to extend the integral to the case $a > b$. In particular, if $f(x)$ is integrable and nonpositive on $[a, b]$, then the area of the region bounded by the curves $y = f(x)$, $y = 0$, $x = a$, and $x = b$ is given by $\int_b^a f(x)dx$.

In the next section we shall use the machinery of upper and lower sums to prove several familiar theorems about the Riemann integral. We close this section with one more result which reinforces the connection between integration and area.

5.16 Theorem. *If $f(x) = \alpha$ is constant on $[a, b]$, then*

$$\int_a^b f(x)\,dx = \alpha(b - a).$$

Proof. By Theorem 5.10, f is integrable on $[a, b]$. Hence, it follows from Theorem 5.15 and Remark 5.5 that

$$\int_a^b f(x)\,dx = (U)\int_a^b f(x)\,dx = \inf_P U(f, P) = \alpha(b - a). \qquad \blacksquare$$

EXERCISES

5.1.0. Suppose that $a < b < c$. Decide which of the following statements are true and which are false. Prove the true ones and give counterexamples for the false ones.

a) If f is Riemann integrable on $[a, b]$, then f is continuous on $[a, b]$.
b) If $|f|$ is Riemann integrable on $[a, b]$, then f is Riemann integrable on $[a, b]$.
c) For all bounded functions $f : [a, b] \to \mathbf{R}$,

$$(L)\int_a^b f(x)\,dx \le \int_a^b f(x)\,dx \le (U)\int_a^b f(x)\,dx.$$

d) If f is continuous on $[a, b)$ and on $[b, c]$, then f is Riemann integrable on $[a, c]$.

5.1.1. For each of the following, compute $U(f, P)$, $L(f, P)$, and $\int_0^2 f(x)dx$, where

$$P = \left\{0, \frac{1}{2}, 1, 2\right\}.$$

Find out whether the lower sum or the upper sum is a better approximation to the integral. Graph f and explain why this is so.

a) $f(x) = x^3$
b) $f(x) = 3 - x^2$
c) $f(x) = \sin(x/5)$

5.1.2. a) Prove that for each $n \in \mathbf{N}$,

$$P_n := \left\{ \frac{j}{n} : j = 0, 1, \dots, n \right\}$$

is a partition of $[0, 1]$.

b) Prove that a bounded function f is integrable on $[0, 1]$ if

$$(*) \qquad I_0 := \lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n),$$

in which case $\int_0^1 f(x)dx$ equals I_0.

c) For each of the following functions, use Exercise 1.4.4 to find formulas for the upper and lower sums of f on P_n, and use them to compute the value of $\int_0^1 f(x)dx$.

α) $$f(x) = x$$

β) $$f(x) = x^2$$

γ) $$f(x) = \begin{cases} 0 & 0 \le x < 1/2 \\ 1 & 1/2 \le x \le 1 \end{cases}$$

5.1.3. Let $E = \{1/n : n \in \mathbf{N}\}$. Prove that the function

$$f(x) = \begin{cases} 1 & x \in E \\ 0 & \text{otherwise} \end{cases}$$

is integrable on $[0, 1]$. What is the value of $\int_0^1 f(x)dx$?

5.1.4. **This exercise is used in Section *14.2.** Suppose that $a < b$ and that $f : [a, b] \to \mathbf{R}$ is bounded.

a) Prove that if f is continuous at $x_0 \in [a, b]$ and $f(x_0) \ne 0$, then

$$(L) \int_a^b |f(x)|\, dx > 0.$$

b) Show that if f is continuous on $[a, b]$, then $\int_a^b |f(x)|dx = 0$ if and only if $f(x) = 0$ for all $x \in [a, b]$.

c) Does part b) hold if the absolute values are removed? If it does, prove it. If it does not, provide a counterexample.

5.1.5. Suppose that $a < b$ and that $f : [a, b] \to \mathbf{R}$ is continuous. Show that

$$\int_a^c f(x)\, dx = 0$$

for all $c \in [a, b]$ if and only if $f(x) = 0$ for all $x \in [a, b]$. (Compare with Exercise 5.1.4, and notice that f need not be nonnegative here.)

