - r_1)(x - r_2)(x - r_3)(x - r_4).$$




T **27. Converging to different zeros** Use Newton's method to find the zeros of $f(x) = 4x^4 - 4x^2$ using the given starting values.

**a.** $x_0 = -2$ and $x_0 = -0.8$, lying in $\left(-\infty, -\sqrt{2}/2\right)$

**b.** $x_0 = -0.5$ and $x_0 = 0.25$, lying in $\left(-\sqrt{21}/7, \sqrt{21}/7\right)$

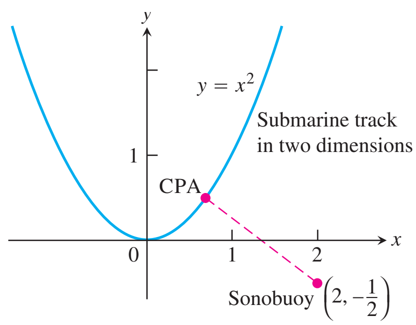
**c.** $x_0 = 0.8$ and $x_0 = 2$, lying in $\left(\sqrt{2}/2, \infty\right)$

**d.** $x_0 = -\sqrt{21}/7$ and $x_0 = \sqrt{21}/7$

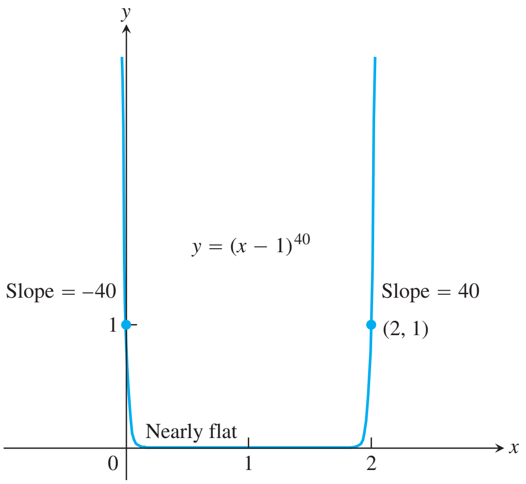
**28. The sonobuoy problem** In submarine location problems, it is often necessary to find a submarine's closest point of approach (CPA) to a sonobuoy (sound detector) in the water. Suppose that the submarine travels on the parabolic path $y = x^2$ and that the buoy is located at the point $(2, -1/2)$.

**a.** Show that the value of $x$ that minimizes the distance between the submarine and the buoy is a solution of the equation $x = 1/(x^2 + 1)$.

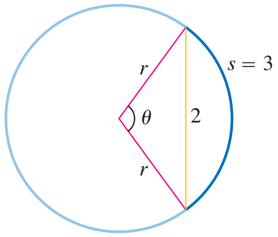
**b.** Solve the equation $x = 1/(x^2 + 1)$ with Newton's method.




T **29. Curves that are nearly flat at the root** Some curves are so flat that, in practice, Newton's method stops too far from the root to give a useful estimate. Try Newton's method on $f(x) = (x - 1)^{40}$ with a starting value of $x_0 = 2$ to see how close your machine comes to the root $x = 1$. See the accompanying graph.




**30.** The accompanying figure shows a circle of radius r with a chord of length 2 and an arc $s$ of length 3. Use Newton's method to solve for $r$ and $\theta$ (radians) to four decimal places. Assume $0 < \theta < \pi$.




# 4.8 Antiderivatives

We have studied how to find the derivative of a function. However, many problems require that we recover a function from its known derivative (from its known rate of change). For instance, we may know the velocity function of an object falling from an initial height and need to know its height at any time. More generally, we want to find a function $F$ from its derivative $f$. If such a function $F$ exists, it is called an *antiderivative* of $f$. We will see in the next chapter that antiderivatives are the link connecting the two major elements of calculus: derivatives and definite integrals.

## Finding Antiderivatives

**DEFINITION** A function $F$ is an **antiderivative** of $f$ on an interval $I$ if $F'(x) = f(x)$ for all $x$ in $I$.

The process of recovering a function $F(x)$ from its derivative $f(x)$ is called *antidifferentiation*. We use capital letters such as $F$ to represent an antiderivative of a function $f$, $G$ to represent an antiderivative of $g$, and so forth.

**EXAMPLE 1** Find an antiderivative for each of the following functions.

**(a)** $f(x) = 2x$ **(b)** $g(x) = \cos x$ **(c)** $h(x) = \frac{1}{x} + 2e^{2x}$

**Solution** We need to think backward here: What function do we know has a derivative equal to the given function?

**(a)** $F(x) = x^2$ **(b)** $G(x) = \sin x$ **(c)** $H(x) = \ln|x| + e^{2x}$

Each answer can be checked by differentiating. The derivative of $F(x) = x^2$ is $2x$. The derivative of $G(x) = \sin x$ is $\cos x$ and the derivative of $H(x) = \ln|x| + e^{2x}$ is $(1/x) + 2e^{2x}$. ■