5.1.6. Let f be integrable on $[a, b]$ and E be a finite subset of $[a, b]$. Show that if g is a bounded function which satisfies $g(x) = f(x)$ for all $x \in [a, b]\backslash E$, then g is integrable on $[a, b]$ and

$$\int_a^b g(x)\,dx = \int_a^b f(x)\,dx.$$

$\boxed{5.1.7}$. This exercise is used in Section 12.3. Let f, g be bounded on $[a, b]$.

a) Prove that

$$(U)\int_a^b (f(x) + g(x))\,dx \le (U)\int_a^b f(x)\,dx + (U)\int_a^b g(x)\,dx$$

and

$$(L)\int_a^b (f(x) + g(x))\,dx \ge (L)\int_a^b f(x)\,dx + (L)\int_a^b g(x)\,dx.$$

b) Prove that

$$(U)\int_a^b f(x)\,dx = (U)\int_a^c f(x)\,dx + (U)\int_c^b f(x)\,dx$$

and

$$(L)\int_a^b f(x)\,dx = (L)\int_a^c f(x)\,dx + (L)\int_c^b f(x)\,dx$$

for $a < c < b$.

$\boxed{5.1.8}$. This exercise is used in Sections *5.5, 6.2, and *7.5.

a) If f is increasing on $[a, b]$ and $P = \{x_0, \dots, x_n\}$ is any partition of $[a, b]$, prove that

$$\sum_{j=1}^{n} (M_j(f) - m_j(f))\,\Delta x_j \le (f(b) - f(a))\,\|P\|.$$

b) Prove that if f is monotone on $[a, b]$, then f is integrable on $[a, b]$. [**Note**: By Theorem 4.19, f has at most countably many (i.e., relatively few) discontinuities on $[a, b]$. This has nothing to do with the proof of part b), but points out a general principle which will be discussed in Section 9.6.]

5.1.9.

5.1.10.

5.2 RIEMANN SU

There is tary calc

5.17 D

Let f :

i) A R gene

ii) The only

for a the r

The foll same as

5.18 Th *f is Rien*

exists, in

5.1.9. Let $a < b$ and $0 < c < d$ be real numbers and $f : [a, b] \to [c, d]$. If f is Riemann integrable on $[a, b]$, prove that $\sqrt{f}$ is Riemann integrable on $[a, b]$.

5.1.10. Let f be bounded on a nondegenerate interval $[a, b]$. Prove that f is integrable on $[a, b]$ if and only if given $\varepsilon > 0$ there is a partition P_ε of $[a, b]$ such that

$$P \supseteq P_\varepsilon \quad \text{implies} \quad |U(f, P) - L(f, P)| < \varepsilon.$$

5.2 RIEMANN SUMS

There is another definition of the Riemann integral frequently found in elementary calculus texts.

5.17 Definition.

Let $f : [a, b] \to \mathbf{R}$.

i) A *Riemann sum* of f with respect to a partition $P = \{x_0, \dots, x_n\}$ of $[a, b]$ generated by samples $t_j \in [x_{j-1}, x_j]$ is a sum

$$\mathcal{S}(f, P, t_j) := \sum_{j=1}^{n} f(t_j)\, \Delta x_j.$$

ii) The Riemann sums of f are said to *converge* to $I(f)$ as $\|P\| \to 0$ if and only if given $\varepsilon > 0$ there is a partition P_ε of $[a, b]$ such that

$$P = \{x_0, \dots, x_n\} \supseteq P_\varepsilon \quad \text{implies} \quad \left|\mathcal{S}(f, P, t_j) - I(f)\right| < \varepsilon$$

for all choices of $t_j \in [x_{j-1}, x_j]$, $j = 1, 2, \dots, n$. In this case we shall use the notation

$$I(f) = \lim_{\|P\| \to 0} \mathcal{S}(f, P, t_j) := \lim_{\|P\| \to 0} \sum_{j=1}^{n} f(t_j)\, \Delta x_j.$$

The following result shows that this definition of the Riemann integral is the same as the one using upper and lower integrals.

5.18 Theorem. *Let $a, b \in \mathbf{R}$ with $a < b$, and suppose that $f : [a, b] \to \mathbf{R}$. Then f is Riemann integrable on $[a, b]$ if and only if*

$$I(f) = \lim_{\|P\| \to 0} \sum_{j=1}^{n} f(t_j)\, \Delta x_j$$

exists, in which case $I(f) = \int_a^b f(x)dx$.

EXERCISES

5.2.0. Suppose that $a < b$. Decide which of the following statements are true and which are false. Prove the true ones and give counterexamples for the false ones.

a) If f and g are Riemann integrable on $[a, b]$, then $f - g$ is Riemann integrable on $[a, b]$.

b) If f is Riemann integrable on $[a, b]$ and P is any polynomial on $\mathbf{R}$, then $P \circ f$ is Riemann integrable on $[a, b]$.

c) If f and g are nonnegative real functions on $[a, b]$, with f continuous and g Riemann integrable on $[a, b]$, then there exist $x_0, x_1 \in [a, b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(x_0)\int_{x_1}^b g(x)\,dx.$$

d) If f and g are Riemann integrable on $[a, b]$ and f is continuous, then there is an $x_0 \in [a, b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(x_0)\int_a^b g(x)\,dx.$$

5.2.1. Using the connection between integrals and area, evaluate each of the following integrals.

a) $$\int_{-2}^{2} |x + 1|\,dx$$

b) $$\int_{-2}^{2} (|x + 1| + |x|)\,dx$$

c) $$\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx, \qquad a > 0$$

d) $$\int_{0}^{2} (5 + \sqrt{2x + x^2})\,dx$$

5.2.2. a) Suppose that $a < b$ and $n \in \mathbf{N}$ is even. If f is continuous on $[a, b]$ and $\int_a^b f(x)x^n dx = 0$, prove that $f(x) = 0$ for at least one $x \in [a, b]$.

b) Show that part a) might not be true if n is odd.

c) Prove that part a) does hold for odd n when $a + b \neq 0$.

5.2.3. Use Taylor polynomials with three or four nonzero terms to verify the following inequalities.

a) $$0.3095 < \int_0^1 \sin(x^2)\,dx < 0.3103$$

(The value of this integral is approximately 0.3102683.)

b)
$$1.4571 < \int_0^1 e^{x^2}\,dx < 1.5704$$

(The value of this integral is approximately 1.4626517.)

5.2.4. Suppose that $f : [0,\infty) \to [0,\infty)$ is integrable on every closed interval $[a,b] \subset [0,\infty)$. If
$$F(x) := \int_0^x e^{-y^2} f(y)\,dy, \quad x \in [0,\infty),$$
then there is a function $g : [0,\infty) \to [0,\infty)$ such that $F(x) = \int_{g(x)}^x f(y)\,dy$ for all $x \in [0,\infty)$.

5.2.5. Prove that if f is integrable on $[0,1]$ and $\beta > 0$, then
$$\lim_{n\to\infty} n^\alpha \int_0^{1/n^\beta} f(x)\,dx = 0$$
for all $\alpha < \beta$.

5.2.6. a) Suppose that $g_n \geq 0$ is a sequence of integrable functions which satisfies
$$\lim_{n\to\infty} \int_a^b g_n(x)\,dx = 0.$$
Show that if $f : [a,b] \to \mathbf{R}$ is integrable on $[a,b]$, then
$$\lim_{n\to\infty} \int_a^b f(x)g_n(x)\,dx = 0.$$

b) Prove that if f is integrable on $[0,1]$, then
$$\lim_{n\to\infty} \int_0^1 x^n f(x)\,dx = 0.$$

5.2.7. Suppose that f is integrable on $[a,b]$, that $x_0 = a$, and that x_n is a sequence of numbers in $[a,b]$ such that $x_n \uparrow b$ as $n \to \infty$. Prove that
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=0}^n \int_{x_k}^{x_{k+1}} f(x)\,dx.$$

5.2.8. Let f be continuous on a closed, nondegenerate interval $[a,b]$ and set
$$M = \sup_{x\in[a,b]} |f(x)|.$$

a) Prove that if $M > 0$ and $p > 0$, then for every $\varepsilon > 0$ there is a nondegenerate interval $I \subset [a, b]$ such that

$$(M-\varepsilon)^p|I| \le \int_a^b |f(x)|^p \, dx \le M^p(b-a).$$

b) Prove that

$$\lim_{p\to\infty} \left(\int_a^b |f(x)|^p \, dx \right)^{1/p} = M.$$

5.2.9. Let $f : [a, b] \to \mathbf{R}$, $a = x_0 < x_1 < \cdots < x_n = b$, and suppose that $f(x_k+)$ exists and is finite for $k = 0, 1, \ldots, n-1$ and $f(x_k-)$ exists and is finite for $k = 1, \ldots, n$. Show that if f is continuous on each subinterval (x_{k-1}, x_k), then f is integrable on $[a, b]$ and

$$\int_a^b f(x)\, dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(x)\, dx.$$

5.2.10. Prove that if f and g are integrable on $[a, b]$, then so are $f \vee g$ and $f \wedge g$ (see Exercise 3.1.8).

5.2.11. Suppose that $f : [a, b] \to \mathbf{R}$.

a) If f is not bounded above on $[a, b]$, then given any partition P of $[a, b]$ and $M > 0$, there exist $t_j \in [x_{j-1}, x_j]$ such that $\mathcal{S}(f, P, t_j) > M$.

b) If the Riemann sums of f converge to a finite number $I(f)$, as $\|P\| \to 0$, then f is bounded on $[a, b]$.

5.3 THE FUNDAMENTAL THEOREM OF CALCULUS

Let f be integrable on $[a, b]$ and $F(x) = \int_a^x f(t)\, dt$. By Theorem 5.26, F is continuous on $[a, b]$. The next result shows that if f is continuous, then F is continuously differentiable. Thus "indefinite integration" improves the behavior of the function.

5.28 Theorem. **[FUNDAMENTAL THEOREM OF CALCULUS].**
Let $[a, b]$ be nondegenerate and suppose that $f : [a, b] \to \mathbf{R}$.

i) *If f is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)\, dt$, then $F \in \mathcal{C}^1[a, b]$ and*

$$\frac{d}{dx} \int_a^x f(t)\, dt := F'(x) = f(x)$$

for each $x \in [a, b]$.

ii) *If f is differentiable on $[a, b]$ and f' is integrable on $[a, b]$, then*

$$\int_a^x f'(t)\, dt = f(x) - f(a)$$

for each $x \in [a, b]$.

EXERCISES

5.3.0. Suppose that $a < b$. Decide which of the following statements are true and which are false. Prove the true ones and give counterexamples for the false ones.

a) If f is continuous and nonnegative on $[a, b]$ and $g : [a, b] \to [a, b]$ is differentiable and increasing on $[a, b]$, then

$$F(x) := \int_a^{g(x)} f(t)\, dt$$

is increasing on $[a, b]$.

b) If f and g are differentiable on $[a, b]$, if f' and g' are Riemann integrable on $[a, b]$, and if $f(a) = 0$ but g is never zero on $[a, b]$, then

$$f(x) = \int_a^x g(t) \left(\frac{f(t)}{g(t)} \right)' dt + \int_a^x \frac{f(t)g'(t)}{g(t)} dt$$

for all $x \in [a, b]$.

c) If f and g are differentiable on $[a, b]$, and if f' and g' are Riemann integrable on $[a, b]$, then

$$\int_a^b f'(x)g(x)\, dx + \int_a^b f(x)g'(x)\, dx = 0$$

if and only if $f(a)g(a) = f(b)g(b)$.

d) If f and g are continuously differentiable on $[a, b]$, and if h is continuous on $[a, b]$, then

$$\int_{g(f(a))}^{g(f(b))} h(x)\, dx = \int_a^b h(g(f(x)))g'(f(x))f'(x)\, dx.$$

5.3.1. If $f : \mathbf{R} \to \mathbf{R}$ is continuous, find $F'(x)$ for each of the following functions.

a) $$F(x) = \int_{x^2}^1 f(t)\, dt$$

b) $$F(x) = \int_{x^2}^{x^3} f(t)\, dt$$

c) $$F(x) = \int_0^{x \cos x} t f(t)\, dt$$

d) $$F(x) = \int_0^x f(t - x)\, dt$$

5.3.2. Suppose that f is nonnegative and continuous on $[1, 2]$ and that $\int_1^2 x^k f(x)dx = 5 + k^2$ for $k = 0, 1, 2$. Prove that each of the following statements is correct.

a) $$\int_1^4 f(\sqrt{x})\,dx \le 20$$

b) $$\int_{\sqrt{2}/2}^1 f\left(\frac{1}{x^2}\right)dx \le \frac{5}{2}$$

c) $$\int_0^1 x^2 f(x+1)\,dx = 2$$

5.3.3. Suppose that f is integrable on $[0.5, 2]$ and that

$$\int_{0.5}^1 x^k f(x)\,dx = \int_1^2 x^k f(x)\,dx + 2k^2 = 3 + k^2$$

for $k = 0, 1, 2$. Compute the exact values of each of the following integrals.

a) $$\int_0^1 x^3 f(x^2+1)\,dx$$

b) $$\int_0^{\sqrt{3}/2} \frac{x^3}{\sqrt{1-x^2}} f\left(\sqrt{1-x^2}\right)dx$$

5.3.4. Suppose that f and g are differentiable on $[0, e]$ and that f' and g' are integrable on $[0, e]$.

a) If $\int_1^e f(x)/x\,dx < f(e)$, prove that

$$\int_1^e f'(x)\log x\,dx > 0.$$

b) If $f(0) = f(1) = 0$, prove that

$$\int_0^1 e^x(f(x) + f'(x))\,dx = 0.$$

c) If $0 \in \{f(0), g(0)\} \cap \{f(e), g(e)\}$, prove that

$$\int_0^e f(x)g'(x)\,dx = -\int_0^e g(x)f'(x)\,dx.$$

5.3.5. Use the First Mean Value Theorem for Integrals to prove the following version of the Mean Value Theorem for Derivatives. If $f \in \mathcal{C}^1[a, b]$, then there is an $x_0 \in [a, b]$ such that

$$f(b) - f(a) = (b - a)f'(x_0).$$

5.3.6. If f is continuous on $[a, b]$ and there exist numbers $\alpha \neq \beta$ such that

$$\alpha \int_a^c f(x)\,dx + \beta \int_c^b f(x)\,dx = 0$$

holds for all $c \in (a, b)$, prove that $f(x) = 0$ for all $x \in [a, b]$.

5.3.7. **This exercise is used in Sections 5.4 and 6.1.** Define $L : (0, \infty) \to \mathbf{R}$ by

$$L(x) = \int_1^x \frac{dt}{t}.$$

a) Prove that L is differentiable and strictly increasing on $(0, \infty)$, with $L'(x) = 1/x$ and $L(1) = 0$.
b) Prove that $L(x) \to \infty$ as $x \to \infty$ and $L(x) \to -\infty$ as $x \to 0+$. (You may wish to prove

$$L(2^n) = \sum_{k=1}^{n} \int_{2^{k-1}}^{2^k} \frac{dt}{t} > \sum_{k=1}^{n} 2^{-k}\left(2^k - 2^{k-1}\right) = \frac{n}{2}$$

for all $n \in \mathbf{N}$.)
c) Using the fact that $(x^q)' = qx^{q-1}$ for $x > 0$ and $q \in \mathbf{Q}$ (see Exercise 4.2.7), prove that $L(x^q) = qL(x)$ for all $q \in \mathbf{Q}$ and $x > 0$.
d) Prove that $L(xy) = L(x) + L(y)$ for all $x, y \in (0, \infty)$.
e) Suppose that $e = \lim_{n\to\infty}(1 + 1/n)^n$ exists. (It does—see Example 4.22.) Use l'Hôpital's Rule to show that $L(e) = 1$. [$L(x)$ is the *natural logarithm* function $\log x$.]

5.3.8. **This exercise was used in Section 4.3.** Let $E = L^{-1}$, where L is defined in Exercise 5.3.7.

a) Use the Inverse Function Theorem to show that E is differentiable and strictly increasing on $\mathbf{R}$ with $E'(x) = E(x)$, $E(0) = 1$, and $E(1) = e$.
b) Prove that $E(x) \to \infty$ as $x \to \infty$ and $E(x) \to 0$ as $x \to -\infty$.
c) Prove that $E(xq) = (E(x))^q$ and $E(q) = e^q$ for all $q \in \mathbf{Q}$ and $x \in \mathbf{R}$.
d) Prove that $E(x + y) = E(x)E(y)$ for all $x, y \in \mathbf{R}$.
e) For each $\alpha \in \mathbf{R}$ define $e^\alpha = E(\alpha)$. Let $x > 0$ and define $x^\alpha = e^{\alpha \log x} := E(\alpha L(x))$. Prove that $0 < x < y$ implies $x^\alpha < y^\alpha$ for $\alpha > 0$ and $x^\alpha > y^\alpha$ for $\alpha < 0$. Also prove that

$$x^{\alpha+\beta} = x^\alpha x^\beta, \qquad x^{-\alpha} = \frac{1}{x^\alpha}, \quad \text{and} \quad (x^\alpha)' = \alpha x^{\alpha-1}$$

for all $\alpha, \beta \in \mathbf{R}$ and $x > 0$.

5.3.9. Suppose that $f : [a, b] \to \mathbf{R}$ is continuously differentiable and 1–1 on $[a, b]$. Prove that

$$\int_a^b f(x)\,dx + \int_{f(a)}^{f(b)} f^{-1}(x)\,dx = bf(b) - af(a).$$

5.3.10. Suppose that ϕ is $\mathcal{C}^1$ on $[a, b]$ and f is integrable on $[c, d] := \phi[a, b]$. If ϕ' is never zero on $[a, b]$, prove that $f \circ \phi$ is integrable on $[a, b]$.

5.3.11. Let $q \in \mathbf{Q}$. Suppose that $a < b$, $0 < c < d$, and that $f : [a, b] \to [c, d]$. If f is integrable on $[a, b]$, then prove that

$$\left(\int_a^x f^q(t)\,dt\right)' = f^q(x)$$

for all $x \in [a, b]$.

5.3.12. For each $n \in \mathbf{N}$, define

$$a_n := \left(\frac{(2n)!}{n!n^n}\right)^{1/n}.$$

Prove that $a_n \to 4/e$.

5.4 IMPROPER RIEMANN INTEGRATION

To extend the Riemann integral to unbounded intervals or unbounded functions, we begin with an elementary observation.

5.37 Remark. *If f is integrable on* $[a, b]$, *then*

$$\int_a^b f(x)\,dx = \lim_{c\to a+}\left(\lim_{d\to b-}\int_c^d f(x)\,dx\right).$$

Proof. By Theorem 5.26,

$$F(x) = \int_a^x f(t)\,dt$$

is continuous on $[a, b]$. Thus

$$\begin{aligned}\int_a^b f(x)\,dx = F(b) - F(a) &= \lim_{c\to a+}(\lim_{d\to b-}(F(d) - F(c)))\\ &= \lim_{c\to a+}\left(\lim_{d\to b-}\int_c^d f(x)\,dx\right).\end{aligned}$$

∎

This leads to the following generalization of the Riemann integral.

To show that $\sin x/x$ is not absolutely integrable on $[1, \infty)$, notice that

$$\begin{aligned}
\int_1^{n\pi} \frac{|\sin x|}{x}\,dx &\geq \sum_{k=2}^{n} \int_{(k-1)\pi}^{k\pi} \frac{|\sin x|}{x}\,dx \\
&\geq \sum_{k=2}^{n} \frac{1}{k\pi} \int_{(k-1)\pi}^{k\pi} |\sin x|\,dx \\
&= \sum_{k=2}^{n} \frac{2}{k\pi} = \frac{2}{\pi} \sum_{k=2}^{n} \frac{1}{k}
\end{aligned}$$

for each $n \in \mathbf{N}$. Since

$$\sum_{k=2}^{n} \frac{1}{k} \geq \sum_{k=2}^{n} \int_k^{k+1} \frac{1}{x}\,dx = \int_2^{n+1} \frac{1}{x}\,dx = \log(n+1) - \log 2 \to \infty$$

as $n \to \infty$, it follows from the Squeeze Theorem that

$$\lim_{n\to\infty} \int_1^{n\pi} \frac{|\sin x|}{x}\,dx = \infty.$$

Thus, $\sin x/x$ is not absolutely integrable on $[1, \infty)$. ■

EXERCISES

5.4.0. Suppose that $a < b$. Decide which of the following statements are true and which are false. Prove the true ones and give counterexamples for the false ones.

a) If f is bounded on $[a, b]$, if g is absolutely integrable on (a, b), and if $|f(x)| \leq g(x)$ for all $x \in (a, b)$, then f is absolutely integrable on (a, b).

b) Suppose that h is absolutely integrable on (a, b). If f is continuous on (a, b), if g is continuous and never zero on $[a, b]$, and if $|f(x)| \leq h(x)$ for all $x \in [a, b]$, then f/g is absolutely integrable on (a, b).

c) If $f : (a, b) \to [0, \infty)$ is continuous and absolutely integrable on (a, b) for some $a, b \in \mathbf{R}$, then $\sqrt{f}$ is absolutely integrable on (a, b).

d) If f and g are absolutely integrable on (a, b), then $\max\{f, g\}$ and $\min\{f, g\}$ are both absolutely integrable on (a, b).

5.4.1. Evaluate the following improper integrals.

a) $$\int_1^{\infty} \frac{1+x}{x^3}\,dx$$

b) $$\int_{-\infty}^{0} x^2 e^{x^3}\,dx$$

c) $$\int_0^{\pi/2} \frac{\cos x}{\sqrt[3]{\sin x}}\, dx$$

d) $$\int_0^1 \log x\, dx$$

5.4.2. For each of the following, find all values of $p \in \mathbf{R}$ for which f is improperly integrable on I.

a) $f(x) = 1/x^p$, $I = (1, \infty)$
b) $f(x) = 1/x^p$, $I = (0, 1)$
c) $f(x) = 1/(x \log^p x)$, $I = (e, \infty)$
d) $f(x) = 1/(1 + x^p)$, $I = (0, \infty)$
e) $f(x) = \log^a x / x^p$, where $a > 0$ is fixed, and $I = (1, \infty)$

5.4.3. Let $p > 0$. Show that $\sin x / x^p$ is improperly integrable on $[1, \infty)$ and that $\cos x / \log^p x$ is improperly integrable on $[e, \infty)$.

5.4.4. Decide which of the following functions are improperly integrable on I.

a) $f(x) = \sin x$, $I = (0, \infty)$
b) $f(x) = 1/x^2$, $I = [-1, 1]$
c) $f(x) = x^{-1} \sin(x^{-1})$, $I = (1, \infty)$
d) $f(x) = \log(\sin x)$, $I = (0, 1)$
e) $f(x) = (1 - \cos x)/x^2$, $I = (0, \infty)$

5.4.5. Use the examples provided by Exercise 5.4.2b to show that the product of two improperly integrable functions might not be improperly integrable.

5.4.6. Suppose that f, g are nonnegative and locally integrable on $[a, b)$ and that

$$L := \lim_{x \to b-} \frac{f(x)}{g(x)}$$

exists as an extended real number.

a) Show that if $0 \le L < \infty$ and g is improperly integrable on $[a, b)$, then so is f.
b) Show that if $0 < L \le \infty$ and g is not improperly integrable on $[a, b)$, then neither is f.

5.4.7. a) Suppose that f is improperly integrable on $[0, \infty)$. Prove that if $L = \lim_{x\to\infty} f(x)$ exists, then $L = 0$.

b) Let

$$f(x) = \begin{cases} 1 & n \le x < n + 2^{-n},\ n \in \mathbf{N} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that f is improperly integrable on $[0, \infty)$ but $\lim_{x\to\infty} f(x)$ does not exist.

5.4.8. Prove that if f is absolutely integrable on $[1, \infty)$, then

$$\lim_{n\to\infty} \int_1^\infty f(x^n)\, dx = 0.$$

5.4.9. Assuming $e = \lim_{n\to\infty} \sum_{k=0}^n 1/k!$ (see Example 7.45), prove that

$$\lim_{n\to\infty} \left(\frac{1}{n!} \int_1^\infty x^n e^{-x}\, dx \right) = 1.$$

5.4.10. a) Prove that

$$\int_0^{\pi/2} e^{-a \sin x}\, dx \le \frac{2}{a}$$

for all $a > 0$.

b) What happens if $\cos x$ replaces $\sin x$?

*5.5 FUNCTIONS OF BOUNDED VARIATION

This section uses no material from any other enrichment section.

In this section we study functions which do not wiggle too much. These functions, which play a prominent role in the theory of Fourier series (see Sections *14.3 and *14.4) and probability theory, are important tools for theoretical as well as applied mathematics.

Let $\phi : [a, b] \to \mathbf{R}$. To measure how much ϕ wiggles on an interval $[a, b]$, set

$$V(\phi, P) = \sum_{j=1}^n |\phi(x_j) - \phi(x_{j-1})|$$

for each partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$. The *variation* of ϕ is defined by

$$\mathrm{Var}(\phi) := \sup\{V(\phi, P) : P \text{ is a partition of } [a, b]\}. \tag{19}$$

5.50 Definition.

Let $[a, b]$ be a closed, nondegenerate interval and $\phi : [a, b] \to \mathbf{R}$. Then ϕ is said to be of *bounded variation* on $[a, b]$ if and only if $\mathrm{Var}(\phi) < \infty$.

The following three remarks show how the collection of functions of bounded variation is related to other collections of functions we have studied.

5.51 Remark. *If $\phi \in \mathcal{C}^1[a, b]$, then ϕ is of bounded variation on $[a, b]$. However, there exist functions of bounded variation which are not continuously differentiable.*

Proof. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. By the Extreme Value Theorem, there is an $M > 0$ such that $|\phi'(x)| \le M$ for all $x \in [a, b]$. Therefore,